RECURSIVE UTILITY FOR THOMPSON AGGREGATORS: UNIQUENESS VIA CONCAVE OPERATOR THEORY AND ITERATIVE APPROXIMATIONS

ROBERT A. BECKER[1,*], JUAN PABLO RINCÓN-ZAPATERO[2]

[1]Department of Economics, Indiana University, Bloomington, IN 47405, USA
[2]Department of Economics, Universidad Carlos III de Madrid, 28903 Getafe (Madrid), Spain

Abstract. An alternative proof of the uniqueness theorem for recursive utility specified by a Thompson aggregator is available by verifying the Koopmans operator is a $u_0 - concave$ operator**.** The Koopmans operator's unique fixed point is a recursive utility. Uniqueness holds only on the interior of the commodity space's positive cone. Consideration of $r - concave$ operators also yields a unique fixed point. An *a posteriori* error bound relates the norm difference between successive approximations of the fixed point and the fixed point.

Keywords. Recursive utility; Thompson Aggregators; Koopmans Equation; $u_0 - concave$ operators; $r - concave$ operators; successive approximations of the fixed point; *a posteriori* error bound.

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1. Introduction

An alternative proof of Marinacci and Montrucchio's [44] uniqueness theorem for recursive utility specified by a Thompson aggregator is available by verifying the Koopmans operator is a $u_0 - concave$ operator as defined by Krasnosel'skiĭ and Zabreĭko [37]. An underlying utility space of admissible solutions to the Koopmans functional equation is specified first along with a determination of the underlying commodity space. The $u_0 - concave$ operator approach shares several structural conditions with Marinacci and Montrucchio's [44] strong subhomogeneous technique. Both proofs show the Koopmans operator has a unique fixed point in the given utility space with the imposition of certain commodity space restrictions. We also consider a stronger Thompson aggregator property: the aggregator is $r - concave$ in its second argument. This strengthening of the Thompson aggregator's properties, proposed by Balbus [10], yields a sequence of successive approximations to the unique fixed point. An *a posteriori* error bound governing the distance between each successive approximation and the fixed point is computable given an $r - concave$ aggregator and appropriate commodity space restrictions. This error bound implies the successive approximation sequence uniformly converges to the unique fixed point.

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*Corresponding author.

E-mail address: becker@iu.edu (R.A. Becker), jrincon@eco.uc3m.es (J.P. Rincón-Zapatero).

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This paper was written in memory of Professor Tapan Mitra, on what would have been his 75th birthday. His works span an extraordinary breadth of economic theory with a special emphasis on fundamental questions in capital theory and economic dynamics.

1.1. **The Problem Situation.** Recursive utility theory focuses on a broad class of intertemporal utility functions that are stationary, time consistent, and time invariant, as well as tractable, in an array of capital theoretic and macrodynamic applications.[1] Optimal growth theory with a recursive utility objective was initiated by Beals and Koopmans [11]. This is the major application arena for discrete time deterministic recursive utility function theories with an infinite horizon setup. Recursive utility theory describes classes of utility functions with many of the attractive properties of stationary exponential discounting time additive utility functions. The underlying commodity space is the set of all nonnegative bounded real-valued sequences, denoted ℓ_∞^+. It is the utility function's domain.

The need for generalizations of the exponential discounting model derive from theoretical considerations. For example, the fixed discount factor assumption is known to drive strong results in optimal growth models and their equilibrium counterparts that may be modified when a particular recursive utility function replaces the exponential discounted utility function.[2] Consequently, the problem of foundations of recursive utility as well as development of methods for creating recursive utility functions are important for promoting applications of recursive utility models.

Contemporary recursive utility function research focuses on proving a recursive utility function solves a particular functional equation, the *Koopmans Equation*.[3] A solution is found as a fixed point of the equation's corresponding nonlinear operator, the *Koopmans operator*. The decision maker may be the planner of optimal growth theory or an infinitely-lived representative household in general equilibrium models. This agent has an underlying intertemporal preference ordering over elements of the commodity space which are termed **consumption sequences** (also known as **streams, profiles** or **bundles)** with generic element $C = \{c_1, c_2, \ldots, c_t, \ldots\}$ where c_t is the time-dated consumption at time $t = 1, 2, \ldots$. A utility function U representing an agent's intertemporal preference over alternative consumption sequences is said to be a **recursive utility function** if there is a function $W(x, y)$ of two real variables such that for each C, $U(C) = W(c_1, U(SC))$, where x is current consumption ($t = 1$), $y = W(x, y)$, $y = U(SC)$ is the utility continuing the consumption stream from time two forward into the future with $SC = \{c_2, c_3, \ldots\}$ and S is the shift operator. The function W is the **aggregator**. This form of the utility representation includes a self-referential property as U appears on the left and right-hand sides of the **Koopmans equation,** $U(C) = W(c_1, U(SC))$ for each C. The utility function on the right side is passed through the aggregator function to form the recursive utility structure.

Koopmans ([31], [32], [33]) derived a recursive utility function representation of an intertemporal preference ordering over alternative consumption sequences and its companion aggregator function based on an axiomatic framework. The axioms embed three economically important utility functional form restrictions. First is *weak separability of the future from the past*.[4] There are two mutually exclusive commodity groups when a consumption sequence is viewed at time

[1] See Halevy [29] on the distinctions between stationarity, time consistency and time invariance. Recursive utility functions possess all three properties.

[2] Epstein and Hynes [26] catalog a range of applications where exponential discounting promotes strong conclusions about steady state optimal growth/equilibrium solutions. Becker and Boyd ([12], Chapter 1) also develops this theme.

[3] There is a parallel literature on dynamic programming with recursive utility functions that is noted in more detail below.

[4] The axiomatic treatment of recursive utility is exposited in Becker and Boyd ([12], pp. 71-75).

zero before consumption begins at time one and continues in each subsequent time period. Decompose $C = (c_1, SC)$ into the first commodity group, $\{c_1\}$, and the second, $\{SC\}$. The weak separability property suggests utility can be written as $U(C) = W(u(c_1), V(SC))$ where u and V are the subutility functions of each commodity group. In the context of a single all purpose consumption good in each period the aggregator's functional form subsumes the utility of the subgroup $\{c_1\}$, and $U(C) = W(c_1, V(SC))$.

The second structural feature is *stationarity*: for $C = \{c_1, SC\}$ and $C' = \{c_1, SC'\}$, $U(C) \geq U(C')$ if and only if $U(SC) \geq U(SC')$. Hence, $U(C) = U(c_1, V(SC)) \geq U(c_1, V(SC')) = U(C')$. Since W depends on SC through $V(SC)$, and the preference order does not depend on calendar time, then $U = V$.

The last feature is a nontriviality or *sensitivity condition*: the utility function is not a constant function on the commodity space. There exist C and C^* such that $U(C) > U(C^*)$ if and only if C is strictly preferred to C^*. This condition would apply to non-recursive utility representations as well. An aggregator strictly increasing in each argument yields a sensitive utility function.

Combining the first two restrictions means U has a *recursively separable structure* — the defining distinction for a recursive utility function representation of a preference order. The aggregator unites the subutility values of the two nonoverlapping commodity groupings to produce an overall utility value when the two commodity groups are reunited as one consumption sequence.

Contemporary recursive utility theory emphasizes an alternative to Koopmans' axiomatic approach. The aggregator is the primitive building block for a recursive utility function. *Given an aggegator, the problem situation is to solve the Koopmans equation by finding a recursive utility function within a specified class of possible or admissible utility functions.* Formulate this problem as follows. Specify an economically motivated aggregator function. Align its qualitative properties with a class of possible real-valued utility functions on the underlying commodity space. This is the utility space. The Koopmans equation is a functional equation in the unknown utility function. The definition of the Koopmans equation and its corresponding operator depend on the joint properties of the aggregator and the underlying utility space/commodity space specifications. For example, a nonnegative aggregator suggests utility functions are nonnegative on the commodity space. Hence, the positive cone of an ordered vector space of functions defined on the commodity space is an appropriate utility space.

The Koopmans operator (belonging to W), denoted T_W, is a self-map on the utility space. Given a U in the utility space, define the Koopmans operator pointwise by the formula:

$$(T_W U)(C) = W(c_1, U(SC)) \text{ for each } C \in \ell_\infty^+.$$

A fixed point of the Koopmans operator solves the Koopmans equation implying $U(C) = T_W U(C)$ for each C defines a recursive utility function. Therefore, the problem is, given an aggregator, prove the Koopmans operator has a solution in the utility space. Moreover, verify it is the unique solution in the specified utility space. The main issues in solving this problem arise from the self-referential utility structure based on the primitive aggregator's specification.

Lucas and Stokey [42] initiated the aggregator as primitive foundation for finding recursive utility functions. Their assumed aggregator properties allowed them to prove the Koopmans equation has a unique solution in the class of sup-norm continuous bounded functions defined on ℓ_∞^+. This utility space is an ordered Banach space. In fact, it is a Banach lattice with unit, the constant function taking the value 1 for each C. They verify Koopmans' operator is a contraction mapping. Their proof depends on the utility function's ordered Banach space structure. The unique fixed point is a bounded continuous recursive utility function. It represents SOME agent's preference order over alternative consumption sequences.

Lucas and Stokey prove the Koopmans operator's contraction mapping satisfies Blackwell's [17] sufficient conditions for a nonlinear operator to be a contraction map. Their verification argument makes no formal use of the commodity and utility spaces are Banach lattices. They assume the aggregator W is a nondecreasing, bounded continuous function and satisfies a global Lipschitz condition in its second argument. This Lipschitz constant is assumed strictly less than one, a type of discounting property. They verify the Koopmans operator is a **monotone operator**, that is $U \leq V$ (pointwise) implies $T_W U \leq T_W V$ (pointwise) and has **Blackwell's contractive property**: $T_W(U+\gamma\varphi) \leq T_W U + \gamma\varphi$ for some constant $0<\gamma<1$ where φ is the constant function $\varphi(C) \equiv \varphi > \theta$ for each C. Becker and Boyd ([12], p. 48) prove a generalized Blackwell theorem for normed Riesz spaces possessing a principal ideal. For example, the principal ideal generated by the function φ is an order unit in the Banach lattice of all bounded real-valued functions on the commodity space (see Section 2).

Aggregators satisfying this Lipschitz condition are classified as **Blackwell aggregators**. Several papers extend Lucas and Stokey's approach to cover other aggregator specifications. Boyd [21] and Becker and Boyd [12] discuss many extensions in the Blackwell family. A number of papers published after Becker and Boyd's monograph extended the Blackwell model in novel ways where the aggregator's global Lipschitz condition fails, may not be bounded, and the Koopmans operator is not a contraction map (see Rincón-Zapatero and Rodriguez-Palmero ([51], [52]), Le Van and Vailakis [39], Martins-da-Rocha and Vailakis ([45], [46])) and Bloise, Le Van, and Vailakis [20].

Reduction of the existence and uniqueness problem for a Koopmans operator to an application of the contraction mapping theorem has another advantage in the Blackwell case. The sequence of iterations of the Koopmans operator with initial input the zero function, $\{T_W^N\theta\}$, has the property that $\{T_W^N\theta\}$ uniformly converges to U_∞, the fixed point of the Koopmans equation and for each C,

$$U_\infty(C) = \lim_{N\to\infty} T_W^N\theta(C) = \lim_{N\to\infty} W(c_1, W(c_2, \ldots, W(c_{N,}, 0))).$$

Here $T_W^N\theta$ is the N^{th} iterate of $T_W\theta$ according to the formula: $T_W^N\theta = T_W\left(T_W^{N-1}\theta\right)$ for $N \geq 1$, with $\theta(C) = 0$ for each C, the zero function, and $T_W^0\theta \equiv \theta$. The Contraction Mapping Theorem also implies for any initial choice of a function in the utility space, the corresponding sequence of iterates converges uniformly to the fixed point U_∞, which is a bounded continuous function. We return to this approximation problem is Section 6.

Marinacci and Montrucchio [43] proposed aggregators that did not fit into the previous literature. They named these examples as members of the **Thompson aggregator** class. For example, the KDW aggregator presented in Section 4.2 may fail to be a Blackwell aggregator for some interesting economic parameterizations. It is a member of their Thompson class in those situations.

They proposed new methods for solving the corresponding Koopmans equation for a given Thompson aggregator. They separated the problems of proving the existence of a solution to the

Koopmans equation from the determination there is a unique solution. The former proof emphasized monotone operator and order theoretic properties attached to the Koopmans operator. A constructive take on that problem is developed by Becker and Rincón-Zapatero ([14]). It is instructive to observe that the Koopmans operator remains a monotone operator in the Thompson case, but fails to satisfy Blackwell's contractive condition (at least in the utility space's natural norm topology). For this reason, monotone methods have been the pathway to both existence and uniqueness theorems in the Thompson aggregator literature. On the existence front monotone operator theories yield a pair of extremal fixed points in combination with underlying order theoretic structures on the utility and commodity space. There is a Least Fixed Point (LFP), U_∞, and a Greatest Fixed Point (GFP), U^∞. Our focus in this paper concerns the uniqueness problem that amounts to proving the LFP and GFP agree.

