Differentiability of the Value Function without Interiority Assumptions *

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This paper studies first–order differentiability properties of the value function in concave dynamic programs. Motivated by economic considerations, we dispense with commonly imposed interiority assumptions. We suppose that the correspondence of feasible choices varies with the vector of state variables, and we allow the optimal solution to belong to the boundary of this correspondence. Under minimal assumptions we prove that the value function is continuously differentiable. We then discuss this result in the context of some economic models, and focus on some examples in which our assumptions are not met and the value function is not differentiable.

KEY WORDS: Constrained optimization, value and policy functions, differentiability, envelope theorem, shadow price.

1 Introduction

Dynamic optimization problems are often analyzed by the methods of dynamic programming which build on properties of the value and policy functions. Although these methods

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have been extensively studied, there is an important gap that places this methodology really behind the static theory of constrained optimization: General results on the differentiability of the value function have essentially been established for interior optimal solutions. This interiority condition is generally subsumed under the following two assumptions: (*i*) The optimal solution lies in the interior of the choice set [e.g., see Benveniste and Scheinkman (1979), and earlier, more limited results by Lucas (1978) and Mirman and Zilcha (1975)], and (*ii*) the choice set does not vary with the vector of states [e.g., see the seminal work of Danskin (1967), and Milgrom and Segal (2002) for further results and economic applications]. Both (*i*) and (*ii*) turn out to be mathematically equivalent if the constraint correspondence is continuous. For concave optimization the differentiability of the value function can then be established by a well–known static argument in which this function is defined as the envelope of differentiable short–run return functions. This static envelope construction breaks down for boundary solutions if the set of feasible choices varies with the vector of state variables. Indeed, in the absence of (*i*) and (*ii*) the derivative of the value function may involve an infinite sum of discounted marginal utilities and returns.

To circumvent the interiority condition we postulate three additional assumptions. First, some optimal choice must lie in the interior of the domain. Second, the matrix of partial derivatives of the saturated constraints must satisfy a full rank condition. The necessity of these two assumptions is well understood from the static theory. The third additional assumption is a new asymptotic condition on the behavior of discounted marginal utilities and returns, and in competitive economies it can be identified with uniqueness of a bubble term over a given equilibrium allocation. We show how our conditions can easily be checked in two specific examples: An extended version of the pure currency model of Lucas (1980), and an optimal growth model with irreversible investment. Under standard assumptions, in the growth economy the bubble term is always equal to zero, and in the monetary economy the bubble term can only be positive in the case of the Friedman rule or zero nominal interest rate.

We also discuss some other examples in the literature where differentiability fails, linking the lack of differentiability back to the necessary assumptions developed in the paper. Thomas and Worrall (1994) study a model of foreign direct investment and find that the value function may not be differentiable for some parameter values. What happens in this model is that the optimal choice may lie in the boundary of the domain.

In a related model by Kocherlakota (1996) the value function may also display some points of non–differentiability [Koeppl (2006)]. We provide a necessary and sufficient condition for the value function to be differentiable which is tied down to a certain number of constraints being binding. If too many constraints are binding then the constraint qualification is not satisfied. In this model it matters for the dynamics as to whether or not the value function is differentiable. Finally, we discuss a limiting case of an optimal growth model with irreversible investment in which the derivative of the value function becomes unbounded even though the production and utility functions have bounded derivatives.

To prove our differentiability result, we first establish a generalized envelope theorem. The main idea is as follows. For a concave function, a superdifferential always exists at interior points. Suppose further that a full rank condition *à la* Arrow–Hurwicz–Uzawa [see e.g. Takayama (1990)] applies as a constraint qualification for the optimal solution. Then, one can use a (generalized) first order condition with respect to tomorrow's choice of the state variable to find values for the Kuhn–Tucker multipliers of the constrained optimization problem at today's state. Our envelope theorem then shows that the superdifferential of the value function can be written in terms of the corresponding multipliers and the superdifferential of the value function at tomorrow's states.

This leads to two formulas for computing the derivative of the value function. First, if eventually (at some finite time T) the policy falls into the interior of the constraint correspondence, by Benveniste and Scheinkman (1979) the derivative of the value function always exists. Then one can iterate backwards to find the derivative of the value function at today's state. A special case of course is the Benveniste and Scheinkman (1979) envelope theorem where $T = 0$. Second, if a transversality condition on marginal utilities and returns holds, one can iterate forward indefinitely to obtain the value of the derivative. In both cases, the derivative will depend on the Kuhn–Tucker multipliers associated with the saturated constraints.

The importance of these results is clear. First, many problems in macroeconomic theory have relied on differentiability of the value function when the policy function does not fall into the interior of the constraint correspondence. As is well known, results in this literature depend crucially on this assumption (see e.g. Kocherlakota (1996)). Hence, having necessary and sufficient conditions for differentiability is important. Second, we show that differentiability of the value function implies that the Kuhn–Tucker multipliers must be unique for concave dynamic programming problems. This entails uniqueness of price systems in decentralized economies or asset pricing models. These shadow values appear as additional state variables in proofs of existence of Markov equilibria for dynamic games of monetary and fiscal policy [Kydland and Prescott (1980) and Phelan and Stacchetti (2001)] and for competitive economies with heterogeneous agents and market frictions [Miao and Santos (2005)].

The paper is structured as follows. In Section 2 we set out an abstract (reduced–form) optimization problem, and recall some basic results from dynamic programming. In Section 3 we present our main results on the differentiability of the value function. In Section 4 we consider several economic applications to illustrate the role of our main assumptions. The Appendix contains all the proofs which are not shown in the main text.

2 The Model and Preliminary Considerations

As in many other related papers, we lay out an abstract stochastic optimization framework that encompasses various economic applications. For convenience of the presentation, we follow closely the formulation of Stokey, Lucas and Prescott (1989) where the revelation of information is given by an exogenous stochastic process; however, our arguments can readily be modified to account for uncertainty as part of optimization.

2.1 Stochastic optimization

Time is discrete, $t = 0, 1, 2, \ldots$. The sequence of choice variables $\{x_t\}_{t\geq 0}$ belongs to a set $X \subset \mathbb{R}^n$, and the realizations of the exogenous stochastic process $\{z_t\}_{t\geq 0}$ lie in a space Z. Let $\mathcal{X}$ be the Borel σ–algebra of X and $\mathcal{Z}$ the σ–algebra of Z. The product space $X \times Z$ is the *state space*, an is endowed with the product σ–algebra $\mathcal{X} \times \mathcal{Z}$. We assume that X is convex with non–empty interior.

The primitive elements of our optimization problem are given by a *constraint correspondence* of feasible choices $\Gamma : X \times Z \longrightarrow 2^X$ with graph $\Omega = \text{Graph}(\Gamma)$, a *one–period return function* $U : \Omega \longrightarrow \mathbb{R}$, a *discount factor* $0 < \beta < 1$, and a *transition probability* or *stochastic kernel* $Q : Z \times \mathcal{Z} \longrightarrow [0, 1]$. We assume that Γ is continuous in x and compact–valued. Moreover, for each z the set $\Omega_z = \{(x, y) : y \in \Gamma(x, z)\}$ is convex, and mapping $U(\cdot, \cdot, z)$ is concave and continuous in (x, y). Transition probability Q satisfies the following standard conditions: For each fixed $z \in Z$ mapping $Q(z,\cdot) : \mathcal{Z} \longrightarrow \mathbb{R}$ is a probability measure, and for each fixed $B \in \mathcal{Z}$ mapping $Q(\cdot, B) : Z \longrightarrow \mathbb{R}$ is a measurable function.

All contingency plans $\{x_t\}_{t\geq 0}$ are constructed from the history of past realizations of the process $\{z_t\}_{t\geq 0}$ so that every choice vector x_{t+1} available at the end of time t must condition on all the information revealed up to that date. More specifically, let Z^t be the space of sequences $z^t = (z_1, z_2, \ldots, z_t)$. For each given z_0 transition probability Q induces a unique probability measure $\mu^t(z_0, \cdot)$ on the product σ–algebra of Z^t. A contingency plan $\{x_t\}_{t\geq 0}$ is feasible if $x_{t+1} : Z^t \longrightarrow X$ is a measurable function and $x_{t+1}(z^t) \in \Gamma(x_t(z^{t-1}), z_t)$, for $z^t \in Z^t$ and $t = 0, 1, \ldots$.

