Topic 1: Matrix diagonalization

1. Review of Matrices and Determinants

Definition 1.1. A matrix is a rectangular array of real numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}.$$

The matrix is said to be of order $n \times m$ if it has n rows and m columns. The set of matrices of order $n \times m$ will be denoted $M_{n\times m}$.
The element a_{ij} belongs to the ith row and to the jth column. Most often we will write in abbreviated form $A = (a_{ij})^{i=1,\dots,n}_{j=1,\dots,m}$ or even $A = (a_{ij})$.
The *main* or *principal*, *diagonal* of a matrix is the diagonal from the upper left– to the lower right–hand corner.

Definition 1.2. The transpose of a matrix A, denoted A^T, is the matrix formed by interchanging the rows and columns of A

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix} \in M_{m\times n}.$$

We can define two operations with matrices, *sum* and *multiplication*. The main **properties of these operations as well as transposition** are the following. It is assumed that the matrices in each of the following laws are such that the indicated operation can be performed and that $\alpha, \beta \in \mathbb{R}$.

(1) $(A^T)^T = A$.
(2) $(A + B)^T = A^T + B^T$.
(3) $A + B = B + A$ (commutative law).
(4) $A + (B + C) = (A + B) + C$ (associative law).
(5) $\alpha(A + B) = \alpha A + \alpha B$.
(6) $(\alpha + \beta)A = \alpha A + \beta A$.
(7) Matrix multiplication is not always commutative, i.e., $AB \neq BA$.
(8) $A(BC) = (AB)C$ (associative law).
(9) $A(B + C) = AB + AC$ (distributive with respect to addition).

1.1. **Square matrices.** We are mainly interested in square matrices. A matrix is *square* if $n = m$. The *trace* of a square matrix A is the sum of its diagonal elements, $\operatorname{trace}(A) = \sum_{i=1}^{n} a_{ii}$.

Definition 1.3. The identity matrix of order n is

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

The square matrix of order n with all its entries null is the *null* matrix, and will be denoted O_n. It holds that $I_n A = A I_n = A$ and $O_n A = A O_n = O_n$.

Definition 1.4. A square matrix A is called regular or invertible if there exists a matrix B such that $AB = BA = I_n$. The matrix B is called the inverse of A and it is denoted A^{-1}.

Theorem 1.5. *The inverse matrix is unique.*

Uniqueness of A^{-1} can be easily proved. For, suppose that B is another inverse of matrix A. Then $BA = I_n$ and

$$B = BI_n = B(AA^{-1}) = (BA)A^{-1} = I_nA^{-1} = A^{-1},$$

showing that $B = A^{-1}$.

Some **properties of the inverse matrix** are the following. It is assumed that the matrices in each of the following laws are regular.

(1) $(A^{-1})^{-1} = A$.
(2) $(A^T)^{-1} = (A^{-1})^T$.
(3) $(AB)^{-1} = B^{-1}A^{-1}$.

1.2. **Determinants.** To a *square* matrix A we associate a real number called the *determinant*, $|A|$ or $\det(A)$, in the following way.

For a matrix of order 1, $A = (a)$, $\det(A) = a$.

For a matrix of order 2, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = ad - bc$.

For a matrix of order 3

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

This is known as the *expansion* of the determinant by the first column, but it can be done for any other row or column, giving the same result. Notice the sign $(-1)^{i+j}$ in front of the element a_{ij}. Before continuing with the inductive definition, let us see an example.

Example 1.6. Compute the following determinant expanding by the second column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 4 & 3 & 5 \\ 3 & 1 & 3 \end{vmatrix} = (-1)^{1+2}2\begin{vmatrix} 4 & 5 \\ 3 & 3 \end{vmatrix} + (-1)^{2+2}3\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} + (-1)^{2+3}1\begin{vmatrix} 1 & 1 \\ 4 & 5 \end{vmatrix}$$
$$= -2\cdot(-3) + 3\cdot(0) - (1)\cdot 1 = 5$$

For general n the method is the same that for matrices of order 3, expanding the determinant by a row or a column and reducing in this way the order of the determinants that must be computed. For a determinant of order 4 one has to compute 4 determinants of order 3.