Counterexamples show that the Koopmans equation attached to a Thompson aggregator may have multiple solutions in the utility function space (and this specification depends on the chosen underlying commodity space). The LFP and GFP disagree at some point(s) in the commodity space. The LFP's value is smaller than the GFP's value in those cases. We show here how to modify the uniqueness problem to work around the known counterexamples.[5]

Their uniqueness theories in ([43], [44]) restrict the commodity space more than the necessary structures for existence theorems. More distinctive is their use of Thompson's [54]) decomposition of the utility space (with its norm topology) into disjoint subsets of the positive cone with its interior as the focus for uniqueness theory. They exclude fixed points in the positive cone's boundary; each fixed point must be in the positive cone's interior. The method for removing boundary fixed points in utility space depends on modifying the underlying commodity space. The commodity space restrictions knock out the troublesome consumption sequences where known examples admit multiple recursive utility functions as solutions to the Koopmans equation. Moreover, we indicate how these examples yield boundary fixed points. For norm interior consumption sequences alone (and corresponding implicit restrictions in the utility space) a uniqueness result becomes available. No boundary fixed point exists. The LFP and GFP are equal to one another in this refined commodity space domain. The Koopmans operator, so restricted in their modified setup, turns out to be a contraction mapping on the implied utility space with respect to the Thompson metric according to Marinacci and Montrucchio's earlier paper, [43].

Their more general methods paper, [44], also uses the Thompson metric. They show the Koopmans operator for a Thompson aggregator is strongly subhomogeneous (see our Section 2 for this concept). Either way, their approach replaces the original norm topology in the utility space, which is the norm interior of an ordered Banach space's positive cone, by a complete metric space using Thompson's metric.[6] They are able to take their Thompson metric space results back to the original ordered Banach space since the Thompson metric is a stronger topology than the norm topology in their modified utility space. The relevant sequences are found by iterating the Koopmans operator over the natural numbers with appropriate initial seeds. Strong subhomogeneity implies these iterates converge in the Thompson metric to a utility function and

[5] In particular, Bloise and Vailakis [18] provide three interesting examples where the Koopmans equation has multiple solutions.

[6] These manipulations are directly worked out in their first paper, [43] and appear in the mathematical results in the second paper [44].

this implies the sequence of iterates norm converges as well. This norm convergence conclusion is a form of successive approximation to the fixed point. Their results, obtained by either the contraction or strong subhomogeneous and Thompson metric techniques, prove the existence of a unique solution to the Koopmans equation with the restricted commodity space. In fact, the Thompson and norm topologies are equivalent in their setting (see Cobzaş and Rus ([22], pp. 240-241 and Guo, et al ([28], pp. 72-73)). A sequence Thompson converges if and only if the corresponding sequence norm converges. *This topological equivalence suggests an alternative uniqueness strategy employing ordered Banach space machinery alone might be available.*[7]

We propose an alternative uniqueness theory in this paper employing a natural order-concave property enjoyed by the Koopmans operator when the Thompson aggegator is a concave function. This order-concavity property is inherited by the Koopmans operator. We prove it is a $u_0 - concave$ operator on the utility space's interior. The utility space's Banach lattice properties are used rather than Thompson metric tools. In this sense, we are working out the way to apply $u_0 - concave$ operator methods as a "bench test" comparison of different solution methodologies in a the same functional equation setup. This approach abstracts and adapts the notion of a concave function to the case of a nonlinear operator acting as a selfmap on the space's positive cone.

Krasonsel'skĭĭ and Zabreĭko's [37] $u_0 - concave$ operator theory for ordered Banach spaces with norm closed positive cones yields at most one solution exists to the given functional equation. Existence theory for that question is treated separately.

Applications of $u_0 - concave$ operator theory presented in Krasonsel'skĭĭ and Zabreĭko's [37] and Guo and Lakshmikantham [27] place a common restriction on the ordered Banach space — it is a Banach lattice with unit. Likewise, Coleman's ([23], [24]) variants of $u_0 - concavity$ operate on subsets of a Banach lattice with unit. This Banach lattice structure is not officially recognized in their works, but is apparent from their choices for the functional equation's underlying space of possible solutions. Marinacci and Montrucchio's [44] recursive utility and dynamic programming applications use Banach lattice with unit domains for their functional equations's domains. We focus attention in Section 3 on the role played by Banach lattice properties in verifying a given nonlinear operator is $u_0 - concave$.

A major advantage of assuming a Banach lattice with unit is the positive cone is solid and normal (Section 2). This enables both Marincacci and Montrucchio and us exploit Thompson's *link (comparable)* binary relation to center uniqueness theory in the utility space's positive cone's interior. We have more to say on this point as our approach simplifies and clarifies the way Thompson's binary relation enters uniqueness theory and how the presence of an order unit is critical for proving the Koopmans operator is a $u_0 - concave$ operator.

In particular, *we exploit all the economically motivated mathematical structures available in the model's utility and commodity spaces in order to arrive at a satisfactory uniqueness theory.* In our approach the Koopmans operator is strictly subhomogeneous, and hence, weaker than Marincacci and Montrucchio's [44] strong subhomogeneous operator property. One important difference with our two theories is we assume the Thompson aggregator is a concave function. This implies the Koopmans operator is order-concave and also subhomogeneous. This creates a way for us to verify the Koopmans operator is $u_0 - concave$ based on the order-concave

[7] Cobzaş and Rus ([22], p. 241) show by an example that a bounded set in one metric may not be bounded in the other.

sufficient conditions developed by Liang et al [40]. Our methodology also helps isolate the source of nonuniqueness problems as cases where the Least Fixed Point (see below) lies in the topological boundary of the utility space's positive cone. Commodity space restrictions are motivated by examination of how nonunique solutions may arise in the commodity space ℓ_∞^+.

There is a constructive prong in taking the Krasonsel'skiĭ and Zabreĭko's [37] $u_0-concave$ operator approach based on iteration of the Koopmans operator over the natural numbers and initiated at the zero function. The details of how this approach constructs the Koopmans' operator's LFP differ in important details from the ones used by us in [14]. Our successive approximation argument here yields two results. First, the sequence $\left\{T_W^N\theta\right\}$ uniformly converges to the LFP as $N\to\infty$. More important, for each N we have an error bound, $B(N,T_W\theta)$ satisfying $\left\|T_W^N\theta-U_\infty\right\|\leq B(N,T_W\theta)\to 0$ as $N\to\infty$. This error bound depends only on the approximate solution at step $N, T_W^N\theta$, the underlying commodity space restrictions detailed in Section 5, and the bound is independent of U_∞. Development of this computable *a posteriori error estimate* is of computational theoretic interest.[8] A strengthening of the Thompson aggregator conditions is needed to develop the successive approximation method with its *a posteriori* error bound. We follow Balbus's [10] approach and assume the Thompson aggregator is an $r-concave$ function in its second argument. This allows us to show the Koopmans operator is an $r-concave$ operator and apply Balbus's theory. The linked relation also enters the verification Balbus's version of $r-concave$ operator theory applies in our Thompson aggregator setup. This iterative scheme and its uniform convergence property is reminiscent of the convergence theory for successive approximations with a contraction map.

The *a posteriori* error bound approximation significantly improves our earlier paper's [14] successive approximation results based on order theoretic structures in the *non-Hausdorff Scott topology*. An operator's continuity in the Scott topology plays a fundamental role in abstract computational theory based on successive approximations of a fixed point. The difficulty is that limits of iterative processes are not unique in this topology. A sequence's (net's) order limit might exist and is unique when it does so. However, in that case any possible utility function less than or equal to it (and not equal to it) with the usual pointwise ordering is also a Scott limit! Our existence theorem shows the Koopmans operator is Scott continuous and the sequence of successive approximations $\left\{T_W^N\theta\right\}$ Scott converges to the LPF, U_∞. However, this convergence to the LFP does not yield *any quantitative information* on how "close" a particular iterate, say $T_W^N\theta$, is to the LFP. Scott convergence only verifies $\left\{T_W^N\theta\right\}$ is eventually in each of the LFP's Scott neighborhoods. The $r-concave$ operator approach, with its *a posteriori* error bounds, puts a metric (norm) measure in play and allows us to say something about how many iterates, in principal, it takes for the iteration over the natural numbers to land in an epsilon norm-neighborhood of the LFP. In that sense, the corresponding error bounds' *quantitative measures* are a significant theoretical improvement over our Scott topology statements regarding convergence to the LFP.

1.2. **Related Literature: Recursive Utility, Dynamic Programming, and Related Applications.** We consider the deterministic Koopmans equation existence and uniqueness theory founded in studying the Koopmans equation for Thompson aggregators. There is a closely related parallel literature devoted to monotone methods for dynamic programming problems when

[8]See Linz [41] and Krasnosel'skiĭ [36] et al on the importance of *a posteriori errot* estimates.

the Bellman operator is monotone, but not a contraction mapping on the underlying space of admissible value function solutions to Bellman's equation. Marinacci and Montrucchio [44] examine the deterministic Bellman equation for a Thompson aggregator sequential optimization problem. Many other papers combine deterministic and stochastic problems (e.g. Marinacci and Montrucchio [43]). Contributions reviewed include Balbus [10], Bloise and Vailakis [18], Bloise, Le Van, and Vailakis ([19], [20]), and Ren and Stachurski [50]. We note that Bertsekas [16] draws general attention to the use of monotone and iterative methods in abstract dynamic programming when contraction mapping techniques are inapplicable.

Balbus [10] takes an aggregator approach. His deterministic features yield existence and uniqueness theory for the Thompson aggregator class satisfying an additional subhomogeneity condition (briefly detailed in Section 3). He uses this to prove the Koopmans operator is *subhomogeneous of degree r* (or, $r-$*concave*) and the Koopmans equation has a unique solution in the utility space's positive cone's interior provided consumption streams are interior to the commodity space's positive cone. He draws on a nonlinear operator uniqueness theorem presented in Guo, et al [28]. This result implies iteration from any initial function in the utility space converges in norm to the Koopman's equation's unique solution and offers "truncation" error estimates as well.[9] He also provides an example of an aggregator that is neither Thompson nor Blackwell, yet his approach applies. This example is excluded in our uniqueness theory.

Bloise and Vailakis [18] derive a uniqueness theorem for deterministic and stochastic dynamic programs with recursive utility objectives. They also prove some novel results for the deterministic Koopmans equation that are relevant for our study.[10] Their focus lies on the Greatest Fixed Point (GFP) belonging to the Koopmans equation for a concave Thompson aggregator in contrast to our later focus on the Least Fixed Point (LFP) in [14]. These solutions are distinct in many examples, at least for some consumption sequences in the commodity space's boundary. Interestingly, they show that the GFP is the unique product topology upper semicontinuous solution to the Koopmans equation on norm-bounded subsets of the commodity space (this includes the consumption space's boundary as well). The GFP is also monotone and concave. These are attractive properties for optimization problems with a recursive utility function objective. So, any other solution, e.g. the LFP, must fail to have this upper semicontinuity property. This underlies their focus on the GFP as the planner's objective.

Bloise, Le Van, and Vailakis [19] address the central issue in a dynamic programming context when the Koopmans equation has multiple solutions. The existence of a distinct LFP and GFP in the Thompson setting (via examples) implies a planner's objective function is ambiguous. They argue in favor of selecting the GFP and its corresponding greatest Bellman equation operator's fixed point — its optimal value function. Their methods also rely on monotone operator techniques as well as the value convexity-concavity methods in Ren and Stachurski [50]. Their message is that uniqueness theory cannot rescue dynamic programmers from the inherent

[9]These are the *a posteriori error estimates* in Krasonsel'skĭĭ et al [36].

[10]Their existence proof for the Koopmans equation follows the auxilliary recursive equation method introduced by Marincacci and Montrucchio [43]. They fix a consumption sequence and iterate the Koopmans equation in utility *values*, which are real numbers. The analog of the Koopmans operator for fixed consumption bundle maps sequences of bounded utility values to bounded utility values. That is, the sequence of utility values lies in the positive cone of ℓ_∞. This operator has a LFP and GFP. They must separately verify the LFP and GFP are recursive utility functions. Our theory both here, and in [14], directly studies iterates of the Koopmans operator as a self map on an order interval of functions in the utility space.

ambiguity of multiple solutions to the Koopmans equation. Instead, the argue for selecting the Bellman equation value function's GFP solution as the one appropriate for optimal growth and related asset pricing problems whether they be deterministic or stochastic.[11]

There are other techniques for proving uniqueness in Thompson aggregator settings besides the one proposed here. For example, our paper [15] verifies the Koopmans operator satisfies the hypotheses of Du's [25] order-concave operator theoretic uniqueness theorem and an iterative approximation theory on an economically important order interval in the utility space's positive cone. The linked relation also plays a prominent role in this approach. This paper uses some techniques that also appear in the present work. Related concave operator methods are developed in Le Van, Morhaim, and Vailakis [38] in the context of deterministic (reduced form) optimal growth models. Their methods include versions of several techniques that show up in more recent work derived from Marinacci and Montrucchio's [43] work. Linked relations and logarithmic (exponential) transformations of the Bellman operator are two examples along with the emphasis on monotone methods and subhomogeneous operator notions.