The stochastic optimization problem can be defined as follows: For each initial condition (x_0, z_0), find the *maximum value* $v(x_0, z_0)$ over the set of all feasible contingency plans $\{x_{t+1}(z^t)\}_{t\geq 0}$ for the following discounted infinite–horizon program

$$v(x_0, z_0) = \sup_{\{x_t\}_{t\geq 1}} \Big\{ U(x_0, x_1, z_0) + \sum_{t=1}^{\infty} \beta^t \int_{Z^t} U(x_t, x_{t+1}, z_t)\, \mu^t(z_0, dz^t) \Big\}. \tag{1}$$

We shall often identify optimization problem (1) with the collection of its primitive elements (Γ, U, β, Q).

2.2 Value and policy functions

Let us write *Bellman's equation*

$$v(x, z) = \sup_{y\in\Gamma(x,z)} \left\{ U(x, y, z) + \beta \int_Z v(y, z')\, Q(z, dz') \right\}. \tag{2}$$

As is well known under standard assumptions the *value function* $v(x, z)$ is measurable, continuous and concave in x for each given z, and the unique fixed point of Bellman's equation. Hence, the *policy correspondence*

$$H(x, z) = \arg \max_{y\in\Gamma(x,z)} \left\{ U(x, y, z) + \beta \int_Z v(y, z')\, Q(z, dz') \right\}$$

is compact valued. We assume that $H(x, z)$ admits a measurable selection $h(x, z)$. Function h is often referred to as a *policy function* and defines a *stationary Markov equilibrium.* The existence of a stationary Markov equilibrium can be derived from various standard technical conditions. For the sake of brevity, we refrain discussion of these underlying conditions since they are not needed for our analysis. Note that we allow for multiple stationary Markov equilibrium solutions as function $U(\cdot,\cdot,z)$ is not strictly concave in (x,y). Multiplicity of solutions does not preclude differentiability of the value function.

Consider the Markov operator $\mathrm{M}v(y,z)=\int_Z v(y,z')\,Q(z,dz')$. It follows that for each z function $\mathrm{M}v(\cdot,z)$ is concave. Hence, at every interior point $y\in\mathrm{int}\,(X)$ the superdifferential $\partial_1(\mathrm{M}v)(y,z)$ of $\mathrm{M}v(y,z)$ with respect to y is a compact set [e.g., Rockafellar (1970)]. The following lemma characterizes $\partial_1(\mathrm{M}v)$ in terms of $\partial_1 v$. We need this characterization for our results below.

Lemma 2.1 *Let (Γ,U,β,Q) be a feasible optimization problem. Then, for every $y\in\mathrm{int}\,(X)$ we have*

$$\partial_1(\mathrm{M}v)(y,z)\equiv\partial_1\int_Z v(y,z')\,Q(z,dz')=\int_Z\partial_1 v(y,z')\,Q(z,dz'). \tag{3}$$

This well–established result can be seen as a non–smooth generalization of the differentiation rule under the integral sign. Clarke (1990, Theorem 2.7.2, p. 76) offers a version of this result for a Lipschitz function defined on an open set, and hence it must hold for a concave function at an interior point of the domain. The interpretation of (3) is as follows: For $z\in Z$ fixed, $q_z\in\partial_1(\mathrm{M}v)(y,z)$ if and only if there is a measurable mapping $z'\mapsto q_z(z')$ with $q_z(z')\in\partial_1 v(y,z')$ almost everywhere (a.e.) in the measure $Q(z,\cdot)$ such that $q_z=\int_Z q_z(z')\,Q(z,dz')$.

3 Differentiability of the Value Function

Constrained optimization is pervasive in economics. Constraints may appear in the form of feasibility and technological restrictions, individual rationality and incentive compatibility conditions, transaction costs, borrowing limits, liquidity and collateral requirements, and many other financial frictions. Standard assumptions on utility and production functions do not prevent these constraints from being binding. Indeed, a vast body of research in economic dynamics has focussed on effects of these constraints on quantitative properties of equilibrium solutions.

3.1 Main results

To establish that function v is of class C^1, we require differentiability of the return function U, and some regularity conditions for boundary solutions. We consider two conditions transported from the static theory: A boundary restriction on the policy function over the domain of definition and a constraint qualification [cf. Takayama (1990)]. We also introduce an asymptotic condition on the expected discounted utility of a marginal unit invested today.

D1: *For every $x \in \mathrm{int}(X)$ and $z \in Z$ function $U(\cdot, \cdot, z)$ is of class C^1 on some open neighborhood $N(x, y)$ of every point (x, y) with $y \in H(x, z)$.*

If (x, y) belongs to the boundary of Ω_z then $D1$ should be read as that $U(\cdot, \cdot, z)$ admits a differentiable extension on some open neighborhood $N(x, y)$.

D2: *For every $x \in \mathrm{int}(X)$ and $z \in Z$ there exists $y \in H(x, z)$ with $y \in \mathrm{int}(X)$.*

This simple assumption says that it is always possible to select an optimal path in the interior of the domain, and should not be confused with the aforementioned interiority condition $y \in \mathrm{int}(\Gamma(x, z))$. Assumption $D2$ may be innocuous if the set X can be appropriately redefined, i.e., the domain could be restricted or expanded so that it is not optimal to reach its boundary. We nevertheless present an example below (Section 4.3) in which $D2$ does not hold and the value function is not differentiable.

D3: *There is a finite collection of functions $g = (\ldots, g^i, \ldots)$, for $i = 1, 2, \cdots, m$, such that $\Omega = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times Z : \; g(x, y, z) \geq 0\}$. Each function $g^i(\cdot, \cdot, z) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is quasiconcave and of class C^1. Let $I(x, y, z) = \{i \; : \; g^i(x, y, z) = 0\}$ denote the set of saturated constraints, and let $s(x, y, z)$ denote the cardinality of $I(x, y, z)$. Then for each (x, z) there exists some optimal solution $y \in H(x, z)$ with $y \in \mathrm{int}(X)$ such that the rank of the matrix of partial derivatives $\{D_2 g^i(x, y, z) \; : \; i \in I(x, y, z)\}$ is equal to $s(x, y, z)$.*

Hence, the graph of the constraint correspondence must be defined by a finite number of constraints. The full rank condition implies that the number of choice variables cannot be less than the number of saturated constraints. The constraint qualification is essential to establish uniqueness of the set of Kuhn–Tucker multipliers as the matrix of partial derivatives $D_2 g$ can be inverted in a certain generalized sense.

We use the following notational conventions. All vectors are column vectors. Then, $D_j g(x, y, z)$ for $j = 1, 2$ is the $m \times n$ matrix of partial derivatives at (x, y, z), and $D_j g_s(x, y, z)$ for $j = 1, 2$ refers to the $s \times n$ matrix of partial derivatives of the saturated constraints $\{g^i(x, y, z) \; : \; i \in I(x, y, z)\}$. By $D3$, $D_2 g_s(x, y, z)$ has a $s \times n$ generalized inverse $D_2 g_s^+ = (D_2 g_s D_2 g_s^\top)^{-1} D_2 g_s$ at every (x, y, z), where $D_2 g_s^\top(x, y, z)$ denotes the transpose matrix. In all our computations below, we let the $n \times n$ matrix $G(x_t, x_{t+1}, z_t) = -D_1 g_s^\top(x_t, x_{t+1}, z_t) D_2 g_s^+(x_t, x_{t+1}, z_t)$. It should be understood that if $x_{t+1} \in \operatorname{int}(\Gamma(x_t, z_t))$ then $G(x_t, x_{t+1}, z_t) = \mathbf{0}_{n\times n}$. To simplify notation, let $\mathcal{G}_{t+1} = \prod_{s=0}^{t} G(x_s, x_{s+1}, z_s)$. Note that $\mathcal{G}_1 = G(x_0, x_1, z_0)$, and $\mathcal{G}_0$ will stand for the identity matrix $\mathbf{I}_{n\times n}$.

D4: *Let $\{x_{t+1}(z^t)\}_{t\geq 0}$ be an optimal contingency plan with $x_0 \in \operatorname{int}(X)$. Then, there is a constant B_0 such that for every measurable selection of supergradients $q_t \in \partial_1 v(x_t, z_t)$ a.e. in the measure $\mu^t(z_0, \cdot)$ we have*

$$\lim_{t\to\infty} \beta^t \int_{Z^t} \mathcal{G}_t q_t \, \mu^t(z_0, dz^t) = B_0. \tag{4}$$

This transversality condition is only required for boundary solutions. Indeed, $\mathcal{G}_t \neq \mathbf{0}_{n\times n}$ implies that for some history $\{z_s\}_{s=0}^t$ every optimal vector $x_{s+1}(z^s)$ belongs to the boundary of $\Gamma(x_s, z_s)$ for all $s = 0, 1, \ldots, t$. As shown in Proposition 3.2 below for most well known models we have $B_0 = 0$, and so the above limit is well defined.