Definition 1.7. Given a matrix A of order n, the complementary minor of element a_{ij} is the determinant of order $n-1$ which results from the deletion of the row i and the column j containing that element. The adjoint A_{ij} of the element a_{ij} is the minor multiplied by $(-1)^{i+j}$.

According to this definition, the determinant of matrix A can be defined as

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \qquad \text{(by row } i\text{)}$$

or, equivalently

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \qquad \text{(by column } j\text{).}$$

Example 1.8. Find the value of the determinant

$$\begin{vmatrix} 1 & 2 & 0 & 3 \\ 4 & 7 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 0 & 2 & 0 & 7 \end{vmatrix}.$$

Answer: Expanding the determinant by the third column, one gets

$$\begin{vmatrix} 1 & 2 & 0 & 3 \\ 4 & 7 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 0 & 2 & 0 & 7 \end{vmatrix} = (-1)^{3+2}2 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 0 & 2 & 7 \end{vmatrix} + (-1)^{3+3}3 \begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 1 \\ 0 & 2 & 7 \end{vmatrix}.$$

The main **properties of the determinants** are the following. It is assumed that the matrices A and B in each of the following laws are square of order n and $\lambda \in \mathbb{R}$.

(1) $|A| = |A^T|$.
(2) $|\lambda A| = \lambda^n |A|$.
(3) $|AB| = |A||B|$.
(4) A matrix A is regular if and only if $|A| \neq 0$; in this case $|A^{-1}| = \frac{1}{|A|}$.
(5) If in a determinant two rows (or columns) are interchanged, the value of the determinant is changed in sign.
(6) If two rows (columns) in a determinant are identical, the value of the determinant is zero.
(7) If all the entries in a row (column) of a determinant are multiplied by a constant λ, then the value of the determinant is also multiplied by this constant.
(8) In a given determinant, a constant multiple of the elements in one row (column) may be added to the elements of another row (column) without changing the value of the determinant.

The next result is very useful to check if a given matrix is regular or not.

Theorem 1.9. *A square matrix A has an inverse if and only $|A| \neq 0$.*

2. Diagonalization of matrices

Definition 2.1. Two matrices A and B of order n are similar if there exists a matrix P such that

$$B = P^{-1}AP.$$

Definition 2.2. A matrix A is diagonalizable if it is similar to a diagonal matrix D, that is, there exists D diagonal and P invertible such that $D = P^{-1}AP$.

Of course, D diagonal means that every element out of the diagonal is null

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}, \quad \lambda_1, \dots, \lambda_n \in \mathbb{R}.$$

Proposition 2.3. *If A is diagonalizable, then for all $m \geq 1$*

$$A^m = PD^mP^{-1}, \tag{2.1}$$

where

$$D^m = \begin{pmatrix} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lambda_2^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^m \end{pmatrix}.$$

Proof. Since A is diagonalizable

$$\begin{aligned} A^m &= (PDP^{-1})(PDP^{-1}) \overset{m}{\cdots} (PDP^{-1}) \\ &= PD(P^{-1}P)D \cdots D(P^{-1}P)DP^{-1} \\ &= PDI_nD \cdots DI_nDP^{-1} = PD^mP^{-1}. \end{aligned}$$

The expression for D^m is readily obtained by induction on m. □

Example 2.4. At a given date, instructor X can teach well or teach badly. After a good day, the probability of doing well for the next class is 1/2, whilst after a bad day, the probability of doing well is 1/9. Let g_t (b_t) the probability of good (poor) teaching at day t. Suppose that at time $t = 1$ the class has been right, that is, $g_1 = 1$, $b_1 = 0$. Which is the probability that the 5th class go fine (bad)?
ANSWER: The data lead to the following equations that relate the probability of a good/bad class with the performance showed by the teacher the day before

$$g_{t+1} = \frac{1}{2}g_t + \frac{1}{9}b_t,$$

$$b_{t+1} = \frac{1}{2}g_t + \frac{8}{9}b_t.$$

In matrix form

$$\begin{pmatrix} g_{t+1} \\ b_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{9} \\ \frac{1}{2} & \frac{8}{9} \end{pmatrix} \begin{pmatrix} g_t \\ b_t \end{pmatrix}.$$