The hypotheses underlying $u_0 - concavity$ theory as well as the approach taken in [15]) are, in our view, easier to state and understand in the context of the Thompson aggregator uniqueness theory compared to the other known uniqueness methods. From a methodological view, we view the Koopmans equation as a testing methodology for learning about the pros and cons of various uniqueness strategies when the underlying operator is monotone, but not a contraction. Experimentation with different tools in a common problem situation, such as that of the Koopmans equation, provides researchers with the means to compare the usefulness of one technique or another in the same setting. For example, it might be interesting to revisit Coleman's [24] Euler equation policy iteration solution from the perspective of $u_0 - concavity$ rather than his pseudo-concave operator variant of a subhomogeneous operator.

1.3. **A Sectional Preview.** Section 2 reviews Riesz space and Banach lattice structures. Section 3 introduces $u_0 - concave$ nonlinear operator solution theory. We follow the Riesz space conventions and definitions in Aliprantis and Border [2] unless otherwise noted. Section 3 covers the basics of Krasnosel'skiĭ and Zabreĭko's [37] $u_0 - concave$ operator uniquness theory as well as Liang et al's [40] sufficient condition. That review is specialized to results applicable in the particular example of a Banach lattice with unit in our application. Thompson aggregators appear in Section 4 which also sets up the underlying commodity space and the vector space of possible utility functions, specifies the Koopmans equation in detail as well as a counterexample to the problem of uniqueness of that equation's solution. Section 5 presents our uniqueness theory as an application of Krasnosel'skiĭ and Zabreĭko [37] theory. The main issue is verifying the Koopmans operator is a $u_0 - concave$ operator. Section 6 examines approximation of the unique Koopmans equation's solution by iterative methods. Two different, but related, strategies are presented. One is based on the utility and commodity spaces' Banach lattice properties in conjunction with $u_0 - concavity$. The second focuses on building a computable *a posteriori* error bound for uniform convergence of successive approximations of the Koopmans equation's unique solution. This result is based on methods used by Balbus [10]. This error bound is derived when the aggregator is $r - concave$ in its second argument and induces an $r - concave$ Koopmans operator. The final section offers concluding comments.

[11] They argue in [20] abstract dynamic programs may have spurious solutions. They produce an example where the GFP is deemed spurious and reject it in favor of the LFP as the economically interesting solution.

2. Mathematical Structures

The uniqueness theory presented in this paper combines order and metric properties of the commodity and utility function vector spaces. The mathematical structures employed are reviewed below. The natural numbers are denoted by $\mathbb{N}=\{1,2,\ldots\}$. A generic element of $\mathbb{N}$ is denoted by n (and sometimes by N in our application).

2.1. Positive Cones and Nonlinear Operators in Riesz Spaces.

A nonempty set X is said to be **partially ordered**, or a **poset**, if it is nonempty and there is a binary relation $x\geq y$ relating certain pairs (x,y) in $X\times X$ which is reflexive, transitive, and antiysmmetric. A poset X is a **lattice** provided each pair of elements has a **supremum (sup, meet)** and an **infimum (inf, join)**. Standard lattice notation for sups and infs is followed: $\sup\{x,y\}=x\vee y$ and $\inf\{x,y\}=x\wedge y$. A **complete lattice** is a lattice in which each nonempty subset Y has a supremum $\bigvee Y$ and an infimum $\bigwedge Y$.

Let E denote a real vector space. It is an **ordered vector space** when it is equipped with a partial order relation, denoted $\geq$, that is reflexive, antisymmetric, transitive, and for each x and y in E: (i) $x\geq y$ implies $x+z\geq y+z$ for each $z\in E$ and (ii) $\alpha x\geq \alpha y$ for each scalar $\alpha\geq 0$. Understand all vector spaces encountered in this paper are ordered vector spaces over the real numbers. A **Banach space** is a complete normed vector space. An ordered vector space that is also a Banach space is an **ordered Banach space**.

Denote the zero element in E by θ unless otherwise noted. A nonempty subset P of E is said to be a **cone** if (i) $P+P\subseteq P,(ii)\,\lambda P\subseteq P$ for each scalar $\lambda\geq 0,(iii)\,P\cap(-P)=\{\theta\}$. We note that a cone as defined here is also a convex set and is said to be a **pointed convex cone**, or more simply, a **convex cone**.[12] A cone induces a partial order on the vectors belonging to E. A vector x is said to be **positive**, written $x\geq\theta$, provided $x\in P$. The cone is then called the **positive cone** of E and is denoted by E^{+} in the sequel. The standard partial relation expressing $x\geq y$ whenever $x,y\in E$ is defined by requiring $x-y\in E^{+}$. Write $x>\theta$ whenever $x\geq\theta$ and $x\neq\theta$. Likewise, $x>y$ provided $x\geq y$ and $x\neq y$.

We reserve the notation P to denote an *arbitrary cone* (which may not be the positive cone corresponding to the vector space's given partial order). The main mathematical theorems signal a formal generality by employing an arbitrary cone, P. Our applications use the positive cone, E^{+}, and we use that notation to signal this is the particular cone in our economic model. These distinctions are important in Section 3.

Let E be an ordered vector space equipped with the partial order derived from the cone E^{+}. It is a **Riesz space** provided it is also a lattice. In particular, each nonempty finite subset of E has a supremum and an infimum in E.[13] For each element $x\in E$, we define its **positive part**, x^{+}, its **negative part** x^{-}, and its **absolute value**, $|x|$, by the formulas:

$$x^{+}=x\vee\theta, x^{-}=x\wedge\theta\,,\text{ and }|x|=x\vee(-x)\,.$$

[12] Our definition follows Aliprantis and Tourky [8]. They note that definitions of cones may vary among authors. Krasnosel'skiĭ and Zabreĭko [37] define a cone more broadly (basically using property (ii) and the direct assumption that the cone is closed and convex. Our definition refines their definition by requiring a pointed cone. The cone's closure property is automatically satisfied in a locally convex-solid Riesz space (Aliprantis and Burkinshaw ([4], p. 163).

[13] Riesz spaces are also known as vector lattices. Consult Aliprantis and Border ([2], Chapter 8) for a thorough review of Riesz spaces. We follow their terminology. All Riesz spaces appearing in our paper are **Archimedean**.

An **order interval** in the Riesz space E is a set of the form $\langle x,y\rangle = \{z \in E : x \leq z \leq y\}$. A subset G of a Riesz space is **order bounded from above** if there is a $y \in E$ such that $z \leq y$ for each $z \in G$. The dual notion that this subset is order bounded from below is defined similarly. A subset of a Riesz space is **order bounded** if it is contained in an order interval. E is **order complete**, or **Dedekind complete**, if every nonempty subset that is order bounded from above has a supremum (and dually, every nonempty subset that is order bounded from below has an infimum). The specific spaces appearing in our economic model are Dedekind complete.

A **lattice norm** on a Riesz space E is a norm, $\|\bullet\|$, satisfying $|x| \leq |y|$ in E implies $\|x\| \leq \|y\|$. A lattice norm is monotonic in the absolute value of a vector. A necessary and sufficient condition for a lattice norm is $\|x\| = \||x|\|$ and $\theta \leq x \leq y$ implies $\|x\| \leq \|y\|$. This norm's induced metric may, or may not be complete. A Riesz space equipped with a lattice norm is a **normed Riesz space**. A subset S of a Riesz space E is **solid** if $|u| \leq |v|$ in E and $v \in S$ implies $u \in S$. A normed Riesz space's topology is **locally-solid** if it is equipped with a base of θ consisting of solid sets. If a locally-solid normed Riesz space's topology is also locally convex, then the topology is said to be **locally convex-solid**.

There is a notion of sequential convergence in a normed Riesz space that is based on order structure alone; it is not a topological concept. A sequence $\{x_n\}$ in E is said to **order converge** to $x \in E$ provided $\liminf_n (x_n) = \sup_n (\inf_{k\geq n}(x_k))$, $\limsup_n (x_n) = \inf_n \left(\sup_{k\geq n}(x_k)\right)$ exist and

$$\liminf_n (x_n) = x = \limsup_n (x_n).$$

In the event that $\{x_n\}$ is monotonic write order convergence as $\{x_n\} \nearrow x$, and dually when $\{x_n\}$ is antitone, $\{x_n\} \searrow x$. If E is Dedekind complete Riesz space, then in the former case, $x = \sup_n (x_n)$ and in the latter, $x = \inf (x_n)$.[14]

A **Banach lattice** is normed Riesz space that is complete in the sense of Cauchy with respect to its lattice norm. The function spaces in our applications turn out to be Banach lattices.[15] A Banach lattice's topology is Hausdorff and locally convex-solid.

Two norms on E, $\|\bullet\|_1$ and $\|\bullet\|_2$, are **equivalent** whenever there exist constants $K, M > 0$ satisfying $K\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$ for each $x \in X$. If E has a lattice norm turning it into a Banach lattice, then Goffman's Theorem says any other lattice norm that turns E into a Banach lattice is an equivalent norm.[16]

Additional restrictions on a Banach space's positive cone are required for our applications. First, the cone must be closed in the norm topology. This condition is automatically satisfied by positive cones in any (complete) normed Riesz space.[17]

Second, the positive cone E^+ is assumed **normal** — there is a constant $\mathcal{N} > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq \mathcal{N}\|y\|$ and $\mathcal{N}$ does not depend on the choices of x and y. The normality constant $\mathcal{N} = 1$ in our setup. The property that a cone is normal is a joint restriction on the space's norm and its positive cone (in particular, the underlying partial ordering). Counterexamples show that an arbitrary partially ordered Banach space's positive cone may not be

[14] Dedekind completeness may be weakened in favor of assuming E is a $\sigma-$*Dedekind complete* Reisz space. This means each order bounded countable set has a supremum and infimum in E.

[15] See Aliprantis and Border [2], Aliprantis and Burkinshaw [4], Meyer-Nieberg [47], and Peressini [48] for details on Riesz spaces and Banach lattices.

[16] See Aliprantis and Burkinshaw ([4], pp. 175-176) or Aliprantis and Border ([2], p. 352).

[17] See Aliprantis and Tourky ([8], p.87).

normal.[18] The positive cone in a normed Riesz space is normal.[19] This implies order intervals in E are norm-bounded.[20]

Consider an abstract nonlinear operator, denoted by A, that is positive on E^+. That is, it is a self-map: $A : E^+ \to E^+$. It is **monotone (isotone, increasing) on** E^+ if $x \leq y, (x, y \in E^+)$ implies $Ax \leq Ay$. The Koopmans operator derived from a Thompson aggregator is monotone. The operator A is **antitone (decreasing) on** E^+ if $x \leq y, (x, y \in E^+)$ implies $Ax \geq Ay$.

Given a nonlinear operator satisfying $AE^+ \subseteq E^+$ we are concerned with proving there is a *unique solution in the cone* E^+ provided a solution exists. The **operator equation** is $Ax = x$ with $x \in E^+$; a solution is a fixed point of the operator, A. In some applications there may be a trivial fixed point, θ. We are only interested in **nontrivial fixed points** $x \in E^+$ with $x \neq \theta$. The Koopmans operator does not admit a trivial fixed point. The set of fixed points belonging to the operator A is denoted $\text{fix}(A)$. A **Least Fixed Point (LFP)**, x^*, is a point in $\text{fix}(A)$ satisfying $x^* \leq x$ for each $x \in \text{fix}(A)$. Dually, a **Greatest Fixed Point (GFP)**, $x^{**} \in \text{fix}(A)$, satisfies $x \leq x^{**}$ for each $x \in \text{fix}(A)$.

The notation $x >> \theta$ means $x \in \text{int}(E^+) \equiv E^{++}$, where $\text{int}(E^+)$ denotes the norm interior of E^+. Of course, this is only meaningful when $\text{int}(E^+) \neq \varnothing$ — a strong topological restriction on the underlying Banach space and its positive cone. An arbitrary cone P contained in E with nonempty interior in its norm topology is said to be a **solid cone**. The positive cones in our model are solid. A solid cone need not be a solid set. A solid cone is a topological concept; a solid set is an order theoretic notion.

