ADVANCED MATHEMATICS FOR ECONOMICS - 2013/2014

Sheet 2: Difference equations

1 Classify the following difference equations

(a) $x_{t+1} = x_t^2 - e^t$;
(b) $x_{t+1} = x_t - e^t$;
(c) $x_{t+1} = 3.2x_t(1 - 0.25x_t)$;
(d) $x_{t+1} - x_t = -\frac{4}{3}x_t$;
(e) $x_{t+1}(2 + 3x_t) = 4x_t$;
(f) $x_{t+2} = 3x_{t+1} - x_t + t$;
(g) $x_{t+4} - x_{t+3} = \sqrt[3]{x_{t+1}}$.

2 Check that the following sequences are solution of the corresponding difference equation

(a) $x_t = 2^t$; $x_{t+2} = x_{t+1} + 2x_t$;
(b) $x_t = \frac{t(t+1)}{2}$; $x_{t+1} = x_t + t + 1$;
(c) $x_t = \cos \pi t$; $x_{t+1} = -x_t$;
(d) $x_t = 2^t + 1$; $x_{t+2} - 3x_{t+1} + 2x_t = 0$;
(e) $x_t = C_1 + C_2 2^t - t$; $x_{t+2} - 3x_{t+1} + 2x_t = 1$.

3 Consider the difference equation $x_{t+1} = \sqrt{x_t - 1}$ with $x_0 = 5$. Compute x_1, x_2. and x_3. What about x_4?

4 Find the solutions of the following difference equations:

(a) $x_{t+1} = 2x_t + 4$, $x_0 = 1$;
(b) $2x_{t+1} + 3x_t + 2 = 0$;
(c) $x_{t+1} - x_t = -\frac{4}{3}x_t$;
(d) $x_{t+2} - 6x_{t+1} + 9x_t = 0$;
(e) $x_{t+2} - 5x_{t+1} = -6x_t$;
(f) $x_{t+2} - 3x_{t+1} + 2x_t = 0$;
(g) $x_{t+2} = -3x_{t+1} - 3x_t = 0$;
(h) $x_{t+4} - 2x_{t+2} + x_t = 0$;
(i) $x_{t+3} = x_t$;

For Cases (a), (b), and (c) obtain the particular solutions when the initial conditions are $x_0 = 1$, $x_0 = -1$, and $x_0 = 3$, respectively. Which is their long run behavior?

5 Find the solution of the following equations

(a) $x_{t+2} - 4x_{t+1} + 4x_t = 3^t$
(b) $x_{t+2} - 2x_{t+1} + 2x_t = t$
(c) $x_{t+2} - 4x_{t+1} + 3x_t = 2^t + t$
(d) $x_{t+2} - 4x_{t+1} + 4x_t = 2^t$
(e) $x_{t+3} - x_{t+2} + x_{t+1} - x_t = 2$
(f) $x_{t+3} - 8x_t = t^2$

(g) $24x_{t+3} - 14x_{t+2} - 9x_{t+1} - x_t = t + 3^t$

(h) $x_{t+4} - x_t = 2^t + 3^t$

(i) $x_{t+3} + x_{t+2} - 5x_{t+1} + 3x_t = t^2 + 1$

(j) $x_{t+2} - 25x_t = t^2 + 1$

6 Investigate the stability of the following equations: (a) $x_{t+1} - \frac{1}{4}x_t = b_t$, (b) $x_{t+2} - x_{t+1} + x_t = c_t$, where $\{b_t\}$ and $\{c_t\}$ are given sequences.

7 Solve the Fibonacci equation $x_{t+2} = x_{t+1} + x_t$, $x_0 = x_1 = 1$ and check that

$$\lim_{t\to\infty} \frac{x_{t+1}}{x_t} = \frac{1+\sqrt{5}}{2} \equiv \varphi, \quad \text{the golden section.}$$

8 The income Y_t evolves according to the equation

$$Y_{t+1} = C_t + I_t,$$

where I_t denotes investment and C_t is consumption. Supposing that $C_t = mY_t + c$, with $0 \leq m < 1$, $c > 0$, and that $I_t = I$ is constant, find a difference equation for income Y_t, solve it, and study the long run behavior of the solution.

9 Let S_0 denotes an initial sum of money. There are two basic methods for computing the interest earned in a period, for example, one year:

(a) S_0 earns *simple interest* at rate r if each period the interest equals a fraction r of S_0.

(b) S_0 earns *compound interest* at rate r if each period the interest equals a fraction r of the sum accumulated at the beginning of that period.

Find a difference equation for the two models above, and find the solution.

10 Given demand and supply for the cobweb model as follows, find the intertemporal equilibrium price, and determine whether the equilibrium is stable:

(a) $Q_d = 18 - 3P$, $Q_s = -3 + 4P$;

(b) $Q_d = 22 - 3P$, $Q_s = -2 + P$;

(c) $Q_d = 16 - 6P$, $Q_s = 6P - 5$;

11 Consider the equation obtained in the multiplier–accelerator model of growth studied in the class notes,

$$Y_{t+2} - a(1+c)Y_{t+1} + acY_t = b,$$

with $a > 0$, $c > 0$ and $a \neq 1$.