Obviously

$$\begin{pmatrix} g_5 \\ b_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{9} \\ \frac{1}{2} & \frac{8}{9} \end{pmatrix}^4 \begin{pmatrix} g_1 \\ b_1 \end{pmatrix}.$$

If the matrix were diagonalizable and we could find matrices P and D, then the computation of the 10th power of the matrix would be easy using Proposition 2.3. We will come back to this example afterwards.

Definition 2.5. Let A be a matrix of order n. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of A and that $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} \neq \mathbf{0}$, is an eigenvector of A associated to λ if

$$A\mathbf{u} = \lambda\mathbf{u}.$$

The set of eigenvalues of A, $\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}$, is called the spectrum of A. The set of all eigenvectors of A associated to the same eigenvalue λ, including the null vector, is denoted $\mathrm{S}(\lambda)$, and is called the eigenspace or proper subspace associated to λ.

The following result shows that an eigenvector can only be associated to a unique eigenvalue.

Proposition 2.6. *Let* $\mathbf{0} \neq \mathbf{u} \in \mathrm{S}(\lambda) \cap \mathrm{S}(\mu)$. *Then* $\lambda = \mu$.

Proof. Suppose $\mathbf{0} \neq \mathbf{u} \in \mathrm{S}(\lambda) \cap \mathrm{S}(\mu)$. Then

$$A\mathbf{u} = \lambda\mathbf{u}$$
$$A\mathbf{u} = \mu\mathbf{u}.$$

Subtracting both equations we obtain $\mathbf{0} = (\lambda - \mu)\mathbf{u}$ and, since $\mathbf{0} \neq \mathbf{u}$, we must have $\lambda = \mu$. □

Recall that for an arbitrary matrix A, *the rank of the matrix* is the number of linearly independent columns or rows (both numbers necessarily coincide). It is also given by the order of the largest non null minor of A.

Theorem 2.7. *The real number* λ *is an eigenvalue of* A *if and only if*

$$|A - \lambda I_n| = 0.$$

Moreover, $\mathrm{S}(\lambda)$ *is the set of solutions (including the null vector) of the linear homogeneous system*

$$(A - \lambda I_n)\mathbf{u} = \mathbf{0},$$

and hence it is a vector subspace, which dimension is

$$\dim \mathrm{S}(\lambda) = n - \mathrm{rank}(A - \lambda I_n).$$

Proof. Suppose that $\lambda \in \mathbb{R}$ is an eigenvalue of A. Then the system $(A-\lambda I_n)\mathbf{u} = \mathbf{0}$ admits some non–trivial solution $\mathbf{u}$. Since the system is homogeneous, this implies that the determinant of the system is zero, $|A - \lambda I_n| = 0$. The second part about $\mathrm{S}(\lambda)$ follows also from the definition of eigenvector, and the fact that the set of solutions of a linear homogenous system is a subspace (the sum of two solutions is again a solution, as well as it is the product of a real number by a solution). Finally, the dimension of the space of solutions is given by the Theorem of Rouche–Frobenius. □

Definition 2.8. The characteristic polynomial of A is the polynomial of order n given by

$$p_A(\lambda) = |A - \lambda I_n|.$$

Notice that the eigenvalues of A are the real roots of p_A. This polynomial is of degree n. The Fundamental Theorem of Algebra estates that a polynomial of degree n has n complex roots (not necessarily different, some of the roots may have multiplicity grater than one). It could be the case that some of the roots of p_A were not real numbers. For us, a root of $p_A(\lambda)$ which is not real is not an eigenvalue of A.