Define the **principal ideal generated by the vector** y in E^+:

$$A_y = \{x \in E : |x| \leq \lambda y \text{ for some } \lambda > 0\}.$$

This is a Riesz subspace of E. The Riesz space E possesses an **order unit** (or, simply a **unit**), $e > 0$, whenever $A_e = E$.

Define the **lattice norm** on A_e by the formula:

$$\begin{aligned} \|x\|_e &= \inf\{\lambda > 0 : |x| \leq \lambda e\} \\ &= \inf\{\lambda > 0 : -\lambda e \leq x \leq \lambda e\}. \end{aligned}$$

The latter equality tells us that $\|\bullet\|_e$ is the **Minkowski functional** of the set $\langle -e, e\rangle$. It is readily confirmed the Minkowski functional of $\langle -e, e\rangle$ defines a lattice norm on A_e. This principal ideal's lattice norm is also known as the **order unit norm**. If a principal ideal's lattice norm is also complete, then the principal ideal is a Banach lattice. This is our particular applied setting. A Banach lattice has an order unit if and only if that unit is an interior point of the space's positive cone; in addition, its positive cone is norm-closed, convex, solid, and normal.[21]

This Banach lattice, which equals the principal ideal A_e, is also an example of an *abstract* $M-space\ with\ unit$, or simply an $AM-space\ with\ unit$: it has the property whenever $x \wedge y = \theta$, then $\|x \vee y\|_e = \max\{\|x\|_e . \|y\|_e\}$.[22] Each $AM-space\ with\ unit$ is lattice isometric to the space

[18] See Guo et al ([28], p. 31 and pp. 40-41) and Khaleelulla ([30], p. 86).

[19] See Aliprantis and Tourky ([8], p.87).

[20] See Aliprantis and Tourky ([8], p. 87).

[21] Aliprantis and Tourky ([8], p. 87).

[22] The norm propery just displayed may hold for Banach lattices which do not possess a unit, such as c_0. These Banach lattices are the *AM-spaces* (Absract M spaces). We distinguish the broader $AM-space$ from an $AM-space\ with\ unit$ in our setting. See Aliprantis and Burkinshaw ([4], pp. 187-188).

of continuous real-valued functions defined on *some* compact Hausdorff space, K, and denoted by $C(K)$.[23] The constant function $e(x) = 1$ for each $x \in K$ is an order unit in A_e and, more generally, in the alternative representative space $C(K)$.[24] An $AM-\mathit{space\ with\ unit}$ is a Banach lattice with a unit. Its positive cone is solid and normal. $AM-\mathit{spaces\ with\ unit}$ form the class of vector spaces that are natural domains for applications of $u_0-\mathit{concave}$ operator theory since an order unit is the obvious choice for u_0.

If A_e is a Banach lattice, then Goffman's Theorem implies there is basically only one lattice norm for which it is a Banach lattice. Our application employs this equivalence property to link two different lattice norms in our application (and see below for more details). The unit ball in A_e is defined by the order interval $\langle -e, e\rangle = \{x \in A_e : -e \le x \le e\}$. Clearly, each point in the unit ball satisfies $\|x\|_e \le 1$ since $\|e\|_e = 1$. The unit ball is both order and norm-bounded, norm-closed, and convex. The unit ball is norm-compact *if and only if* it is a finite dimensional subset of A_e. The topology of A_e induced by the order unit norm is a locally convex-solid topology.

Krasnosel'skiĭ and Zabreĭko's [37] theory only assumes that E is an ordered Banach space with a closed positive cone. We verify a nonlinear operator acting as a selfmap on a Banach space's positive cone that is also solid. We assume u_0 is an order unit in the positive cone in order to verify the operator is $u_0-\mathit{concave}$. This implies the underlying ordered Banach space is an $AM-\mathit{space\ with\ unit}$. The uniform approximation theories we develop in Section 6 assume the positive cone is solid and normal, a feature of any $AM-\mathit{space\ with\ unit}$.

There is an equivalence relation on the positive cone that is useful in formulating uniqueness theorems (e.g. Marinacci and Montrucchio [44]). Thompson's [54] **comparability** relation states: vectors $x, y \in E^+$, both non-zero, are **comparable (linked)**, written $x \sim y$, provided there exist strictly positive scalers α and β (depending on x) such that:

$$\alpha y \le x \le \beta y. \tag{2.1}$$

Evidently this binary relation is an equivalence relation partioning the positive cone into disjoint *components*, or *constituents*, and defines a quotient set $E^+/\sim$. Set $Q(x) = \{y \in E^+ : x \sim y\}$. The zero vector does not belong to any constituent. Each component forms a complete metric space (in Thompson's metric) suitable for application of the Contraction Mapping Theorem and thereby solve some nonlinear operator existence and uniqueness problems. Our uniqueness theory avoids the Thompson metric while clarifying the ways in which the constituents enter the Krasnosel'skiĭ and Zabreĭko [37] $u_0-\mathit{concavity}$ theory.

The vector space of all real-valued functions defined on a non-empty set X is designated by $\mathbb{R}^X$. It is a Riesz space with the usual pointwise partial order. It is not a *normed* Riesz space. The corresponding positive cone is denoted by $\left(\mathbb{R}^X\right)^+$. There are two subspaces of particular interest. First is the Banach space (and Dedekind complete Riesz space) of all bounded real-valued functions defined on X :

$$\mathrm{B}(X) = \left\{ f : X \to \mathbb{R} : \|f\|_\infty = \sup_{x \in X} |f(x)| < \infty \right\}$$

[23] Schaefer ([53], p.106) finds the promised lattice isometric representation $C(K)$ by compactification arguments in several examples of AM-spaces with units. Schaefer discusses how the set K arises by compactification of X in two important cases: $\mathrm{B}(X)$, the bounded real-valued functions on a nonempty set X with its sup norm, and the related subspace, $C_b(X)$, of all bounded continuous functions defined on X.

[24] See Aliprantis and Border ([2], pp. 358-356) and Aliprantis and Burkinshaw ([4], p. 194).

with its usual pointwise partial order and sup-norm $\|\bullet\|_\infty$, which is also a lattice norm based on the order unit e, the constant function $e : X \to \mathbb{R}_+$ defined by $e(x) = 1$ for each $x \in X$. Therefore, $\mathrm{B}(X)$ is an $AM-space\ with\ unit$. Its positive cone is denoted by $\mathrm{B}^+(X)$. Here $e >> \theta$ and $\mathrm{B}^+(X)$ has a nonempty norm interior. Set $\mathrm{B}^{++}(X) = \mathrm{int}(\mathrm{B}^+(X))$. We abbreviate $\mathrm{B}(X)$ as B, $\mathrm{B}^+(X)$ by B^+, and $\mathrm{B}^{++}(X)$ by B^{++} when the underlying set X is understood.

Second, there is the principal ideal, A_e, corresponding to Endow A_e with its usual pointwise partial order and corresponding order unit norm, $\|\bullet\|_e$. Since e is an order unit in A_e, it follows that $\mathrm{B}(X) = A_e$ and the order unit norm is equivalent to the sup-norm $\|\bullet\|_\infty$ by Goffman's Theorem.

An important example of $\mathrm{B}(X)$ occurs for $X = \mathbb{N}$; it is the vector space of all bounded real-valued sequences with its sup-norm. The sequence $e = (1,1,1,\ldots)$ is an order unit. A constant vector is usually denoted as $x_{con} = \{x,x,x,\ldots\}$. The zero vector in this space is denoted by 0 instead of 0_{con} as the meaning is clear. Standard notation for $\mathrm{B}(\mathbb{N})$ is ℓ_∞. The sup-norm for $x \in \ell_\infty$ is $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$, where $x = \{x_1, x_2, \ldots\}$ and sometimes abbreviated as (x_n) or $\{x_n\}$. The positive cone is ℓ_∞^+ and its norm interior, ℓ_∞^{++}, is nonempty.

$\mathrm{B}(X)$ is the prototype for the (weighted) vector spaces encountered in our application. We work in principal ideals of $\mathbb{R}^X$ generated by weight functions that turn out to be order units in those principal ideals.

There are unbounded functions in $\mathbb{R}^X$ (and they arise in our application), so we transform *some* of those functions to bounded functions by applying a **weight function**, which can be any function $\varphi : X \to \mathbb{R}_+$ such that $\varphi(x) > 0$ for each $x \in X$. Define the φ **– weighted sup-norm** (or simply φ **– norm**) by the formula: for $f : X \to \mathbb{R}$,

$$\|f\|_\infty^\varphi = \sup_{x\in X} \frac{|f(x)|}{\varphi(x)}. \tag{2.2}$$

The φ – weighted sup-norm is easily verified to be a lattice norm on $\mathrm{B}_\varphi(X)$ where

$$\mathrm{B}_\varphi(X) = \left\{f : X \to \mathbb{R} : \|f\|_\infty^\varphi < \infty\right\}.$$

The space $\mathrm{B}_\varphi(X)$ with its φ – weighted sup-norm is a normed Riesz space. It is a Banach lattice: $|f| \le |g|$ and $\varphi(x) > 0$ for each x implies $|f|/\varphi \le |g|/\varphi$; take the supremum to verify $\|\bullet\|_\infty^\varphi$ is a lattice norm. Clearly $\|\varphi\|_\infty^\varphi = 1$, φ is an order unit and $\|\bullet\|_\infty^\varphi$ is an order unit norm.

The principal ideal $A_\varphi(X)$ is well-defined and its corresponding Minkowski functional, $\|f\|_\varphi$, is an order-unit norm on $A_\varphi(X)$, where:

$$\begin{aligned}\|f\|_\varphi &= \inf\{\lambda > 0 : |f| \le \lambda\varphi\} \\ &= \inf\{\lambda > 0 : -\lambda\varphi \le f \le \lambda\varphi\}.\end{aligned} \tag{2.3}$$

$A_\varphi(X)$ is a normed Riesz space since $\|\bullet\|_\varphi$ is a lattice norm. Evidently $\|\varphi\|_\varphi = 1$. Suppose $f, g \in A_\varphi(X)$ and $|f| \le |g| \le \lambda_g\varphi$ for some $\lambda > 0$. Set $\|g\|_\varphi = \lambda_g$. Then,

$$\|f\|_\varphi = \inf\{\lambda > 0 : |f| \le \lambda\varphi\} \le \|g\|_\varphi = \lambda_g$$

implies $\|f\|_\varphi \le \|g\|_\varphi = \lambda_g$ verifying $\|\bullet\|_\varphi$ is a lattice (and order unit) norm. The positive cone, $A_\varphi^+(X)$ is automatically a normal cone with normality constant $\mathscr{N}=1$. Since φ is an order unit, $\mathrm{B}_\varphi(X) = A_\varphi(X)$ and Goffman's Theorem implies the lattice norms $\|\bullet\|_\infty^\varphi$ and $\|\bullet\|_\varphi$ are equivalent. $\mathrm{B}(X)$ and $\mathrm{B}_\varphi(X)$ are lattice (Riesz) isomorphic. Define the positive linear operator $T : \mathrm{B}_\varphi(X) \to \mathrm{B}(X)$ by $Tf = f/\varphi$. T is positive since $Tf \ge \theta$ for each $f \ge \theta$. It is also monotone:

$f \leq g$ implies $Tf \leq Tg$; moreover, T is a one-to-one onto mapping and invertible (its left and right inverses are bounded functions). For each $f, g \in \mathrm{B}_{\varphi}(X)$, $T(f \vee g) = (Tf) \vee (Tg)$ verifying the lattice isomporhism connection.[25]

Every positive linear operator from a Banach lattice to a normed Riesz space is continuous.[26] Hence, the inverse mapping $T^{-1} : (\mathrm{B}(X), \|\bullet\|_{\infty}) \to (\mathrm{B}_{\varphi}(X), \|\bullet\|_{\varphi})$ sends points in the Banach lattice $(\mathrm{B}(X), \|\bullet\|_{\infty})$ to the normed Riesz space $(\mathrm{B}_{\varphi}(X), \|\bullet\|_{\varphi})$ is continuous. Moreover, T^{-1} carries (f/φ) into f and $\mathrm{B}_{\varphi}(X)$ inherits the complete norm structure from $\mathrm{B}(X)$. A sequence $\{f^n\}$ in $\mathrm{B}_{\varphi}(X)$ is a Cauchy sequence if and only if $\{f^n/\varphi\}$ is a Cauchy sequence in $\mathrm{B}(X)$. Hence, $\mathrm{B}_{\varphi}(X)$ is a Banach lattice as well. Thus, the positive operator T maps a Banach lattice to another Banach lattice and is a homeomorphism. The Banach lattice features of $\mathrm{B}_{\varphi}(X)$ and $\mathrm{B}(X)$ are the same from lattice theoretic and topological perspectives.