In order to prove our main result, we now derive an envelope theorem for constrained, non–smooth optimization. Let $\varphi(x, y, z) = U(x, y, z) + \beta \mathrm{M}v(y, z)$. Let $\partial_{1,2}\varphi$ be the superdifferential of φ with respect to the first two component variables (x, y). For convex set Ω_{z_0} at point (x_0, y_0), the normal cone $N_{\Omega_{z_0}}(x_0, y_0)$ is defined as

$$N_{\Omega_{z_0}}(x_0, y_0) = \{\xi \in \mathbb{R}^{2n} \,:\, \xi \cdot (x - x_0, y - y_0) \leq 0, \forall (x, y) \in \Omega_{z_0}\}. \tag{5}$$

Proposition 3.1 *Consider a constrained optimization problem (Γ, U, β, Q). Let D1–D3 be satisfied. Then, for any $x_0 \in \operatorname{int}(X)$ and $z_0 \in Z$, $q_0 \in \partial v(x_0, z_0)$ if and only if there exists $q_1(z_1) \in \partial_1 v(x_1, z_1)$ a.e. in the measure $Q(z_0, \cdot)$ such that*

$$q_0 = D_1 U(x_0, y_0, z_0) + G(x_0, y_0, z_0)\left(D_2 U(x_0, y_0, z_0) + \beta \int_Z q_1(z_1) Q(z_0, dz_1)\right), \tag{6}$$

where $y_0 = h(x_0, z_0)$.

Proof. By $D1$, function $\varphi(x, y, z)$ is differentiable with respect to x. Hence, $\partial_1\varphi(x, y, z) = \{D_1\varphi(x, y, z)\}$, and so $\partial_{1,2}\varphi(x, y, z) = \{D_1 U(x, y, z)\} \times \partial_2\varphi(x, y, z)$ for all (x, y, z). Again, by $D1$–$D2$ the superdifferential $\partial_2\varphi(x, y, z) = \{D_2 U(x, y, z)\} + \beta\partial_1(\mathrm{M}v)(y, z)$. Then, by a standard technical argument (e.g., see Lemma 5.1 in the Appendix below), we obtain $q_0 \in \partial_1 v(x_0, z_0)$ if and only if there exists $q_1 \in \partial_1(\mathrm{M}v)(y_0, z_0)$ such that

$$\big(q_0 - D_1U(x_0, y_0, z_0), -D_2U(x_0, y_0, z_0) - \beta q_1\big) \in -N_{\Omega_{z_0}}(x_0, y_0). \tag{7}$$

Moreover, as is well known [e.g., Clarke (1990, Corollary 2, p. 56)] by $D3$ we must have

$$\begin{aligned} &- N_{\Omega_{z_0}}(x_0, y_0) = \\ &\qquad \Big\{(q, p) \in \mathbb{R}^{2n} \,:\, (q, p) = \sum_{i \in I(x_0, z_0)} \lambda^i (D_1 g^i(x_0, y_0, z_0), D_2 g^i(x_0, y_0, z_0)), \lambda^i \geq 0\Big\}. \end{aligned}$$

Now, combining this expression with (7) we get

$$q_0 - D_1U(x_0, y_0, z_0) = \sum_{i \in I(x_0, z_0)} \lambda^i D_1 g^i(x_0, y_0, z_0), \tag{8}$$

$$-D_2U(x_0, y_0, z_0) - \beta q_1 = \sum_{i \in I(x_0, z_0)} \lambda^i D_2 g^i(x_0, y_0, z_0) \tag{9}$$

for some $\lambda^i \geq 0$, for all $i \in I(x_0, z_0)$. Let $\lambda = (\ldots, \lambda^i, \ldots)$. Now, from (9) we obtain that $\lambda = -D_2 g_s^+(x_0, y_0, z_0)(D_2U(x_0, y_0, z_0) + \beta q_1)$. To complete the proof we substitute this expression for λ into (8) and let $G(x_0, y_0, z_0) = -D_1 g_s(x_0, y_0, z_0)^\top D_2 g_s^+(x_0, y_0, z_0)$. Finally, recall from Lemma 2.1 that for $z \in Z$ fixed, $q_z \in \partial_1(\mathrm{M}v)(y, z)$ if and only if there is a measurable mapping $z' \mapsto q_z(z')$ with $q_z(z') \in \partial_1 v(y, z')$ a.e. in the measure $Q(z, \cdot)$ such that $q_z = \int_Z q_z(z')\, Q(z, dz')$. ∎

This envelope theorem applies to non–differentiable objective functions with boundary solutions. Indeed, (6) defines the superdifferential of the value function at the current state in terms of the superdifferentials of the instantaneous utility and of the expected value function evaluated at the optimal path. For decentralized economies, this result reduces to the fundamental theorem of finance with binding constraints: The price of an asset is equal to the sum of all expected discounted gross returns where these returns may include the shadow values of the binding constraints.

It is easy to see that a repeated iteration of Lemma 2.1 and Proposition 3.1 yields

$$\begin{aligned} q_0 = &\sum_{t=0}^{T-1} \beta^t \int_{Z^t} \mathcal{G}_t \Big(D_1U(x_t, x_{t+1}, z_t) + G(x_t, x_{t+1}, z_t) D_2U(x_t, x_{t+1}, z_t)\Big)\, \mu^t(z_0, dz^t) \\ &+ \beta^T \int_{Z^T} \mathcal{G}_T\, q_T(z^T)\, \mu^T(z_0, dz^T). \end{aligned} \tag{10}$$

Our strategy of proof is then to show that at every interior point x_0 the superdifferential $q_0 \in \partial_1 v(x_0, z_0)$ is a singleton, and hence by concavity [Rockafellar (1970)] function $v(\cdot, z_0)$ is differentiable of class C^1 on $\mathrm{int}(X)$.

Theorem 3.1 *Consider a constrained optimization problem* (Γ, U, β, Q)*. Let* $\{x_{t+1}(z^t)\}_{t\geq 0}$ *be an optimal contingency plan satisfying D1–D4 with* $x_0 \in \mathrm{int}(X)$*. Then, function* $v(\cdot, z_0) : \mathrm{int}(X) \longrightarrow \mathbb{R}$ *is differentiable at* x_0 *and the partial derivative*

$$D_1 v(x_0, z_0) = \sum_{t=0}^{\infty} \beta^t \int_{Z^t} \mathcal{G}_t \Big(D_1 U(x_t, x_{t+1}, z_t) + G(x_t, x_{t+1}, z_t) D_2 U(x_t, x_{t+1}, z_t) \Big) \, \mu^t(z_0, dz^t) + B_0. \tag{11}$$

Proof. As already pointed out, we just need to show that $\partial_1 v(x_0, z_0)$ is a singleton. By way of contradiction, let q_0, $\widetilde{q}_0 \in \partial_1 v(x_0, z_0)$. Then, after subtracting common terms in (10) there exist $q_T(z_T), \widetilde{q}_T(z_T) \in \partial_1 v(x_T, z_T)$ for $T \geq 0$ such that

$$q_0 - \widetilde{q}_0 = \beta^t \int_{Z^T} \mathcal{G}_T (q_T - \widetilde{q}_T) \, \mu^T(z_0, dz^T).$$

By Assumption $D4$ this expression converges to zero as T goes to ∞. This proves that $\partial_1 v(x_0, z_0)$ is a singleton at every point $x_0 \in \mathrm{int}(X)$. Hence, function $v(\cdot, z_0) : \mathrm{int}(X) \longrightarrow \mathbb{R}$ is differentiable at x_0. The value of the derivative $D_1 v(x_0, z_0)$ in (11) is obtained by letting T go to ∞ in the above expression (10). ■

For a given optimal contingency plan $\{x_{t+1}(z^t)\}_{t\geq 0}$, let $E_T = \{z^T \in Z^T : \mathcal{G}_{T+1} \neq \mathbf{0}_{n\times n}\}$. That is, E_T is the set of histories at time T such that the optimal solution $\{x_{t+1}(z^t)\}_{t\geq 0}$ lies always in the boundary. If with probability one there is a time T such that E_T is empty, then $D4$ trivially holds.