Example 2.9. Find the eigenvalues and the proper subspaces of

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Answer:

$$A - \lambda I = \begin{pmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}; \quad p(\lambda) = (1-\lambda)\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2+1).$$

The characteristic polynomial has only one real root, hence the spectrum of A is $\sigma(A) = \{1\}$. The proper subspace $\mathrm{S}(1)$ is the set of solutions of the homogeneous linear system $(A - I_3)\mathbf{u} = \mathbf{0}$, that is, the set of solutions of

$$(A - I_3)\mathbf{u} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the above system we obtain

$$\mathrm{S}(1) = \{(0,0,z) \,:\, z \in \mathbb{R}\} =< (0,0,1) > \text{ (the subspace generated by } (0,0,1)).$$

Notice that $p_A(\lambda)$ has other roots that are not reals. They are the *complex* numbers $\pm i$, that are not (real) eigenvalues of A. If we would admit complex numbers, then they would be eigenvalues of A in this extended sense.

Example 2.10. Find the eigenvalues and the proper subspaces of

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}.$$

Answer: The eigenvalues are obtained solving

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0.$$

The solutions are $\lambda = 3$ (simple root) and $\lambda = 2$ (double root). To find $\mathrm{S}(3) = \{\mathbf{u} \in \mathbb{R}^3 \,:\, (B-3I_3)\mathbf{u} = \mathbf{0}\}$ we compute the solutions to

$$(B - 3I_3)\mathbf{u} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which are $x = y$ and $z = -2y$, and hence $\mathrm{S}(3) =< (1, 1, -2) >$. To find $\mathrm{S}(2)$ we solve the system

$$(B - 2I_3)\mathbf{u} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

from which $x = y = 0$ and hence $\mathrm{S}(2) =< (1, 0, 0) >$.

Example 2.11. Find the eigenvalues and the proper subspaces of

$$C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}$$

Answer: To compute the eigenvalues we solve the characteristic equation

$$0 = |C - \lambda I_3| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 0-\lambda \end{vmatrix}$$

$$= |2 - \lambda| \begin{vmatrix} 1-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(3-\lambda)$$

So, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. We now compute the eigenvectors. The eigenspace $\mathrm{S}(1)$ is the solution of the homogeneous linear system whose associated matrix is $C - \lambda I_3$ with $\lambda = 1$. That is, $\mathrm{S}(1)$ is the solution of the following homogeneous linear system

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the above system we find that

$$\mathrm{S}(1) = \{(-2z, 0, z) : z \in \mathbb{R}\} =< (-2, 0, 1) >$$

On the other hand, $\mathrm{S}(2)$ is the set of solutions of the homogeneous linear system whose associated matrix is $C - \lambda I_3$ with $\lambda = 2$. That is, $\mathrm{S}(2)$ is the solution of the following

$$\begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$$\mathrm{S}(2) = \{(2y, y, -3y) : y \in \mathbb{R}\} =< (2, 1, -3) >$$

Finally, $\mathrm{S}(3)$ is the set of solutions of the homogeneous linear system whose associated matrix is $A - \lambda I_3$ with $\lambda = 3$. That is, $\mathrm{S}(3)$ is the solution of the following

$$\begin{pmatrix} -2 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and we obtain

$$\mathrm{S}(3) = \{(0, 0, z) : z \in \mathbb{R}\} =< (0, 0, 1) >$$

We now start describing the *procedure to diagonalize a matrix*. Fix a square matrix A. Let

$$\lambda_1, \lambda_2, \ldots, \lambda_k$$

be distinct *real* roots of the characteristic polynomial $p_A(\lambda)$ an let m_k be the multiplicity of each λ_k (Hence $m_k = 1$ if λ_k is a simple root, $m_k = 2$ if it is double, etc.). Note that $m_1 + m_2 + \cdots + m_k \leq n$.

The following result estates that the number of independent vectors in the subspace $\mathrm{S}(\lambda)$ can never be bigger than the multiplicity of λ.