Since $\varphi \in \mathrm{B}_{\varphi}(X)$ is an order unit it follows that $\varphi \in \mathrm{int}_{\|\bullet\|_{\varphi}}\left(\mathrm{B}_{\varphi}^{+}(X)\right)$. That is, $\varphi >> \theta$ and $\mathrm{B}_{\varphi}^{+}(X)$ is solid. Evidently, it is also a closed, normal and convex cone. The unit ball in $\mathrm{B}_{\varphi}(X)$ coincides with the order interval $\langle -\varphi, \varphi \rangle$ where $\|\varphi\|_{\varphi} = 1$. The unit ball is a φ – norm closed and bounded set. It is not a compact set in our application. $\mathrm{B}_{\varphi}(X)$ is Dedekind complete Riesz space since $\mathrm{B}(X)$ is one. The vector space $\mathrm{B}_{\varphi}(X) = \mathrm{B}(X)$ if and only if $\varphi \in \mathrm{B}^{+}(X)$, i.e. $\sup_x \varphi(x) < +\infty$. This weight function may, or may not, belong to $\mathrm{B}^{+}(X)$ depending on the specification of X. Hence, $\mathrm{B}_{\varphi}(X)$ and $\mathrm{B}(X)$ are always lattice isomorphic even when they are otherwise distinct vector spaces (e.g. weighted φ-norm versus usual sup-norm topologies).

Three X domains are used to define distinct bounded function spaces. The first is $X = \ell_{\infty}^{+}$, second, $X = \ell_{\infty}^{++}$, and last, $X = \langle ae, be \rangle \subset \ell_{\infty}^{++}$ when $0 < a < b < \infty$. Suppose the weight function is $\varphi(x) = \eta\,(1 + \|x\|_{\infty})^{1/\gamma}$ for some $\gamma, \eta > 0$. This weight function choice is central to our Thompson aggregator applications. Evidently, $1 \leq \varphi(a) \leq \varphi(x) \leq \varphi(b) < \infty$ implies $\varphi \in \mathrm{B}_{\varphi}(\langle ae, be \rangle)$. Hence, $\mathrm{B}_{\varphi}(\langle ae, be \rangle) = \mathrm{B}(\langle ae, be \rangle)$ for this particular weight function. Since $\ell_{\infty}^{++} = \cup_{0<a<b<\infty} \langle ae, be \rangle$ and φ is unbounded in $\mathrm{B}(\ell_{\infty}^{++})$ (and an order unit) it will be useful to formally work in $\mathrm{B}_{\varphi}(\langle ae, be \rangle)$ even though that space equals $\mathrm{B}(\langle ae, be \rangle)$.

Our application stresses the φ – norm rather than the Minkowski functional when we define the utility function space. However, the order structure foundation of $\mathrm{B}_{\varphi}(X)$ as the principal ideal A_{φ} in $\mathbb{R}^X$ generated by $\{\varphi\}$ provides the basis for deriving the order-theoretic properties of $\mathrm{B}_{\varphi}(X)$ with its φ-norm. Interpret the elements of the φ-weighted space $\mathrm{B}_{\varphi}(X)$ as "functions which cannot grow faster than φ grows." This restricts which functions in $\mathbb{R}^X$ are considered when proving theorems about the operator A, a nonlinear self-map on the positive cone $\mathrm{B}_{\varphi}(X)^{+}$.

The following lemma applies to the Banach lattices $\mathrm{B}(X)$ and $\mathrm{B}_{\varphi}(X)$. These spaces are endowed with their standard sup-norms, which are lattice norms, or their equivalent order unit lattice norms. The sequence appearing in the lemma is assumed lattice norm convergent. Lattice norm convergence and uniform convergence are the same for these Banach lattices. The lemma is stated for this special case that includes our intended applications. Aliprantis and Burkinshaw ([7], p. 57) prove a general version holds for any Hausdorff locally solid Riesz space E (with nets replacing sequences). This is the minimal structure sufficient to insure lattice operations in E are continuous.

[25] See the discussions in Aliprantis and Tourky ([8], pp. 18-19).

[26] Aliprantis and Burkinshaw ([4], p. 175).

Lemma 2.1. *Assume that a sequence $\{x_n\}$ lattice norm-converges to x in a Banach lattice and $x_n \leq x_{n+1}$ for each n. Then x is the least upper bound of the set $\{x_n\}$ and the sequence $\{x_n\}$ order-converges to x.*

A proof may be found in Aliprantis ([1], p. 110). The completion property of a Banach lattice does not play a role since the given sequence is assumed to be lattice norm convergent. A direct argument implies $x = \sup_n \{x_n\}$ in this special case — even when the underlying Riesz space is not Dedekind complete.

This lemma connects order theoretic LFP theory in [14] and successive approximations results reported in Section 6**.** It tells us that a lattice norm monotonic sequence in a Banach lattice with uniform limit x is also *order convergent* with $x = \sup_{n\in\mathbb{N}}(x_n)$. This is an important use of Banach lattice structure and allows us to connect sequential approximations in the $u_0 - concave$ operator setting to the LPP construction in our earlier work [14]. This is interesting since the converse is false in the Banach lattices under consideration: order convergence does not imply norm convergence.[27]

3. $u_0 - Concave$ Operator Theory

Demonstration that an operator equation's fixed point is uniquely determined without relying on a (generalized) contraction mapping theorem is important in applications. The approach taken here draws on features derived from the presence of a partial order on elements in the operator's underlying real Banach space domain and range. Krasnosel'skiĭ's [35] presents his original work on concave operator theory in a Banach space's positive cone and also introduces $u_0 - concave$ operators. Krasnosel'skiĭ and Zabreĭko's [37] theory establishes suitable conditions showing at most one solution exists. Guo and Lakshmikantham ([27], pp.59-69) provide additional theory and examples of $u_0 - concave$ operators.

3.1. The Krasnosel'skiĭ and Zabreĭko Theorem.

Fix a *non-zero* element, denoted by u_0, in a cone P contained in a Banach space. Maintain the assumption that this cone is nonempty, norm-closed, and convex. Refer to P as the **cone** P in our general mathematical overview; E^+ features in our applications.

Let θ denote the zero element in P. An operator $A : P \to P$ is called $u_0 - concave$ on P if for each non-zero element $x \in P$ there are positive scalars $a(x)$ and $b(x)$ such that

$$a(x)\,u_0 \leq Ax \leq b(x)\,u_0, \tag{3.1}$$

and if for each $x \in P$ with $a_1(x)\,u_0 \leq x \leq b_1(x)\,u_0$, for some $a_1, b_1 > 0$, we have

$$A(tx) \geq [1+\eta(t,x)]\,tAx \text{ for } 0 < t < 1, \tag{3.2}$$

where $\eta(t,x) > 0$.

The restrictions $a_1(x)\,u_0 \leq x \leq b_1(x)\,u_0$ and $u_0 \neq \theta$ taken together mean that $x > \theta$, but not necessarily that $x >> \theta$ as u_0 is not assumed to be an order unit in P. Inequality(3.2) implies $A(tx) > tAx$ holds. This implies A is *strictly subhomogeneous*, a weaker property than *strong subhomogeneity* as featured in Marinacci and Montrucchio [44].

The inequalities in (3.1) express the link relation holds between u_0, Ax, and x. That is, (3.1) is equivalent to $Ax \sim u_0$ for each non-zero $x \in P$ with $x \sim u_0$. In our application P is a solid cone

[27] Peressini ([48], p. 91) has an illustrative example.

and u_0 is an order unit in P; hence, u_0 is an element of the norm-interior of P. Transitivity of the link relation implies $Ax \sim x$ as well. This insures that if x is a norm-interior point of P, then so is Ax. Since the norm-interior of P is a constituent of $P/\sim$ both Ax and x belong to $Q(u_0)$.

The main Krasnosel'skiĭ and Zabreĭko result ([37], Theorem 46.1, p. 290) is stated below.[28]

Theorem 3.1. *(Krasnosel'skiĭ and Zabreĭko). Let A be a monotone operator which is u_0- concave on P. Then the equation*

$$Ax = x$$

has at most one non-zero solution in P.

Their theorem emphasizes order theoretic ideas based on the partial ordering of the underlying function space as determined by the elements of the cone P (E^+ in our model). Topological considerations enter through the underlying Banach space structure (which explicitly determines the norm-topology) whereby P is a closed set. It is of some interest to note that this result does not require the cone P to be a normal cone. However, the positive cones are normal in our setup and this property plays an important structural role in iterative approximation theory.

Krasnosel'skiĭ and Zabreĭko's Theorem imposes two conditions on the operator A — it is *monotone* and $u_0-concave$. The Koopmans operator is monotone, so the only task in applying their theorem in our setting concerns verification that it is a $u_0-concave$ operator. We verify this property taking a roundabout approach using Liang et al's [40] sufficient conditions for a $u_0-concave$ operator.

3.2. **Subhomogeneous and order-concave Operators.** The Koopmans operator belonging to a Thompson aggregator is a self-map on the positive cone in a real ordered Banach space. An order-concave operator is a crucial criterion in Liang et al's ([40]) sufficient condition for $u_0-concavity$. General related subhomogeneity conditions along with order-concavity are reviewed below.

Fix an ordered Banach space (denoted E) as well. Set $E^+ = \{x \in E : x \geq \theta\}$, where θ denotes the zero element in E^+. Assume it is a nonempty, norm-closed and convex. Fix the monotone operator $A : E^+ \to E^+$. Then, A is called:[29]

- **(i): subhomogeneous** on E^+ if $A(tx) \geq tAx$ for each $t \in [0,1]$ and each $x \in E^+$;
- **(ii): strictly subhomogeneous** if A is subhomogeneous and the inequality $A(tx) > tAx$ for each $t \in (0,1)$ and each $\theta \neq x \in E^+$
- **(iii): strongly subhomogeneous** on E^+ if $A(tx) \geq \varphi(t,x)Ax$ for each $x \in E^+, x \neq \theta$, and $t \in (0,1)$, and some $\varphi(t,x)$ where $t < \varphi(t,x) < 1$.
- **(iv): subhomogeneous of order** $r \in (0,1)$ (*also known as* $r-$ **concave**) on E^+ if for $A(tx) \geq t^r Ax$ for each $t \in (0,1)$ and each $x \in E^+$, $x \neq \theta$;
- **(v): order-concave** if for each $t \in [0,1]$ and each $x, y \in E^+$ with $y \geq x$:

$$A(ty + (1-t)x) \geq tAy + (1-t)Ax. \tag{3.3}$$

[28] Krasnosel'skiĭ ([35], p. 188) presents the first formulation of this result within the context of nonlinear eigenvalue problems.

[29] Amman [9] introduced the notion of an order-concave operator. Krasnosel'skiĭ [35] introduced subhomogeneous operators; also see Krasnosel'skiĭ and Zabreĭko [37]. Potter [49] introduced subhomogeous operators of order r as $r-concave$, the name we will use in sequel for this subhomogeneity property. An order concave operator is also an h-subconcave operator for each $\theta \leq h \leq A\theta$ as noted by Marinacci and Montrucchio [44], a property that implies A is subhomogeneous.

A nonempty subset S of P is **order-convex** if $x,y\in S$, $y\geq x$ implies $ty+(1-t)x\in S$ for each $t\in[0,1]$. The positive cone E^{+} is order convex since that cone is a convex set by assumption. Set $x=\theta$ and let $y\geq\theta$ and suppose A is order-concave on E^{+}. Then, A is subhomogenous. Order-concavity is a stronger property than subhomogeneity, but it is weaker than assuming A is concave (i.e. inequality (3.3) holds for arbitrary $x,y\in E^{+}$, not just ordered pairs with $y\geq x$). Some authors (e.g. [27] and [28]) identify the terms concave and order-concave operators. We keep the prefix "order" in place as a reminder the condition $y\geq x$ has an important role to play in this theory. order-concavity alone does not imply $u_0-concavity$. However, order-concave operators are monotone operators (Guo et al [28], Lemma 3.1.6).

Krasnosel'skiĭ and Zabreĭko's [37] theory combines minimal order theoretic ideas based on the positive cone's defining partial order and topological properties derived its Banach space structures. Their formal assumptions are parsimonious. The underlying ordered Banach space need not be a Riesz space, Dedekind complete, or a Banach lattice. The positive cone may be neither normal nor solid. We verify the Koopmans operator acts on ordered vector spaces which are Banach lattices with units in order to verify it is $u_0-concave$. The corresponding positive cones are normal and solid. These properties play an indispensable role in our proof it is $u_0-concave$ as well as in a construction showing there is an iterative approximation uniformly converging to the Koopmans equation's unique solution.

Refinement of Krasnosel'skiĭ and Zabreĭko's [37] theory to the case of an operator acting on a $AM-space\ with\ unit$ has another benefit. This property enables a connection between lattice-norm convergent monotone sequences and their order sequence limits. This links iterative methods based on $u_0-concavity$ to order theoretic LFP theory [14]. The role of an order unit in this approach implies each element of $\text{fix}\,(A)$ is an interior point of the space's positive cone, which is nonempty when an order unit exists. Verification that the model satisfies Liang et al's [40] sufficient condition exploits the existence of order units and the linked relation in a nontrivial manner.