(a) Find a particular solution of this equation;

(b) Discuss whether the solutions of the characteristic equation are real or complex.

(c) Find the general solution in each of the following cases.

i. $a = 4$, $c = 1$;

ii. $a = \frac{3}{4}$, $c = 3$;

iii. $a = 0.5$, $c = 1$.

12 Let C_t denotes consumption, K_t capital stock, Y_t net national product. We suppose that these variables are related as

$$C_t = cY_{t-1},$$
$$K_t = \sigma Y_{t-1},$$
$$Y_t = C_t + K_t - K_{t-1},$$

where c and σ are positive constants.

(a) Give an economic interpretation of the equations.

(b) Derive a second order difference equation for Y_t.

(c) Find necessary and sufficient conditions for the solution of the equation in (b) to have explosive oscillations.

13 Solve the systems of difference equations $X_{t+1} = AX_t + B$, where the matrix A and the vector B are given below:

(a) $A = \begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, B = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

(c) $A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

(d) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(e) $A = \begin{pmatrix} 0 & -1 & 3 \\ -3 & 2 & 3 \\ 1 & -1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

14 For Cases (a), (b), and (c) of the previous exercise, obtain the solutions when $X_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

15 K. is a student with the following habit: once she studies one day, it is likely that she will not study the following day with probability 0.7. On the other hand, the probability that K. did not study two consecutive days is 0.6. Assuming that today K. has promised to study, with which probability does K. study in the long run?

16 A psychologist places a mouse inside a jail with two doors, A and B. Going through door A, the mouse receive an electrical shock, but some food is waiting behind door B. At the beginning of the experiment (Monday), the mouse chooses A or B with the same probability. After choosing B, the probability of returning to B in the following day is 0.6.

(a) With which probability does the mouse choose door A on Thursday?

(b) Which is the stationary distribution of this experiment?

(c) What do you think the mouse thinks about the psychologist?

17 Prove that a quadratic equation $\lambda^2 - p\lambda + q = 0$ has roots satisfying $|\lambda| < 1$ iff

$$|p| < 1 + q \text{ and } q < 1.$$

The condition is called the *Jury condition*.

18 **Phillips curve I**.

The *Phillips curve* relates negatively the rate of growth of money wage w and the unemployment rate U,

$$w = f(U), \qquad f'(U) < 0. \tag{1}$$

This was justified empirically by A.W. Phillips for the U.K. in a very influential paper[1]. Later, the relation was postulated to affect also to the rate of inflation, p, since a growing–money wage

[1] A.W. Phillips (1956) "The relationship between unemployment and the rate of change of money wage rates in the United Kingdom," *Economica*, November 1958, pp. 283–299.

cots would had inflationary effects[2],

$$p = w - T.$$

Here, T denotes an exogenous increase in labor productivity (hence inflation appears only if the salary grows faster than productivity). Assuming a linear form of function f, $f(U) = \alpha - \beta U$, we will have that at every $t \geq 1$

$$p_t = \alpha - T - \beta U_t, \qquad \alpha, \beta > 0. \tag{2}$$

On the other hand, the theory links the unemployment rate and the rate of inflation according to

$$U_{t+1} - U_t = -k(m - p_t), \qquad 0 < k \leq 1, \tag{3}$$

where m is the rate of growth of the nominal money balance[3]. Noticing that $m - p$ is the rate of growth of real money, Eqn. (3) establishes that the rate of growth of unemployment is negatively related with the rate of growth of real money.

Find a difference equation for U_t and study the stability properties of the solution.

19 **Phillips curve II**.

Continuing with the Phillips' model, we analyze now the modification introduced by Friedman[4], considering the *expected–augmented* version of the Phillips relation

$$w = f(U) + g\pi, \qquad (0 < g \leq 1), \tag{4}$$

where π denotes the expected rate of inflation. The idea is that if an inflationary trend has been observed long enough, people form certain inflation expectations, which they attempt to incorporate into their money–wage demands. Then, (2) results in the equation

$$p_t = \alpha - T - \beta U_t + g\pi_t, \quad t \geq 0. \tag{5}$$

How is formed inflation expectations? Commonly is is assumed the *adaptive expectations* hypothesis

$$\pi_{t+1} - \pi_t = j(p_t - \pi_t), \qquad 0 < j \leq 1. \tag{6}$$

This means that when the actual rate of inflation p turns out to exceed the expected rate π, the latter, having now been proven to be too low, is revised upward. Conversely, if p falls short of π, then π is revised in the downward direction. The speed of adjustment is j.

Consider the model given by Eqs. (3), (5) and (6).

(a) Eliminate p_t and write a system of linear difference equations for the variables (U_t, π_t).

(b) Using the Jury condition, determine whether the system is g.a.s.

(c) Find and interpret the fixed or equilibrium points of the system.

[2] The rate of growth of money wage is $(W_{t+1} - W_t)/W_t$, where W_t is wage at time t; the rate of inflation is the rate of the general price level, $p = (P_{t+1} - P_t)/P_t$.

[3] That is, $m = (M_{t+1} - M_t)/M_t$, where M_t is the nominal money balance, fixed by the monetary authority, and supposed here to be constant, independent of t.

[4] M. Friedman (1968) "The role of monetary policy," *American Economic Review*, pp. 1–17.