Proposition 2.12. *For each* $j = 1, \ldots, k$

$$1 \leq \dim \mathrm{S}(\lambda_j) \leq n_j.$$

The following theorem gives necessary and sufficient conditions for a matrix A to be diagonalizable.

Theorem 2.13. *A matrix A is diagonalizable if and only if the two following conditions hold.*

(1) Every root, $\lambda_1, \lambda_2, \ldots, \lambda_k$ of the charateristic polynomial $p_A(\lambda)$ is real.
(2) For each $j = 1, \ldots, k$

$$\dim \mathrm{S}(\lambda_j) = n_j.$$

Corollary 2.14. *If the matrix A has n distinct real eigenvalues, then it is diagonalizable.*

Theorem 2.15. *If A is diagonalizable, then the diagonal matrix D is formed by the eigenvalues of A in its main diagonal, with each λ_j repeated n_j times. Moreover, a matrix P such that $D = P^{-1}AP$ has as columns independent eigenvectors selected from each proper subspace $\mathrm{S}(\lambda_j)$, $j = 1, \ldots, k$.*

Comments on the examples above.

- Matrix A of Example 2.9 is not diagonalizable, since p_A has complex roots.
- Although all roots of p_B are real, B of Example 2.10 is not diagonalizable, because $\dim \mathrm{S}(2) = 1$, which is smaller than the multiplicity of $\lambda = 2$.
- Matrix C of Example 2.11 is diagonalizable, since p_C has 3 different real roots. In this case

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad P = \begin{pmatrix} -2 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix}.$$

Example 2.16. Returning to Example 2.4, we compute

$$\begin{vmatrix} \frac{1}{2} - \lambda & \frac{1}{9} \\ \frac{1}{2} & \frac{8}{9} - \lambda \end{vmatrix} = 0,$$

or $18\lambda^2 - 25\lambda + 7 = 0$. We get $\lambda_1 = 1$ and $\lambda_2 = \frac{7}{18}$. Now, $\mathrm{S}(1)$ is the solution set of

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{9} \\ \frac{1}{2} & -\frac{1}{9} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We find $y = \frac{9}{2}x$, so that $\mathrm{S}(1) =< (2, 9) >$. In the same way, $\mathrm{S}(\frac{7}{18})$ is the solution set of

$$\begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We find $y = -x$, so that $\mathrm{S}(\frac{7}{18}) =< (1, -1) >$. Hence the diagonal matrix is

$$D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{7}{18} \end{pmatrix}$$

and

$$P = \begin{pmatrix} 2 & 1 \\ 9 & -1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{11} \begin{pmatrix} 1 & 1 \\ 9 & -2 \end{pmatrix}.$$

Thus,

$$A^n = \frac{1}{11} \begin{pmatrix} 2 & 1 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (\frac{7}{18})^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 9 & -2 \end{pmatrix}.$$

In particular, for $n = 4$ we obtain

$$A^4 = \begin{pmatrix} 0.1891 & 0.1802 \\ 0.8111 & 0.8198 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} g_4 \\ b_4 \end{pmatrix} = A^4 \begin{pmatrix} g_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0.1891 & 0.1802 \\ 0.8111 & 0.8198 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.1891 \\ 0.8111 \end{pmatrix}.$$

This means that probability that the 5th class goes right, conditioned to the event that the first class was also right is of 0.1891.
We can wonder what happens in the long run, that is, supposing that the course lasts forever (oh no!). In this case

$$\lim_{n\to\infty} A^n = P(\lim_{n\to\infty} D^n)P^{-1} = P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} \frac{2}{11} & \frac{2}{11} \\ \frac{9}{11} & \frac{9}{11} \end{pmatrix},$$

to find that the *stationary distribution of probabilities is*

$$\begin{pmatrix} g_\infty \\ b_\infty \end{pmatrix} = \begin{pmatrix} 0.1818 \\ 0.8182 \end{pmatrix}.$$