Corollary 3.1 *Consider a constrained optimization problem* (Γ, U, β, Q)*. Let* $\{x_{t+1}(z^t)\}_{t\geq 0}$ *be an optimal contingency plan satisfying D1–D3 with* $x_0 \in \mathrm{int}(X)$*. Assume that there exists a first time* $T \geq 0$ *such that* $\mu^T(z_0, E_T) = 0$*. Then, function* $v(\cdot, z_0) : \mathrm{int}(X) \longrightarrow \mathbb{R}$ *is differentiable at* x_0 *and the partial derivative*

$$\begin{aligned} D_1 v(x_0, z_0) = & \sum_{t=0}^{T-1} \beta^t \int_{Z^t} \mathcal{G}_t \big(D_1 U(x_t, x_{t+1}, z_t) + G(x_t, x_{t+1}, z_t) D_2 U(x_t, x_{t+1}, z_t) \big) \, \mu^t(z_0, dz^t) \\ & + \beta^T \int_{Z^T} \mathcal{G}_T D_1 U(x_T, x_{T+1}, z_T) \, \mu^T(z_0, dz^T). \end{aligned} \tag{12}$$

Proof. By the envelope theorem [Benveniste and Scheinkman (1979)] for $x_{T+1} \in \mathrm{int}(\Gamma(x_T, z_T))$ function $v(\cdot, z_T)$ is differentiable and the derivative $D_1 v(x_T, z_T) = D_1 U(x_T, x_{T+1}, z_T)$. Moreover, $\mathcal{G}_{T+1} = \mathbf{0}_{n\times n}$. Then, expression (12) follows directly from (10). Therefore, the superdifferential $\partial_1 v(x_0, z_0)$ is unique and $v(\cdot, z_0)$ is differentiable at x_0. ■

Of course, for $T = 0$ we get the standard envelope theorem $D_1 v(x_0, z_0) = D_1 U(x_0, x_1, z_0)$. But it should be noted that for $T > 0$ this result requires constraint qualification $D3$ at those optimal vectors (x_t, x_{t+1}, z_t) with $x_{t+1} \in \mathrm{bd}(\Gamma(x_t, z_t))$.

3.2 Duality theory

As an application of Theorem 3.1 we now show that for every optimal path there exists a *unique* set of Kuhn–Tucker multipliers satisfying the Euler equations and the transversality condition. The existence of these multipliers can be established in a simple way by an induction argument on Bellman's equation [Weitzman (1973)], but uniqueness has remained an open issue because of the complexity involved in these equations. By the welfare theorems, the uniqueness of the multipliers entails that an optimal allocation is just supported by a unique price system.

Let $\lambda(x, z)$ be a non–negative vector of Kuhn–Tucker multipliers. As shown in the proof of Proposition 3.1, the derivative

$$D_1 v(x, z) = D_1 U(x, h(x, z), z) + D_1 g^\top(x, h(x, z), z)\lambda(x, z) \tag{13}$$

for every $\lambda(x, z)$ such that

$$D_2 U(x, y, z) + \beta D_1 \mathrm{M} v(y, z) + D_2 g^\top(x, y, z)\lambda(x, z) = 0. \tag{14}$$

Observe that the above expression (6) readily follows from (13) after substituting out $\lambda(x, z)$ from (14). Moreover, from these equations we can see informally the role of assumption $D3$: If the matrix of derivatives of the saturated constraints $D_2 g_s^\top(x, y, z)$ has full rank then (14) implies that the vector of multipliers $\lambda(x, z)$ is unique. Consequently, if $v(\cdot, z)$ is differentiable at some $y = h(x, z)$ then there is a unique multiplier $\lambda(x, z)$ and so the superdifferential $\partial_1 v(x, z)$ must contain a unique vector.

Let $\{x_{t+1}(z^t)\}_{t\geq 0}$ be an optimal contingency plan and write $\lambda_t = \lambda(x_t, z_t)$. If $v(\cdot, z)$ is differentiable, then by conditions (13) and (14) evaluated over these optimal values we can derive the following system of Euler equations

$$
\begin{aligned}
&D_2U(x_{t-1}, x_t, z_{t-1}) + D_2g^\top(x_{t-1}, x_t, z_{t-1})\lambda_{t-1} \\
&\qquad + \beta\int_Z \Big(D_1U(x_t, x_{t+1}, z_t) + D_1g^\top(x_t, x_{t+1}, z_t)\lambda_t\Big)\, Q(z_{t-1}, dz_t) = 0,
\end{aligned}
\tag{15}
$$

where $\lambda_t \geq 0$ and $g(x_t, x_{t+1}, z_t) \geq 0$ with $\lambda_t^\top g(x_t, x_{t+1}, z_t) = 0$, for all $t = 1, 2, \ldots$. To this system of equations we also need to append a transversality condition. For simplicity, let us assume that X is a compact set. Then, let

$$
\lim_{T\to\infty} \beta^T \int_{Z^T} \big(D_1U(x_T, x_{T+1}, z_T) + D_1g^\top(x_T, x_{T+1}, z_T)\lambda_T\big)\mu^T(z_0, dz^T) = 0. \tag{16}
$$

As is well known [cf. Benveniste and Scheinkman (1982)], both (15)–(16) are sufficient conditions[1] for the characterization of an optimal path $\{x_t\}_{t\geq 0}$.

Theorem 3.2 *Assume that X is a compact set. Under the conditions of Theorem 3.1, for every optimal contingency plan $\{x_{t+1}(z^t)\}_{t\geq 0}$ with $x_0 \in \mathrm{int}(X)$ there exists a unique system of Kuhn–Tucker multipliers $\{\lambda_t\}_{t\geq 0}$ satisfying (15)–(16).*

This result can be viewed as an envelope theorem for concave infinite-horizon optimization. Indeed, by (13)–(14) we can construct a system of Kuhn–Tucker multipliers $\{\lambda_t\}_{t\geq 0}$ that satisfies the Euler equations (15) and the transversality condition (16). Then, Theorem 3.2 completes the other direction: The system of multipliers $\{\lambda_t\}_{t\geq 0}$ satisfying the Euler equations (15) and the transversality condition (16) is unique and corresponds to the derivative of the value function D_1v as given by (13)–(14).

3.3 Sensitivity

In many economic applications it is of interest to establish that the derivative of the value function varies continuously with perturbations of the model. For simplicity, we focus on perturbations of the return function U under the sup norm. For given two functions U and U_n let $\|U - U_n\| = \sup_{(x,y,z)\in\Omega} |U(x, y, z) - U_n(x, y, z)|$. Note that convergence in the sup norm amounts to uniform convergence in the space of functions. Let v_n

[1] The extension of our uniqueness result below to an unbounded domain X requires some further mild regularity conditions. The non–negativity conditions of Proposition 3.2 allow for a simple extension of the transversality condition to unbounded domains, e.g., see Benveniste and Scheinkman (1982).

refer to the value function of an optimization problem (Γ, U_n, β, Q). By Theorem 3.2 the following continuity result applies to the unique system of Kuhn–Tucker multipliers $\{\lambda_t\}_{t\geq 0}$ satisfying the Euler equations (15) and the transversality condition (16).

Theorem 3.3 *Let all feasible optimization problems* (Γ, U, β, Q) *and* $\{(\Gamma, U_n, \beta, Q)\}_{n\geq 0}$ *satisfy assumptions D1–D4. Assume that the sequence of functions* $\{U_n\}_{n\geq 0}$ *converges uniformly to function* U. *Then, the sequence of value functions* $\{v_n\}_{n\geq 0}$ *converges uniformly to the original value function* v, *and for each* z *the sequence of derivative functions* $\{D_1 v_n(\cdot, z)\}_{n\geq 0}$ *converges uniformly to the derivative function* $D_1 v(\cdot, z)$ *on every compact set* $K \subset \mathrm{int}(X)$.

3.4 Differentiability and bubbles

The decomposition given in Theorem 3.1 of the derivative of the value function into the fundamental value and a bubble term is common in asset pricing, where B_0 can be identified with the bubble term of some existing assets. It is possible to construct examples with a non–null bubble component [e.g. Montrucchio and Privileggi (2001), and the cash-in-advance model below], but given that bubbles occur in general equilibrium models under rather pathological circumstances [Santos and Woodford (1997)], in what follows we will focus on cases in which the bubble term $B_0 = 0$. In our next result we provide some simple conditions that require the value function to be bounded and increasingly monotone.