3.3. **The Liang, Wang and Li Order-Concave Operator Theorem.** Fix the function $u_0>\theta$ with $u_0\in P$. Define

$$Q(u_0)=\{x\in P:x\sim u_0\}\,. \tag{3.4}$$

The constituent $Q(u_0)$ is readily seen to be a convex subset of P. It may not be a solid cone since u_0 need not be an order unit. Hence, $Q(u_0)$ may not equal its norm-interior. Indeed, the norm-interior of P may be empty at this level of generality.

Let $P'=range\,(A)\subseteq P$ denote the range of the operator A acting on the cone P. This range set P' *may* also be a cone, but this is not a formal requirement. What matters is that P' inherits the partial order induced by the cone, P as well as be a convex subset of P.

Condition (3.1) differs from (3.4). The former uses the image of x under the mapping A, whereas the latter employs the domain point, x. The set $Q(u_0)$ is the operator's "target." This is why we must establish (3.1) in order to verify A has the u_0- concavity property. There is a subtle point to add — (3.1) excludes the function θ. Liang et al's [40] sufficient condition (below) applies to θ as well as all other points in P. This observation underlies our application of Liang et al's [40] verification technique.

The convexity of the domain and range spaces is critical for the inequality above to make sense. The former set is convex since the cone is a convex set. The latter is an assumption that must be checked in applications.

The following lemma is crucial. It is implicit in Liang et al's proof ([40], Lemma 4, p. 579).

Lemma 3.2. *Suppose $A : P \to Q(u_0)$. Then for each $x \in P$ there is a positive number $\mu(x)$ such that*

$$A\theta \geq \mu(x) Ax. \tag{3.5}$$

Proof. Let $x \in P$. Then $Ax \in Q(u_0)$ by assumption. By the definition of $Q(u_0)$ there are numbers $a(Ax) > 0$, $b(Ax) > 0$ such that

$$a(Ax)u_0 \leq Ax \leq b(Ax)u_0.$$

In particular, for $x = \theta \in P$, the above inequality is true:

$$a(A\theta)u_0 \leq A\theta \leq b(A\theta)u_0.$$

Combining both sets of inequalities as follows we have

$$\begin{aligned} A\theta &\geq a(A\theta)u_0 = \left(\frac{a(A\theta)}{b(Ax)}\right) b(Ax)\, u_0 \\ &\geq \mu(x)Ax, \end{aligned}$$

where

$$\mu(x) \equiv \left(\frac{a(A\theta)}{b(Ax)}\right) > 0.$$

□

We notice that the function $\mu(x)$ depends on the fixed operator, A, and the zero function, θ as x varies in P.

Theorem 3.3. *(Liang et al [40]) Suppose $A : P \to Q(u_0)$ is an order-concave operator. Then A is a $u_0 - concave$ operator.*[30]

Proof. For each $0 < t < 1$: the order-concavity condition implies, since $x \geq \theta$, that

$$\begin{aligned} A(tx) &= A(tx + (1-t)\theta) \\ &\geq tAx + (1-t)A\theta \\ &\geq tAx + (1-t)\mu(x)Ax \text{ (by (3.5))} \\ &= t\left(1 + \frac{(1-t)\mu(x)}{t}\right)Ax. \end{aligned}$$

Now set

$$\eta(t,x) = \frac{(1-t)\mu(x)}{t} > 0$$

since $\mu(x) > 0$. Hence, A is a $u_0 - concave$ operator as we have shown

$$A(tx) \geq (1 + \eta(t,x))\, tAx.$$

□

[30] We include their proof in order to show how the previous lemma implies an ordered concave operator is also a $u_0 - concave$ operator.

The practical impact of this result is plain — check that an operator is a $u_0 - concave$ operator on the cone P by verifying the sufficient condition in Liang, Wang, and Li's Theorem! This effectively means checking that the given operator is order-concave and satisfies condition (3.5). As the cone is convex, it is also an order convex set.

Marinacci and Montrucchio ([44], Lemma 1 and Theorem 2) prove existence and uniqueness theorems for monotone order-concave operators (and, hence subhomogeneous operators) when the underlying ordered vector space has an order unit and either it is Dedekind complete or the nonlinear operator is order continuous on the space's positive cone. In either case, it is important to clarify that sups and infs of finitely many elements must be well-defined. Hence, the actual ordered vector space structure assumed must be a Riesz space. In that situation, the principal ideal generated by the order unit is well-defined. But, it is NOT assigned its usual lattice norm topology (in fact, any topology). Order-convex sets in Riesz spaces are order intervals. Their detailed arguments apply to operators that leave certain order intervals invariant; they use the order unit to exclude lower perimeter fixed points ([44], Theorem 2). The lower perimeter is an order theoretic analog of the positive cone's norm boundary when the vector space is a Riesz space. The lower perimeter is nonempty when an order unit is present ([44], Proposition 3). They comment that for a normed Riesz space the lower perimeter is the positive cone's lattice norm topological boundary. If a normed Riesz space has a unit, then the positive cone's topological boundary is nonempty.

Their main order theoretic uniqueness result (Theorem 2) does not combine order and topological structures possible when the underlying ordered vector space is a normed Riesz space. Marinacci and Montrucchio [44] combine order and topological perspectives into their uniqueness theorems for recursive utility and dynamic programming applications. We also take this combined perspective in applying the $u_0 - concave$ operator theory exposited here to proving the Koopmans equation based on a Thompson aggregator has a unique solution. This additional topological structure is the source for improving on the mere existence and uniqueness available from their purely order theoretic structure (e.g. the possibility of an iterative approximation theory).

The Koopmans operator is shown by us in [14] to be order continuous on the positive cone of an $AM - space\ with\ unit$. It is also invariant on the order intervals required by Marinacci and Montrucchio's order theoretic result ([44], Theorem 2). The main issue is proving the Koopmans operator has no fixed points in the positive cone's lower perimeter. The latter point is better addressed in the topological setting as that is where full advantage of our Banach lattice machinery is available and gives an intuitive analytical description of the positive cone's topological boundary in Section 3.4 below. Combining order and topological methods sharpens results and expands them (see Section 6) beyond what order theory alone provides.

3.4. **The AM-Space with Unit Case.** The application of $u_0 - concave$ operator theory to the Koopmans operator, T_W, turns on checking Liang et al's [40] sufficient condition. Doing so means there must be a clear choice of the cone, P, and selection of the point u_0. Verification that $T_W U$ belongs to the corresponding set $Q(u_0)$ whenever $U \in P$, and checking T_W is an order-concave operator are critical steps. One important consideration is that we must show $T_W \theta \in Q(u_0)$. Additional restrictions on the underlying commodity space must be imposed to show that inclusion holds and the Koopmans operator is a $u_0 - concave$ operator. We restrict the commodity space domain and employ Banach lattice properties when a unit is present. This restriction allows us to prove $T_W\theta \in Q(u_0)$ when u_0 is an order unit. We offer an economically motivated condition towards this end in Section 4. The key is to fully employ the commodity and utility space's economically motivated Banach lattice with units properties.

Let E be an $AM-space\ with\ unit$ in the positive cone, E^+. Our model places utility functions that might solve the Koopmans operator equation in the space $\mathrm{B}(X)$ with its usual partial ordering and appropriate specifications of the underlying commodity space, X. The advantage of assuming E has order units is the positive cone's norm interior is nonempty. Hence, the positive cone's boundary, defined by $\partial_0 E^+ = E^+ \diagdown E^{++}$, is nonempty. In the case of $\mathrm{B}(X)$ a function $f : X \to \mathbb{R}_+$ is a boundary point if $\inf_{x\in X} f(x) = 0$. The positive cone, $\mathrm{B}^+(X)$ has an order unit, namely the constant function $e(x) = 1$ for each $x \in X$. Hence, its norm-interior is nonempty. Evidently $\theta \in \partial_0 \mathrm{B}^+(X)$, so the boundary is also nonempty. Marinacci and Montrucchio [44] characterize the positive cone's boundary when the underling Riesz space has an order unit.

Lemma 3.4. *An element $x \in E^+$ **does not belong** to $\partial_0 E^+$ if and only if x is comparable to u for **some** order unit, u, in E^+.*

In particular, when u is an order unit in E, then $u >> \theta$ and E^+ is solid and the linked relation yields $Q(u) = \mathrm{int}(E^+)$. The lemma immediately yields:

Corollary 3.5. *An element $x \in E^+$ **belongs** to $\partial_0 E^+$ if and only if x is NOT comparable to **any** order unit u in E^+.*

Clearly $\theta \in \partial_0 E^+$ by this test. The next lemma applies to a monotone operator where E is a Banach lattice (AM space) with unit, u. The hypothesis $A(E^+) \subset Q(u)$ is critical. Order-concavity does not enter into this lemma and its corollary.

Lemma 3.6. *Let $A : E^+ \to Q(u)$ be a monotone operator. Then*

- **(a):** $A\theta \sim u$;
- **(b):** $Ax^* = x^*$ *and* $x^* > 0$ *implies* $x^* \sim u$*, i.e.,* $x^* >> \theta$.

Proof. (a) follows by the lemma's maintained hypotheses.

(b) A monotone, $\theta \le x$ and $\theta < x$ imply $A\theta \le Ax$. As $A\theta$ is linked to u, it follows $Ax^* = x^* > \theta$ is also linked to u, i.e. $Ax^* \sim u$. Hence, $x^* \sim u$ as well and $x^* \in E^{++}$. □

Each of the operator's fixed points (that are non-zero) are norm-interior points of the positive cone. There is an immediate corollary: *A has no fixed points in the positve cone's boundary.*

Corollary 3.7. *$Ax \neq x$ for each $x \in \partial_0 E^+$.*

Proof. If $x \in \partial_0 E^+$, then x is not linked to u. But $Ax \in Q(u)$ by assumption. Therefore $Ax \neq x$. □

This corollary has an important implication. Suppose that $A(E^+) \subseteq E^+$ and $A\theta \in \partial_0 E^+$. That is, $A\theta \notin Q(u)$. Suppose further $A^{n+1}\theta = A(A^n\theta) \in \partial_0 E^+$ with $n = 1, 2, \ldots$. If, in addition, E is a Dedekind complete Riesz space, A is order continuous, $\{A^n\theta\}_{n=1}^{\infty}$ is increasing, and order bounded from above, then $\sup_n \{A^n\theta\} = A(\bar{x}) = \bar{x} \in \partial_0 E^+$.[31] This situation does not exclude the possibility A has another fixed point in $Q(u)$. Hence, the assumption that $A(E^+) \subset Q(u)$ (underlying inequality (3.5)) is important in showing the operator has, at most, one fixed point in E^+ by ruling out the possibility of a fixed point in $\partial_0 E^+$. Hence, $\mathrm{fix}(A) \subset Q(u)$.

[31] See Becker and Rincon-Zapatero [14] for details of this possibility in the Koopmans operator setting.

The Koopmans operator has this feature when we consider the utility space is $\mathrm{B}^{+}(\ell_{\infty}^{+})$ since $T_W\theta \in \partial_0\mathrm{B}^{+}(\ell_{\infty}^{+})$ as $\inf_{C\in\ell_{\infty}^{+}} T_W\theta(C) = 0$ is achieved when $C = 0_{con}$ provided $W(0,0) = 0$ (see the formal definition of the Koopmans operator below). Moreover, its $N-$*stage iterate*, $T_W^N\theta \in \partial_0\mathrm{B}^{+}(\ell_{\infty}^{+})$. In this instance, we have the situation outlined above and the operator's Least Fixed Point belongs to $\partial_0\mathrm{B}^{+}(\ell_{\infty}^{+})$. It cannot be comparable to any order unit. There is also a Greatest Fixed Point, which resides in $\mathrm{B}^{++}(\ell_{\infty}^{+})$, and is comparable to an order unit. The uniqueness theory based on Krasnosel'skiĭ and Zabreĭko's [37] theory does not apply. Hence, we must exclude the LFP as an element of $\partial_0\mathrm{B}^{+}(\ell_{\infty}^{+})$. The order unit structure can be exploited to do so, but *we must restrict the commodity space*. This eliminates the boundary points in $\partial_0\ell_{\infty}^{+}$ as domain elements in the utility space. Thus, we reset the utility space to $\mathrm{B}^{+}(\ell_{\infty}^{++})$ by setting $X = \ell_{\infty}^{++}$, the norm interior of ℓ_{∞}^{+}. We build this structure by first replacing $\mathrm{B}^{+}(\ell_{\infty}^{+})$ by $\mathrm{B}^{+}(\langle ae, be\rangle)$. Here, $\langle ae, be\rangle$ is the order interval determined by e, the order unit in ℓ_{∞}^{+}, and the choice of scalars a, b satisfying $0 < a < b < \infty$. The Koopmans operator acting on the zero function in $\mathrm{B}^{+}(\langle ae, be\rangle)$ produces an order unit in the range space, $\mathrm{B}^{+}(\langle ae, be\rangle)$. Liang et al's inequality (3.5) is verified at $u_0 = T_W\theta$ (chosen for the u_0-*concavity* property) as it is an order unit in $\mathrm{B}^{+}(\langle ae, be\rangle)$. The Koopmans operator will be shown to have a fixed point in this positive cone's norm-interior. Once this is established, we can extend the result to $\mathrm{B}^{+}(\ell_{\infty}^{++})$ by recognizing that ℓ_{∞}^{++} is the union of all the order intervals $\langle ae, be\rangle$ as a and b vary over the positive scalars with $a < b$.