Proposition 3.2 *Consider a constrained optimization problem* (Γ, U, β, Q). *Assume that* U *is a bounded function. Let* $\{x_{t+1}(z^t)\}_{t\geq 0}$ *be an optimal contingency plan with* $x_0 \in \mathrm{int}(X)$ *such that there is a constant* $\alpha > 0$ *with* $x_{jt} \geq \alpha$ *for each coordinate* j *and all* t. *Let* $G(x, y, z) \geq 0$ *and* $D_1 U(x, y, z) + G(x, y, z) D_2 U(x, y, z) \geq 0$ *for all feasible* (x, y, z). *Then, under D1–D3 we have* $B_0 = 0$ *so that expression* (4) *becomes*

$$\lim_{t\to\infty} \beta^t \int_{Z^t} \mathcal{G}_t \, q_t \, \mu^t(z_0, dz^t) = 0.$$

Note that this proposition focuses on non–negative fundamental values in assets in positive supply [cf. Santos and Woodford (1997)]. Under some mild regularity assumptions this result can be extended to unbounded return functions. As shown below, the non–negativity conditions are satisfied in standard models of economic growth. If monotonicity does not hold, the bubble term will also vanish in economies in which the fundamental pricing equation can be generated by a contraction mapping in a bounded space of functions [Lucas (1978)]. In our case, the following conditions ensure this contraction property: (i) Function $U(x,y,z)$ and its partial derivatives $D_jU(x,y,z)$, $j=1,2$, are uniformly bounded on $\Omega \times Z$, and (ii) $\sup_{(x,y,z)\in\Omega} \beta \left\| \int_Z G(x,y,z')\ \mu(z,dz') \right\| < 1$.

3.5 Checking our assumptions: Two applications

3.5.1 A pure currency model

We write the recursive formulation of the pure currency model as

$$v(m,z) = \max_{c,\ m'\geq 0} \left\{ u(c,z) + \beta \int_Z v(m',z')\, \mu(dz') \right\}$$

$$\text{s.t.} \quad m'(1+\pi) + c - m - y \leq 0,$$
$$c - m \leq 0.$$

The utility function $u : \mathbb{R}_+ \times Z \longrightarrow \mathbb{R}$ is continuous and Z is a compact interval. Moreover, for each z the utility function $u(\cdot,z) : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is strictly increasing, concave and of class C^1. Here, m denotes current real balances of the household, m' real balances at the beginning of next period, π is an exogenous inflation tax, c is current consumption, and y is real income from the sale of goods and government transfers. For this problem, $U(m,m',z) = u(m+y-m'(1+\pi),z)$, $\Gamma(m) = [y/(1+\pi),(y+m)/(1+\pi)]$. Thus, the constraints are $g^1(m,m') = m'(1+\pi) - y \geq 0$, and $g^2(m,m') = y + m - m'(1+\pi) \geq 0$. Let us check conditions $D1$–$D4$ in this example. $D1$ and $D2$ hold for interior points, $m > 0$, and $m > 0$ is satisfied along any equilibrium solution of the Lucas' model; $D3$ is satisfied since g^1, g^2 can never be both active except at $m = 0$; $D4$ is also satisfied for $\beta/(1+\pi) < 1$. Indeed, if g^1 is binding then $G = -D_1g^1/D_2g^1 = 0$ since g^1 is independent of m, and if g^2 is binding then $G = -D_1g^2/D_2g^2 = 1/(1+\pi)$. Therefore, $\beta G < 1$ and (4) holds since real balances are bounded below.

Note that the optimal quantity of money or Friedman rule occurs for $\beta/(1+\pi) = 1$. In this borderline case, money may simply be a bubble, and we cannot generally show that the limit in $D4$ is satisfied. As is well known, the Friedman rule may not be compatible with existence of a monetary equilibrium [Bewley (1983)] or there could be a continuum of equilibrium real balances [Grandmont and Younes (1973)].

Therefore, it is immediate to check our conditions in this model – without computing the left and right side limits of the derivative at the point where one of the constraints becomes active.

3.5.2 A simple growth model with irreversible investment

We provide a simple proof of the differentiability of the value function in the stochastic one-sector growth model. Here boundary solutions occur when either consumption or investment are equal to zero. These extreme values may be triggered by the existence of large shocks. We shall simply check that all our assumptions are satisfied.

Consider the recursive formulation of the problem

$$v(x,z) = \max_{c,\, y \geq 0} \left\{ u(c,z) + \beta \int_Z v(y,z')\, \mu(z,dz') \right\}$$

$$\text{s.t.} \quad c + y = f(x,z),$$
$$y \geq (1-\delta)x.$$

We assume that functions $u : \mathbb{R}_+ \times Z \longrightarrow \mathbb{R}$ and $f : \mathbb{R}_+ \times Z \longrightarrow \mathbb{R}_+$ are continuous and Z is a compact set. Moreover, for each z the utility function $u(\cdot, z) : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and the production function $f(\cdot, z) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are both increasing, strictly concave and of class C^1. Let $f(0,z) = 0$, and $\lim_{x \to 0} \beta D_1 f(x,z) > 1$ and $(1-\delta) + \epsilon \leq \lim_{x \to \infty} D_1 f(x,z) < \frac{1}{\beta}$ for all z and $\epsilon > 0$.

From these primitive functions, the return function $U : \mathbb{R}_+ \times \mathbb{R}_+ \times Z \longrightarrow \mathbb{R}$ is defined as $U(x,y,z) = u(f(x,z) - y, z)$ and the constraint correspondence $\Gamma : \mathbb{R}_+ \times Z \longrightarrow 2^{\mathbb{R}_+}$ as $\Gamma(x,z) = [(1-\delta)x, f(x,z)]$. For each z the graph of correspondence $\Gamma(x,z)$ is bordered by two concave functions $g^1(x,y,z) = f(x,z) - y \geq 0$ and $g^2(x,y,z) = -(1-\delta)x + y \geq 0$. Both constraints depend on state variable x. Hence, the aforementioned interiority condition does not hold if any of these constraints is saturated.

For $x_0 > 0$ assumption $D2$ trivially holds for an optimal y with $g^1(x,y,z) = 0$ since $y = f(x_0, z) > 0$, and $D2$ also holds for an optimal y with $g^2(x,y,z) = 0$ since $y = (1-\delta)x_0 > 0$. The full rank condition in $D3$ is always satisfied since functions g^1 and g^2 are additively separable in y. Finally, as it is well known the transversality condition $D4$ must also hold [Santos and Woodford (1997)] at every optimal path.

Using contractive arguments, Sargent (1980) provides a proof of differentiability for boundary solutions in which $g^2(x,y,z) = 0$. In this case, $G(x,y,z) = (1-\delta) < 1$. Our proof allows for optimal solutions to reach the upper boundary $g^1(x, y, z) = 0$ without imposing a restriction of the type $\beta G(x, y, z) = \beta D_1 f(x, z) < 1$.

Therefore, $D1$–$D4$ are all satisfied and for each z the derivative $D_1 v(x_0, z_0)$ exists at all interior points x_0. And it seems natural to allow for boundary solutions since the interiority of the optimal solution is not easy to check.

4 Some counterexamples

We study now the necessity of assumptions $D2$–$D4$ in some models of economic growth, finance, and dynamic contracts. We show that the value function fails to be differentiable when each of these assumptions is not met. Our results are of interest in other areas of economics using recursive optimization such as monetary theory, taxation, labor, and industrial organization.

4.1 A simple growth model with irreversible investment

We show that in the above growth model studied in Section 3.5.2, the derivative of the value function may become unbounded as the stock of capital approaches zero – even if the derivatives of the utility and production functions are bounded. We already discussed that under mild regularity conditions all our assumptions are satisfied and for each z the derivative $D_1 v(x_0, z)$ exists at all interior points x_0. The situation is quite different for $x = 0$ as $f(0, z) = 0$. Then, $D2$ is not satisfied. Moreover, Figure 1 depicts a simple deterministic example where $D4$ does not hold at $x = 0$. Here, the derivative $f'(0) > \frac{1}{\beta}$, and the policy function lies at the upper boundary $h(x_0) = f(x_0)$ for x_0 near 0.[2] From Corollary 3.1 for x_0 near 0 the derivative $v'(x_0) = \beta^{T(x_0)} \mathcal{G}_{T(x_0)} v'(x_{T(x_0)})$ where $T(x_0)$ is the first time T such that $c_T > 0$ and $\mathcal{G}_T = f'(x_0) f'(x_1) \cdots f'(x_{T(x_0)-1})$. Then $T(x_0)$ goes to ∞ as x_0 goes to 0 and $\beta^{T(x_0)} \mathcal{G}_{T(x_0)} = \beta^{T(x_0)} f'(x_0) f'(x_1) \cdots f'(x_{T(x_0)-1})$ goes to ∞. Consequently, in this example the derivative $v'(0^+)$ will be unbounded even if the utility and production functions have bounded derivatives.

[2]For $f'(0) > \frac{1}{\beta}$ this optimal policy arises for a linear utility function and for related utility functions sufficiently close to the linear utility. Condition $f'(0) > \frac{1}{\beta}$ is required for the existence of an interior steady–state solution.

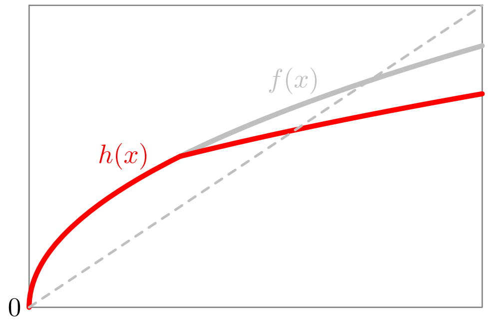

Figure 1: The derivative of the value function may be unbounded even if the utility and production functions have bounded derivatives. If for all x in a small neighborhood of $x = 0$ the policy function $h(x) = f(x)$ and $\beta f'(0) > 1$ then $v'(x)$ gets unbounded as x converges to 0. This example does not satisfy $D2$ and $D4$ at $x = 0$.

4.2 Constrained efficient allocations

We study the differentiability of the Pareto frontier for the model of Kocherlakota (1996). Koeppl (2006) presents an example in which the value function fails to be differentiable. He also provides some sufficient conditions for differentiability. Here we offer a complete analysis of differentiability: We provide a *necessary* and *sufficient* condition which is directly linked to the constraint qualification in assumption $D3$. We thus identify the lack of differentiability of the value function with a failure of assumption $D3$. Under the special structure of the optimization problem, assumption $D4$ is not needed. Indeed, in pure exchange economies with no real assets the transversality condition holds trivially.

The recursive formulation of Kocherlakota's exchange economy with two agents is as follows:

$$
\begin{array}{lll}
V(U_0) = \max_{\{c_s, U_s\}} & \sum_{s=1}^{S} \pi_s[u(\overline{\omega} - c_s) + \beta V(U_s)] & \\
\text{s.t.} & \sum_{s=1}^{S} \pi_s[u(c_s) + \beta U_s] \geq U_0, & (P_1) \\
 & u(c_s) + \beta U_s \geq u(\omega_s^1) + \beta U_{\text{aut}} \text{ for all } s, & (P_2) \\
 & u(\overline{\omega} - c_s) + \beta V(U_s) \geq u(\overline{\omega} - \omega_s^1) + \beta U_{\text{aut}} \text{ for all } s, & (P_3) \\
 & U_s \in [U_{\text{aut}}, U_{\max}] \text{ for all } s, &
\end{array}
$$

where the value function $V(U_0)$ assigns the maximum utility to an agent (say agent 2) over all possible utility levels U_0 of the other agent. Aggregate output is constant, and denoted by $\overline{\omega}$, and individual endowments $\omega_s^i > 0$, for $i = 1, 2$, are subject to idiosyncratic shocks, symmetrically distributed, that follow an *iid* process. The utility function u is bounded, increasing, concave and differentiable.

In the sequel we assume that the Pareto frontier $(U_0, V(U_0))$ is non-degenerate, that is, $U_{\max} > U_{\text{aut}}$. Let $U_{\max} = V(U_{\text{aut}})$. One readily checks that function $V : [U_{\text{aut}}, U_{\max}] \longrightarrow \mathbb{R}$ is well defined, decreasing, concave and continuous. Moreover, even though the value function is in the constraint set, in the Appendix below we extend the above arguments so that our differentiability analysis can be applied: The superdifferential of the value function can be characterized recursively in terms of fundamental values and superdifferentials of the value function at future states.

For given U_0, let λ, $\{\mu_s\}$ and $\{\nu_s\}$ be a set of Kuhn–Tucker multipliers corresponding to constraints (P_1), (P_2) and (P_3), respectively. Let $S_2(U_0)$ be the subset of states s where constraint (P_2) is saturated, and $S_2^b(U_0)$ the subset of states s where constraint (P_2) is binding. That is, $S_2(U_0)$ is the subset of states s where constraint (P_2) holds with equality at the optimal solution $\{c_s, U_s\}$, and $S_2^b(U_0)$ is the subset of states s with $\mu_s > 0$. Analogously, let $S_3(U_0)$ be the subset of states s where constraint (P_3) is saturated and $S_3^b(U_0)$ be the subset of states s where constraint (P_3) is binding. Note that (P_1) will always be binding. Also, it is easy to show that the intersection $S_2(U_0) \cap S_3(U_0)$ is empty [cf., Kocherlakota (1996)]. Hence, for each s there is at most one constraint (P_2) or (P_3) that is saturated.

D3': *$S_2(U_0) \cup S_3^b(U_0) \neq S$ and $S_2^b(U_0) \cup S_3(U_0) \neq S$.*

Theorem 4.1 *Let $U_0 \in (U_{\text{aut}}, U_{\max})$. Then, $S_2(U_0) \neq S$ and $S_3(U_0) \neq S$. The value function V is differentiable at U_0 if and only if $D3'$ is satisfied. At points of differentiability, the derivative*

$$V'(U_0) = -\frac{u'(\overline{\omega} - c_s)}{u'(c_s)},$$

for every state s where none of the constraints (P_2)–(P_3) are binding.

As one can see from the method of proof of Theorem 4.1, assumption $D3'$ is a necessary and sufficient condition. Under this latter condition there is a unique multiplier λ and so the derivative $V'(U_0) = -\lambda$. Note that $D3'$ can be defined as: (i) $S_2(U_0) \cup S_3(U_0) \neq S$ and (ii) for $S_2(U_0) \cup S_3(U_0) = S$ there must be some non–binding constraint in $S_2(U_0)$ and some other non–binding constraint in $S_3(U_0)$. Hence, $D3'$ is slightly weaker than $S_2(U_0) \cup S_3(U_0) \neq S$. We would like to emphasize that the condition $S_2^b(U_0) \cup S_3^b(U_0) \neq S$

leaves out some cases in which V is not differentiable, and so this is merely a sufficient condition. By concavity the value function V is differentiable at almost all U_0. Hence, $D3'$ must be satisfied at almost all U_0.

4.3 A model of foreign direct investment

We briefly discuss a model of foreign direct investment with risk of expropriation by the host country [Thomas and Worrall (1994)]. In this model the value function may fail to be differentiable if $D2$ is not satisfied.

For illustrative purposes, we focus on the deterministic case. Using Bellman's equation, the optimal contracting problem is written as

$$\begin{array}{lll} V(U_0) = \max_{I,\tau,U} & \{-I + r(I) - \tau + \beta V(U)]\} & \\ \text{s.t.} & \tau + \beta U \geq U_0 & (R_1) \\ & \tau - r(I) + \beta U \geq 0 & (R_2) \\ & V(U) \geq 0 & (R_3) \\ & r(I) - \tau \geq 0 & (R_4) \\ & \tau \geq 0. & (R_5) \end{array}$$

If none of the constraints are saturated, then optimal investment I^* achieves the first–best efficient level, $r'(I^*) = 1$. Therefore, for first–best Pareto–efficient allocations (V, U) the value function V is differentiable. And since both parties are risk neutral the derivative $V'(U) = 1$.

Suppose now that a first–best Pareto–efficient allocation cannot be achieved. Then, constraint (R_2) must be binding; moreover, optimality requires that the optimal transfer $\tau_s = 0$ and so (R_5) is also saturated. Then, by (R_1) the reservation utility U grows over time. As in Thomas and Worrall (1994), we consider two cases:

(i) *The optimal path eventually reaches an interior, first–best optimal solution.* Then, Proposition 3.1 applies as constraint qualification $D3$ is always satisfied. Therefore, the value function V is differentiable at the initial reservation utility U_0.

(ii) *The optimal path never reaches an interior, first–best optimal solution.* As $\tau_s = 0$, by (R_1) there is a finite time T such that $V(U_T) = V(U_{\max}) = 0$. Following the method of proof of Proposition 3.1 and (10), one can check that the value function is not differentiable at the smallest point U_{T-1} such that the optimal outcome $U_{\max} = h(U_{T-1})$. As a matter of fact, we just need to evaluate the directional derivatives of the value function V at U_{T-1} from Bellman's equation. To see that these derivatives are different we first note that the optimal policy $U_1 = h(U_0)$ is increasing. Then, for the right–hand side derivative the term $\beta V(U_T)$ is constant as $V(U_T) = V(U_{\max})$, whereas for the left–hand side derivative this term $\beta V(U_T)$ is decreasing with negative slope bounded away from zero. Therefore, both directional derivatives cannot have the same value.