4. Recursive Utility Theory for The Thompson Aggregator Class

Concave Thompson aggregators are defined in this section. Our assumptions strengthen those in Marinacci and Montrucchio [43] by imposing joint concavity and continuity properties on the aggregator. These augmented restrictions on the class of Thompson aggregators imply the Koopmans operator is order-concave, a critical ingredient in applying Liang et al's [40] technique.

4.1. **Concave Thomson Aggregators.**

The class of Thompson aggregators covered by our uniqueness theory is delineated by the following four basic assumptions.

Definition 4.1. $W : \mathbb{R}_{+}^{2} \to \mathbb{R}$ is said to be a **concave Thompson aggregator** if it satisfies properties (T1) – (T6):

- **(T1):** $W \geq 0$, continuous, and monotone: $(x,y) \leq (x',y')$ implies $W(x,y) \leq W(x',y')$.
- **(T2):** $W(x,y) = y$ has at least one nonnegative solution for each $x \geq 0$.
- **(T3):** W is a concave function of (x,y).
- **(T4):** $W(x,0) > 0$ for each $x > 0$.
- **(T5):** *W is $\gamma-$ subhomogeneous — there is some $\gamma > 0$ such that:*
$$W(\mu^{\gamma}x, \mu y) \geq \mu W(x,y)$$
for each $\mu \in (0,1]$ and each $(x,y) \in \mathbb{R}_{+}^{2}$. If the defining inequality in $(T5)$ is an equality, then we say W is $\gamma-$homogeneous.
- **(T6):** *W satisfies the **MM-Limit Condition**:*
$$\lim_{t\to\infty} \frac{W(1,t)}{t} < 1, \tag{4.1}$$
with $t > 0$.

The definition of a concave Thompson aggregator builds in it is jointly continuous and concave in (x,y) over $\mathbb{R}^2_+$. This differs somewhat from the formal assumptions given in Marinacci and Montrucchio ([43], [44]). In the sequel understand the shortened expression *Thompson aggregator refers to concave Thompson aggregators unless otherwise indicated.*

Our strengthened axioms are appropriate for solving the existence and uniqueness problem for the Koopmans operator equation by $u_0 - concavity$ theory. Our maintained conditions are satisfied in the major examples of Thompson aggregators.

$(T6)$ is a *joint restriction* on the aggregator function and the underlying commodity space. For example, it admits productive technologies *exhibiting diminishing marginal returns to capital accumulation while excluding sustainable growth production models.*

4.2. **Examples of Thompson Aggregators.** There are two important sources for examples. The Koopmans, Diamond, and Williamson [34]), or KDW aggregator (defined below) has parametrizations placing it outside the Blackwell class and firmly in the Thompson family. There are also many new examples based on the functional form for utility functions and production functions commonly studied in microeconomic theory. For example, both the CES (Constant Elasticity of Substitution) and KDW aggregators satisfy $(T1)-(T6)$.

The class of **CES aggregators** are defined parametrically by the formula:

$$W(x,y) = (1-\beta)x^{\rho} + \beta y^{\rho}, \text{ for } 0 < \rho < 1. \tag{4.2}$$

The parameter β is restricted — $0 < \beta < 1$. Note that this family of functions is positively homogeneous of degree ρ. The **elasticity of substitution** is $\sigma := 1/(1-\rho); \rho \neq 1$. All six Thompson aggregator criteria are met when $\sigma > 1$. A**ssume** this restriction applies without further notice. The CES Thompson aggregators are readily shown to satisfy $(T5)$ with $\gamma = 1$.

Other Thompson aggregators are available as variations on the CES theme (see Marinacci and Montrucchio [43]). For example, the **quasi-linear aggregator** $W(x,y) = x + \beta y^{\rho}$ is Thompson for the same parameter restrictions imposed on the CES family.

The **KDW aggregator** is defined by the formula

$$W(x,y) = \frac{\delta}{d}\ln\left(1 + ax^b + dy\right)$$

where $a,b,d,\delta > 0$ and $b < 1$. This is a Thompson aggregator; it satisfies $(T5)$ with $\gamma = b^{-1}$. It is a Blackwell aggregator when "discounting" is assumed using the restriction $\delta < 1$. Similarly, "upcounting." or "no discounting," is permissible in the Thompson class. These cases correspond to $\delta > 1$ and $\delta = 1$, respectively.

Bloise and Vailakis ([18], Example 3) define the concave Thompson aggregator $W(x,y) = x + \min\{y, \alpha + \beta y\}$ where the parameters satisfy $\alpha > 1$ and $0 < \beta < 1$. Our existence and uniqueness theories apply to this functional form. The CES, KDW and Bloise-Vailakis aggregators satisfy $W(0,0) = 0$.

Balbus [10] assumes the given aggregator is strictly increasing in (x,y), jointly continuous and $W(x,y) = 0$ if and only if $x = 0$ and $y = 0$. He assumes neither our concavity assumption nor the weaker concave at the origin condition required by Marinacci and Montrucchio ([43], [44]). Instead, Balbus imposes the aggregator restriction: W is *r-concave in y for each x*: there is a number $r \in (0,1)$ such that for each $t \in (0,1)$, $W(tx,ty) \geq t^r W(x,y)$. This property holds, for example, in the case of CES aggregator (4.2) when for $0 < \rho < 1$. Balbus's assumptions are sufficient to verify the Koopmans operator is an *r-concave operator on the norm interior* *of the utility space's positive cone.* This yields an existence and uniqueness theorem as an application of Guo, et al ([28], Theorem 3.1.7, p. 94). We apply the $r-$*concavity* property in Section 6 on the uniqueness of the fixed point obtained as the uniform limit of the sequence of iterations of the Koopmans operator over the natural numbers and the closely related existence of *a posteriori* error estimates for each iteration.

4.3. **The Commodity Space.** The commodity space is the positive cone, ℓ_∞^+, of the vector space ℓ_∞, which is a Banach lattice with unit $e=(1,1,1,\ldots)$. This is a typical commodity space in deterministic optimal growth models where all feasible consumption paths are bounded by diminishing marginal returns in the model's production sector. There is also a technical reason for choosing this commodity space — its order intervals underlying our proofs must be *shift invariant* in the recursive utility framework. Marinacci and Montrucchio [43] use a different shift invariant domain that, unlike ours, accommodates growing economies. However, they use the same order interval constructions in their sequel [44] on uniqueness. Our choice of domain is motivated by its intuitive economic basis and for its technical convenience in demonstrating the Koopmans operator satisfies hypotheses supporting u_0-*concavity* theory.

The positive cone's norm interior, ℓ_∞^{++}, has a prominent place in Thompson aggregator uniqueness theories. Counterexamples show uniqueness of the Koopmans operator's fixed point can fail at consumption sequences in the positive cone's boundary.[32] The positive cone's norm interior is characterized as $\ell_\infty^{++}=\{x\in\ell_\infty^+ : \inf_t x_t>0\}$. It consists of order units. Each order interval $\langle ae,be\rangle$ with $0<a<b<\infty$ is a subset of ℓ_∞^{++}. Moreover, $\ell_\infty^{++}=\cup_{a,b>0}\langle ae,be\rangle$; evidently $\liminf_t x_t\geq\inf_t x_t$.

The **boundary** of the positive cone, denoted $\partial_0\ell_\infty^+$, is the complement of the interior relative to the positive cone. That is, $\partial_0\ell_\infty^+=\ell_\infty^+\diagdown\ell_\infty^{++}$, and it is a normed-closed subset of ℓ_∞^+.

A sequence $x\in\partial_0\ell_\infty^+$ if and only if $\inf_t x_t=0$. A sequence belongs to c_0 if it converges to zero. A convergent sequence is also bounded, and hence an element of ℓ_∞. The set c_0 is the set of **null sequences**. Its positive cone, c_0^+, is the corresponding set of nonnegative null sequences. Assign c_0 the sup-norm topology. It is a Banach lattice that *contains no order units*.[33] Each point in $c_0^+\subset\ell_\infty^+$ must be a boundary point in ℓ_∞^+. There are other boundary points. Any sequence in ℓ_∞^+ with finitely many components equal to zero is also a boundary point. For instance, the sequence $\{0,1,1,\ldots\}\in\partial_0\ell_\infty^+$. Likewise, each sequence in ℓ_∞^+ that has a convergent null subsequence must be a boundary point as well.

Consumption sequences in the commodity space are generally denoted by $C=\{c_t\}_{t=1}^\infty$; we write $C=\{c_t\}$ when the meaning is clear. Define the **shift operator** $S:\ell_\infty^+\to\ell_\infty^+$ according to the rule $C=\{c_1,c_2,c_3,\ldots\}\mapsto SC=\left\{c_2,c_{3,}\ldots\right\}$.[34] The definition of the shift operator says that the positive cone is invariant under its action: $S(\ell_\infty^+)\subseteq\ell_\infty^+$. There are other sets which are invariant with respect to the shift and they are important in our uniqueness theory. Evidently ℓ_∞^{++} is shift invariant. However, the boundary set, $\partial_0\ell_\infty^+$, is NOT shift invariant: $S(\{0,1,1,\ldots\})=\{1,1,1,\ldots\}\notin\partial_0\ell_\infty^+$.

[32]See Becker and Rincón-Zapatero [13] and Bloise and Vailakis [18].

[33]See Aliprantis and Border ([2], p. 529).

[34]The shift operator can be defined as a self-map on the vector space ℓ_∞. However, the shift operator is invariant on that space's positive cone and this is the feature we utilize.

We prove our uniqueness theorem for consumption sequences belonging to an arbitrary order interval of the form $\langle ae, be\rangle$ with $0 < a < b < \infty$. Interpret ae as the *minimum consumption sequence* where each term delivers the consumption necessary to sustain life within a time period. Regard the constant sequence be where the scalar b is the *maximum consumption goods* output produced using the maximum sustainable capital stock input in a one or two-sector growth model with diminishing returns. The sequence be overstates prospective consumption in the sense that it bounds output available for consumption goods at the production level of the maximum sustainable capital stock. An agent consuming that level at some time would not be able to continue production as its capital would be zero thereafter (e.g. assuming a standard Cobb-Douglas production function). The sequence be places a weak upper bound on the *maximum consumption in any one period* given the technology. This suffices to set an upper bound on consumption in each period. The restriction of consumption to an order intervals of mimimal and maximal consumption sequences is a proof device designed to build an adequate uniqueness theory for the Koopmans operator. It first appears in Martins-da Rocha and Vailakis [46]. Subsequently, Marinacci and Montrucchio [44] as well as us [13] use this restricted commodity space in our uniqueness theories.

4.4. **The Space of Possible Utility Functions.** Utility functions corresponding to a Thompson aggregator may be unbounded from above. We introduce a weighted space of possible utility functions in order to work within a space of suitably bounded real-valued functions.

First, define a **weight function**, φ_γ following Marinacci and Montrucchio's [43] specification. For each $C \in \ell_\infty^+$ define φ_γ by the formula:

$$\varphi_\gamma(C) = (1 + \|C\|_\infty)^{1/\gamma}. \tag{4.3}$$

This weight function is uniformly continuous and convex on ℓ_∞^+ with respect to the sup-norm topology.[35] Here, the parameter $\gamma > 0$ appearing in the weight function is taken from $(T5)$. Note that This weight function as well the sup-norm entangle preference and technology parameters — the parameter γ comes from the model's preference side and the presence of a maximum sustainable capital stock in standard one and two-sector models subject to diminishing returns sup-norm bounds consumption sequences. This function is monotone: $C \leq C'$ implies $\varphi_\gamma(C) \leq \varphi_\gamma(C')$.