In conclusion, the lack of differentiability of the value function for this model appears as a failure of $D2$. In Kocherlakota's model the problem stems from $D3$, and in the model with irreversible investment the derivative becomes unbounded as $D4$ fails for boundary point $x = 0$.

5 Appendix

In our first preliminary result we apply basic arguments from convex analysis to the Bellman equation

$$v(x,z) = \max_{y \in \Gamma(x,z)} \{U(x,y,z) + \beta \mathrm{M} v(y,z)\}$$

for all $x \in X$. Recall that $\varphi(x,y,z) = U(x,y,z) + \beta \mathrm{M} v(y,z)$ and $\partial_{1,2}\varphi$ denotes the superdifferential of φ with respect to the first two component variables (x,y). In what follows x_0 refers to an interior point. The normal cone $N_{\Omega_{z_0}}(x_0,y_0)$ of the convex projection set Ω_{z_0} at point (x_0,y_0) was defined in (5). Throughout the Appendix, y_0 means $y_0 = h(x_0,z_0)$.

Lemma 5.1 *$q_0 \in \partial_1 v(x_0,z_0)$ if and only if there exists $(\xi_1,\xi_2) \in \partial_{1,2}\varphi(x_0,y_0,z_0)$ such that $(q_0 - \xi_1, -\xi_2) \in -N_{\Omega_{z_0}}(x_0,y_0)$.*

Proof. We follow some well-established arguments, e.g. see Aubin (1993, Problem 35). Define the indicator function of set Ω_z as

$$\delta(x,y,z) = \begin{cases} 0, & (x,y) \in \Omega_z \\ -\infty, & (x,y) \notin \Omega_z. \end{cases}$$

Note that function δ is concave and upper semicontinuous in (x,y) for z fixed. Now, rewrite Bellman's equation as

$$v(x,z) = \max_{y \in \mathbb{R}^n} \big\{\varphi(x,y,z) + \delta(x,y,z)\big\}$$

for all $x \in X$, $z \in Z$. This is an unconstrained optimization problem. By Aubin (1993, Prop. 4.3), $q_0 \in \partial_1 v(x_0, z_0)$ if and only if $(q_0, 0) \in \partial_{1,2}\big(\varphi + \delta\big)(x_0, y_0, z_0)$. Moreover,

$$\begin{aligned}\partial_{1,2}\big(\varphi + \delta\big)(x_0, y_0, z_0) &= \partial_{1,2}\varphi(x_0, y_0, z_0) + \partial_{1,2}\delta(x_0, y_0, z_0) \\ &= \partial_{1,2}\varphi(x_0, y_0, z_0) - N_{\Omega_{z_0}}(x_0, y_0).\end{aligned}$$

Therefore, $q_0 \in \partial v(x_0, z_0)$ if and only if there exists $(\xi_1, \xi_2) \in \partial_{1,2}\varphi(x_0, y_0, z_0)$ such that $(q_0 - \xi_1, -\xi_2) \in -N_{\Omega_{z_0}}(x_0, y_0)$. ■

Proof of Theorem 3.2. Suppose that $\{\overline{x}_t\}_{t\geq 0}$ is an optimal contingency plan starting at $\overline{x}_0$. For this optimal path assume that there are two sequences of Kuhn–Tucker multipliers $\{\lambda_t\}_{t\geq 0}$ and $\{\lambda'_t\}_{t\geq 0}$ that satisfy equations (15) and (16). For z_0 fixed, all initial conditions x_0, all possible realizations z_t, and all feasible sequences $x_{t+1} \in \Gamma(x_t, z_t)$ for $t = 1, 2, \ldots, n$, let

$$v_{\lambda,n}(x_0, z_0) = \max_{\{x_t\}_{t=1}^{n+1}} \sum_{t=0}^{n} \beta^t \int_{Z^t} U(x_t, x_{t+1}, z_t)\, \mu^t(z_0, dz^t) + \beta^{n+1} \int_{Z^{n+1}} \alpha_{n+1} x_{n+1}\, \mu^{n+1}(z_0, dz^{n+1}), \tag{17}$$

where $\alpha_{n+1} = D_1U(\overline{x}_{n+1}, \overline{x}_{n+2}, z_{n+1}) + D_1g^\top(\overline{x}_{n+1}, \overline{x}_{n+2}, z_{n+1})\lambda_{n+1}$, and

$$v_{\lambda',n}(x_0, z_0) = \max_{\{x_t\}_{t=1}^{n+1}} \sum_{t=0}^{n} \beta^t \int_{Z^t} U(x_t, x_{t+1}, z_t)\, \mu^t(z_0, dz^t) + \beta^{n+1} \int_{Z^{n+1}} \alpha'_{n+1} x_{n+1}\, \mu^{n+1}(z_0, dz^{n+1}), \tag{18}$$

where $\alpha'_{n+1} = D_1U(\overline{x}_{n+1}, \overline{x}_{n+2}, z_{n+1}) + D_1g^\top(\overline{x}_{n+1}, \overline{x}_{n+2}, z_{n+1})\lambda'_{n+1}$. Note that the added linear parts α_{n+1} and α'_{n+1} are chosen so that at point $\overline{x}_0$ the optimal solution is $\{\overline{x}_t\}_{t=0}^{n+1}$ for both optimization problems, and for this optimal solution $\{\lambda_t\}_{t=0}^{n}$ is the sequence of associated Kuhn–Tucker multipliers under (17), and $\{\lambda'_t\}_{t=0}^{n}$ is the sequence of associated Kuhn–Tucker multipliers under (18). By $D3$, each sequence of multipliers is unique.

By the same methods as the proof of Theorem 3.1, we can readily see that functions $v_{\lambda,n}$ and $v_{\lambda',n}$ are concave and of class C^1 in x. Moreover, by (16) and the definitions of α_{n+1}, α'_{n+1}, the sequences of functions $\{v_{\lambda,n}\}_{n\geq 1}$ and $\{v_{\lambda',n}\}_{n\geq 1}$ converge uniformly to function v. Hence, the sequences of derivative functions $\{D_1v_{\lambda,n}\}_{n\geq 1}$ and $\{D_1v_{\lambda',n}\}_{n\geq 1}$ converge uniformly to function D_1v on every compact set $K \subset \text{int}(X)$ [Rockafellar (1970, Theorem 25.7)]. Observe that $D_1v_{\lambda,n}(\overline{x}_0, z_0) = D_1U(\overline{x}_0, \overline{x}_1, z_0) + D_1g^\top(\overline{x}_0, \overline{x}_1, z_0)\lambda_0$, and $D_1v_{\lambda',n}(\overline{x}_0, z_0) = D_1U(\overline{x}_0, \overline{x}_1, z_0) + D_1g^\top(\overline{x}_0, \overline{x}_1, z_0)\lambda'_0$, for all n. The convergence of these derivatives to a unique common value implies that $D_1g^\top(\overline{x}_0, \overline{x}_1, z_0)\lambda_0 = D_1g^\top(\overline{x}_0, \overline{x}_1, z_0)\lambda'_0$.

Moreover, by the same argument it follows that $D_1 g^\top(\overline{x}_1, \overline{x}_2, z_1)\lambda_1 = D_1 g^\top(\overline{x}_1, \overline{x}_2, z_1)\lambda_1'$ almost everywhere. Then, by condition $D3$ applied to (15) we get uniqueness of the multiplier, $\lambda_0 = \lambda_0'$. ■

PROOF OF THEOREM 3.3. Let us first prove that $\{v_n\}_{n\geq 0}$ converges uniformly to v. The proof is standard. Pick an initial condition x_0. For this initial condition x_0, let $\{x_t\}_{t\geq 0}$ be an optimal contingency plan for optimization problem (Γ, U, β, Q), and let $\{x_{nt}\}_{t\geq 0}$ be an optimal contingency plan for optimization problem $(\Gamma, , U_n, \beta, Q)$. Without loss of generality, assume that $v(x_0, z_0) > v_n(x_0, z_0)$. Then,

$$\begin{aligned}
v(x_0, z_0) - v_n(x_0, z_0) &= \sum_{t=0}^{\infty} \beta^t \int_{Z^t} \left(U(x_t, x_{t+1}, z_t) - U_n(x_{nt}, x_{nt+1}, z_t)\right) \mu^t(z_0, dz^t) \\
&\leq \sum_{t=0}^{\infty} \beta^t \int_{Z^t} \left(U(x_t, x_{t+1}, z_t) - U_n(x_t, x_{t+1}, z_t)\right) \mu^t(z_0, dz^t) \\
&\leq \frac{1}{1-\beta} \|U - U_n\|.
\end{aligned}$$

Hence, $\{v_n\}_{n\geq 0}$ converges uniformly to v. Therefore, for each z the sequence of mappings $\{v_n(\cdot, z)\}_{n\geq 0}$ converges to $v(\cdot, z)$. By Theorem 3.1, all functions $v_n(x, z)$ and $v(x, z)$ are differentiable in x. As these functions are also concave in x, by Rockafellar (1970, Theorem 25.7) the sequence of derivative functions $\{D_1 v_n(\cdot, z)\}_{n\geq 0}$ converges uniformly to $D_1 v(\cdot, z)$ on every compact set $K \subset \mathrm{int}(X)$. ■

PROOF OF PROPOSITION 3.2. Under the stated non–negativity conditions it is easy to see that at every point (x_0, z_0) the superdifferential $\partial_1 v(x_0, z_0)$ must be composed of non–negative numbers. Then this optimization problem can be reconverted into an asset pricing model with real assets along the lines of Santos and Woodford (1997); see especially their footnote 10. This asset pricing model considers a matrix of returns – which in this case it is given by the vector $D_1 U(x, y, z) + G(x, y, z) D_2 U(x, y, z)$ – and a matrix of transformation of securities – which in this case it is given by the matrix $G(x, y, z)$. Hence, interior solutions correspond to one–period assets, and boundary solutions lead to multiperiod assets. The bubble term belongs to long–lived assets starting at $t = 0$. Then, for every optimal path $\{x_{t+1}(z^t)\}_{t\geq 0}$ we can generate a sequence of asset prices $q(z_t) \in \partial_1 v(x_t, z_t)$ so that the asset pricing equation (6) is always satisfied. We can also introduce a single consumption good at each date with relative price equal to unity, and assume that the marginal utility of consumption at the optimal point is equal to one.

End–of–period asset holdings can be defined in a rather arbitrary way, as the agent can be endowed with new securities at the beginning of each period so as to replicate the optimal path $\{x_{t+1}(z^t)\}_{t\geq 0}$. Hence, under the stated assumptions it follows from Santos and Woodford (1997) that the bubble term $B_0 = 0$. ■

An Extension of Theorem 3.1 for non–smooth data. Lemma 5.1 remains true if the functions g^i are concave but not necessarily smooth. Then, our results can be extended to concave problems with non–smooth data. This extension relies on the characterization of the normal cone to Ω under the much weaker Slater's condition [cf. Rockafellar (1970)]:

$$-N_\Omega(x_0, y_0, z_0) = \Big\{(q,p) \in \mathbb{R}^{2n} \,:\, (q,p) \in \sum_{i\in I(x_0,z_0)} \lambda^i \partial_{1,2}\, g^i(x_0, y_0, z_0),\ \lambda^i \geq 0\Big\}. \qquad (19)$$

Therefore, our next result will be useful in the model of Kocherlakota (1996), where the value function appears in the constraints, and may not be differentiable. This result should be of interest for some other models with incentive and participation constraints [e.g., the unemployment insurance model of Hopenhayn and Nicolini (1996) and several other models discussed in Ljungqvist and Sargent (2004)].

For the following result we modify $D3$ as follows.

D3": *There is a finite collection of functions $g = (\ldots, g^i, \ldots)$, for $i = 1, 2, \ldots, m$, such that $\Omega = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times Z \,:\, g(x, y, z) \geq 0\}$. Each function $g^i(\cdot, \cdot, z) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is concave.*

The following result is a simple consequence of Lemma 5.1.

Corollary 5.1 *Consider a feasible optimization problem (Γ, U, β, Q). Let D1–D3$''$ be satisfied. For any $x_0 \in \operatorname{int}(X)$ and $z_0 \in Z$, we must have: $q_0 \in \partial_1 v(x_0, z_0)$ if and only if there exists $(\xi_1, \xi_2) \in \partial_{1,2}\varphi(x_0, y_0, z_0)$ such that $(q_0 - \xi_1, -\xi_2) \in -N_{\Omega_{z_0}}(x_0, y_0)$ for every $y_0 \in \operatorname{int}(X)$.*

Proof of Theorem 4.1. First we show that $S_2(U_0) \neq S$ and $S_3(U_0) \neq S$. Suppose not. For concreteness, assume that $S_2(U_0) = S$. Then, summing over (P_2) we get that $U_0 = \sum_{s=1}^{S} \pi_s[u(c_s) + \beta U_s] = U_{\text{aut}}$. But this is in contradiction with (P_1) as $U_0 > U_{\text{aut}}$.

By Corollary 5.1 and the characterization of $-N_\Omega(x_0, y_0, z_0)$ in (19), we now have to prove that the superdifferential of V at U_0 is unique. As this is an exchange economy, one should expect the derivative of the value function to depend on current utilities. Indeed, following Kocherlakota (1996) for any $U_0 \in (U_{\text{aut}}, U_{\max})$, we get the following system of first–order conditions

$$0 = -\pi_s u'(\overline{\omega} - c_s) + \lambda \pi_s u'(c_s) + \mu_s u'(c_s) - \nu_s u'(\overline{\omega} - c_s), \tag{20}$$

$$0 = \beta \pi_s q'_s + \lambda \beta \pi_s + \mu_s \beta + \nu_s \beta q_s, \tag{21}$$

for some $q'_s \in \partial V(U_s)$ for all s, and $\lambda, \{\mu_s\}_s$, and $\{\nu_s\}_s$ are Kuhn–Tucker multipliers corresponding to the constraints (P_1), (P_2), and (P_3), respectively. It turns out that the first equation (20) suffices to pin down the multiplier λ, and hence the derivative of the value function depends on the consumption allocation in the first period.

To study the solutions of system (20)–(21) we distinguish four cases that encompass all possibilities. In the first two cases we consider that $q'_s = q_s$ in (21). This does not entail any loss of generality, since the case $q'_s \neq q_s$ may add other multiple solutions.

(i) $S_2^b(U_0) \cup S_3^b(U_0) = S$. That is, for every s we have that either $\mu_s > 0$ or $\nu_s > 0$. Note that $\lambda > 0$. Assume that $q'_s = q_s$ in (21). Multiplying (21) by $-\frac{u'(c_s)}{\beta}$, and adding up (20) and (21) we get that $q_s = -\frac{u'(\overline{\omega} - c_s)}{u'(c_s)}$. Hence, in this case (20) and (21) are always collinear. Therefore, to see the determinacy of these multipliers it suffices to consider the system of equations (20). There are then S equations in $S + 1$ unknowns, λ, μ_s and ν_s. (Note that μ_s and ν_s appear only in the equation associated with state s and either $\mu_s > 0$ or $\nu_s > 0$.) It follows that there are multiple solutions, and so there is a continuum of λ that satisfy (20) and (21).

(ii) $S_2^b(U_0) \cup S_3(U_0) = S$ or $S_2(U_0) \cup S_3^b(U_0) = S$, but $S_2^b(U_0) \cup S_3^b(U_0) \neq S$. For simplicity, let us just consider $S_2^b(U_0) \cup S_3(U_0) = S$, where $\mu_s > 0$ for all $s \in S_2^b(U_0)$ and $\nu_s \geq 0$ for all $s \in S_3(U_0)$ with $\nu_s = 0$ for some s. As in *(i)*, indeterminacy of the solutions does exist, but the multiplier λ can only be increased from the original value as some $\nu_s = 0$. (If λ is decreased then some ν_s is forced to be a negative number.)

(iii) $S_2^b(U_0) \cup S_3(U_0) \neq S$ and $S_2(U_0) \cup S_3^b(U_0) \neq S$ but $S_2(U_0) \cup S_3(U_0) = S$. In this case, the multiplier λ is unique, and it is just determined by (20). It cannot be increased because in the above equation system (20) there is some $\mu_s = 0$, and it cannot be decreased because there is some $\nu_s = 0$.

(iv) $S_2(U_0) \cup S_3(U_0) \neq S$. In this case there is some state s for which neither (P_2) nor

(P_3) are saturated. Then, for some s we have that both μ_s and ν_s are equal to zero. It then follows from (20) that λ is unique.

Therefore, in cases (iii)–(iv) the multiplier λ is unique and the value of the derivative $V'(U_0) = -\lambda$ is as stated in the theorem. ■

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