A function $U : \ell_\infty^+ \to \mathbb{R}$ is $\varphi_\gamma-$ *bounded* provided

$$\|U\|_\gamma := \sup_{C \in \ell_\infty^+} \frac{|U(C)|}{(1 + \|C\|_\infty)^{1/\gamma}} < +\infty.$$

The set of all $\varphi_\gamma-$ bounded real-valued functions with domain ℓ_∞^+ is denoted by $\mathrm{B}_\varphi(\ell_\infty^+)$. This space is also lattice isomorphic with $\mathrm{B}(\ell_\infty^+)$. **As the weight function remains fixed we abbreviate this space using the notation** B given these spaces are lattice isomorphic and there is no ambiguity about the underlying commodity space**.** Then, its positive cone is B^+. Put differently, B is the set of $\|\bullet\|_\gamma -$ *bounded* real-real valued functions defined on ℓ_∞^+.

The **zero function,** θ, is defined by $\theta(C) = 0$ for each C. The zero function is the origin in the vector space B; it is a Dedekind complete Riesz space with the usual pointwise partial

[35] The norm $\|\bullet\|$ is a uniformly continuous real-valued function defined on the set ℓ_∞. See Aliprantis and Burkinshaw ([5], p. 218). Hence, the function $\varphi_\gamma(C)$ is continuous as the composition of the continuous functions $1 + \|C\|$ and $\phi(x) = x^{1/\gamma}$ for $x > 0$.

ordering. Clearly the weight function φ_γ satisfies $\varphi_\gamma(\theta) = 1$ and $\varphi_\gamma(C) \geq 1$ for each C. Moreover, $\left\|\varphi_\gamma\right\|_\gamma = 1$ as well and φ_γ is an order unit in B. Hence, B is a Banach lattice with unit equipped with the lattice norm $\|\bullet\|_\gamma$. The cone B^+ is $\|\bullet\|_\gamma$– closed, solid, convex, and a normal cone (with normal constant, 1).

The commodity space domain ℓ_∞^+ may be replaced by the order interval $\langle ae, be\rangle \subset \ell_\infty^{++}$ on economic grounds. The $\|\bullet\|_\gamma$ remains well-defined; set $\mathrm{B}(a,b)$ is the set of $\|\bullet\|_\gamma -$ *bounded* real-valued functions on $\langle ae, be\rangle$ and $\mathrm{B}^+(a,b)$ is its positive cone. The weight function φ_γ is an order unit in $\mathrm{B}^+(a,b)$ and $\mathrm{B}^+(a,b)$ is a $\|\bullet\|_\gamma$– closed, solid, convex, and normal cone. It is a Banach lattice with unit.

Set $\mathrm{B}(\ell_\infty^{++})$ equal to the $\|\bullet\|_\gamma -$ bounded real-valued functions defined on ℓ_∞^{++} with positive cone $\mathrm{B}^+(\ell_\infty^{++})$ and is a Banach lattice with units. Here, consumption sequences in the positive cone's boundary are formally excluded from possible consumption. The solution to the uniqueness problem turns out to be a function in $\mathrm{B}^+(\ell_\infty^{++})$. This use of ℓ_∞^{++} over the domain ℓ_∞^+ amounts to imposing a **regularity condition** on admissible on possible solutions to the Koopmans equation by restricting the utility space.[36]

4.5. **The Koopmans Equation.** The aggregator approach to recovering recursive utility representations of an underlying preference relation defined on the given commodity space is expressed in terms of a functional equation. This equation takes the aggregator function as the primitive concept. The **Koopmans equation for recursive utility** is defined for each $C \in \ell_\infty^+$ by the formula (where S is the shift operator):

$$U(C) = W(c_1, U(SC)). \tag{4.4}$$

A **solution** of this equation is a recursive utility function representation of the preference relation. Of course, it all depends on what is meant by a solution. Proving this functional equation has a solution turns on recasting the problem as demonstrating a corresponding nonlinear operator, known as the **Koopmans operator** (denoted by T_W) has a fixed point in the desired function space of possible solutions. Formally define the Koopmans operator given a function$U \in \mathrm{B}^+$ by the following equation for each $C \in \ell_\infty^+$:

$$(T_W U)(C) = W(c_1, U(SC)). \tag{4.5}$$

The Koopmans operator is a self-map on $\mathrm{B}^+(\ell_\infty^+)$. In fact, there is function $U^T \in \mathrm{B}^+(\ell_\infty^+)$ with $U^T >> \theta$ such that for the order interval $\left\langle\theta, U^T\right\rangle \subset \mathrm{B}^+(\ell_\infty^+)$, it can be shown that $T_W\left(\left\langle\theta, U^T\right\rangle\right) \subset \left\langle\theta, U^T\right\rangle$.

The function U^T is defined by

$$U^T(C) = W(1, y^*)\,\varphi_\gamma(C).$$

Here, the element $y^* > 0$ is the unique solution to $W(1, y^*) = y^*$ (which exists since W is a concave Thompson aggregator and $(T6)$ holds). Evidently $U^T \in \mathrm{B}^+(\ell_\infty^+)$, $U^T \geq \theta, U^T(C) > 0$ for each C, and $\left\|U^T\right\|_\gamma = W(1, y^*) < +\infty$. As $\inf_{C \in \ell_\infty^+} U^T(C) > 0$ it follows that $U^T \in \mathrm{B}^{++}(\ell_\infty^+)$ since it is comparable to the order unit, $\varphi_\gamma \in \mathrm{B}^+(\ell_\infty^+)$.

[36]Bertsekas ([16], pp. 141-164; pp. 265-282.) develops notions of regularity conditions for solving dynamic programming problems. The notion of a restriction on possible solutions in the Thompson aggregator model is similar in spirit as the regularity ideas presented by Bertsekas.

The Koopmans operator has extremal fixed points in the order interval $\left\langle \theta, U^T \right\rangle \subset \mathrm{B}^+ \left(\ell_\infty^+\right)$ according to the existence theory in our paper [14]. That is, there is a LFP, denoted U_∞, and a GFP, U^∞, and any $U \in \operatorname{fix}\left(T_W\right) \subset \left\langle \theta, U^T \right\rangle$ satisfies $U_\infty \leq U \leq U^\infty$.[37] There is a unique fixed point whenever $U_\infty = U^\infty$.

For an aggregator satisfying $W\left(0,0\right) = 0$, as is the case in the concave Thompson aggregator examples,

$$\inf_{C \in \ell_\infty^+} T_W \theta\left(C\right) = 0$$

since $T_W \theta\left(C\right) = W\left(c_1, 0\right)$ and for C with $c_1 = 0$, $T_W \theta\left(C\right) = 0$. Hence, $T_W \theta \in \partial_0 \mathrm{B}^+\left(\ell_\infty^+\right)$ and $T_W^N \theta \in \partial_0 \mathrm{B}^+\left(\ell_\infty^+\right)$ for each natural number N as $W\left(0,0\right) = 0$. LFP existence theory shows that $T_W^N \theta \nearrow U_\infty$ pointwise and in order with $U_\infty = \sup_n \left(T_W^N \theta\right)$. Clearly $W\left(0,0\right) = 0$ implies $U_\infty\left(0_{con}\right) = 0$ and $U_\infty \in \partial_0 \mathrm{B}^+\left(\ell_\infty^+\right)$. A parallel argument for iteration of the Koopmans operator over the natural numbers initiated at U^T yields $T_W^N U^T \searrow U^\infty$. Each $T_W^N U^T$ is comparable with U^T, so each iterate $T_W^N U^T$ is a point in the norm-interior of $\mathrm{B}^+\left(\ell_\infty^+\right)$ as well: $U^T\left(0_{con}\right) = y^*$ and $U^T\left(C\right) \geq y^* > 0$ for each $C \in \ell_\infty^+$. Hence, for each N,

$$\inf_{C \in \ell_\infty^+} T_W^N U^T\left(C\right) > 0,$$

and U^∞ is a norm interior point of $\mathrm{B}^+\left(\ell_\infty^+\right)$.

There are several counterexamples to the Koopmans operator has a unique solution when evaluated at boundary consumption bundles. One important case is the Bloise and Vailakis [18] concave Thompson aggregator case. Its corresponding recursive utility function is not uniquely determined. Uniqueness fails for an uncountable number of consumption sequences in the positive cone of c_0. These points of failure are boundary points in the larger commodity space's positive cone. Their uniqueness theory, and ours, exclude such consumption bundles. We must exclude the entire boundary of ℓ_∞^+ including all positive null sequences.

The implications of non-uniqueness for boundary consumption sequences has an important consequence when viewed in the utility space, $\mathrm{B}^+ = \mathrm{B}^+\left(\ell_\infty^+\right)$. Both extremal fixed points are elements in B^+, but exhibit a fundamental difference. The LFP is a boundary point in B^+; the GFP is a norm-interior point of B^+.

The common strategy among the uniqueness papers by Martins-da-Rocha and Vailakis [46], Marinacci and Montrucchio ([43],[44]), Bloise and Vailakis [18], Balbus [10], and Becker and Rincón-Zapatero [13] is place additional restrictions on the commodity space together with appropriate utility space modifications. What constitutes a solution to the Koopmans equation differs between the utility spaces $\mathrm{B}^+\left(\ell_\infty^+\right)$ and $\mathrm{B}^+\left(\ell_\infty^{++}\right)$ with their sup norms. The change from $\mathrm{B}^+\left(\ell_\infty^+\right)$ to the utility space, $\mathrm{B}^+\left(\ell_\infty^{++}\right)$ eliminates the commodity sequences where the LFP and GFP are known, by examples, to differ (i.e $\partial_0 \ell_\infty^+$). Since the LFP and GFP are each order units and belong to $\mathrm{B}^{++}\left(\ell_\infty^{++}\right)$, and if a unique solution exists, then the LFP and GFP agree on ℓ_∞^{++}.[38] A unique solution exists in $\left\langle \theta, U^T \right\rangle \subset \mathrm{B}^{++}\left(\ell_\infty^{++}\right)$. The overarching point is to use modified utility spaces, based on their underlying commodity spaces, to remove any utility function boundary fixed points belonging to the corresponding Koopmans operator. Success

[37] See Becker and Rincón-Zapatero [14] for details.

[38] Balbus [10] also excludes the boundary of the positive cone of B^+ from the operator's domain and assumes it is a selfmap on the interior of B^+. His operator is not formally restricted to be a selfmap on an order intervals in B^+.

on this front implies the GFP and LFP are equal in the modified fixed point problem's solution which lies in the interior of $\mathrm{B}^{++}(\ell_\infty^{++})$.

We prove our uniqueness theorem for the restricted commodity space $\langle ae, be\rangle$ with $0 < a < b < \infty$ where a and b are arbitrarily chosen. This restriction has an economic motivation (see Section 4.3). However, the mathematical approach works on a general order interval in the interior of ℓ_∞^{+}. This leads to extension of the uniqueness theorem to $\mathrm{B}^{+}(\ell_\infty^{++})$. The union of these order intervals covers the positive cone's norm interior. Details are given in the following sections. Once the order interval $\langle ae, be\rangle$ is specified, the resulting utility space is:

$$\mathrm{B}^{+}(a,b) = \{U : \langle ae, be\rangle \to \mathbb{R}_{+}\}.$$

This space, endowed with the $\|\bullet\|_\gamma$ — norm, equals the space of nonnegative functions on $\langle ae, be\rangle$ bounded by the usual $\|\bullet\|_\infty$ — norm, since the weight function $\varphi_\gamma(C)$ is bounded from above by $\varphi_\gamma(be)$. We choose to maintain the original weighting structure in our calculations to readily extend results obtained for $\mathrm{B}^{+}(a,b)$ to $\mathrm{B}^{+}(\ell_\infty^{++})$.

Given consumption sequences are in $\langle ae, be\rangle$, there are upper and lower bounds of the following type (by W monotone and Thompson $(T4)$):

$$\begin{aligned} \inf_{C\in\langle ae,be\rangle} T_W\theta(C) &= W(a,0) > 0; \\ \sup_{C\in\langle ae,be\rangle} T_W\theta(C) &= W(b,0) < \infty. \end{aligned}$$

The first equation above implies $T_W\theta > \theta$ holds for each $C \in \langle ae, be\rangle$ and $SC \in \langle ae, be\rangle$ as well. This implies $T_W\theta$ is an order unit in $\mathrm{B}(a,b)$. In addition, this implies each iterate, $T_W^N\theta$, is also an order unit in that space as $\left\{T_W^N\theta\right\}$ is a monotone sequence in $\left\langle \theta, U^T\right\rangle$ taken as a subset of $\mathrm{B}^{+}(a,b)$. Constructive existence theory (see [14]) and the norm-convergence of $\left\{T_W^N\theta\right\}$ to some point U^* by $u_0 - concave$ operator approximation theory (Section 6). This implies $U^* = \sup_N \left\{T_W^N\theta\right\}$ since $\left\{T_W^N\theta\right\}$ is increasing and order convergent by Lemma 1. As an order convergent sequence has at most one limit point, we conclude that $U^* = U_\infty = \sup_N \left(T_W^N\theta\right)$ obtains. Once this result is in place, $u_0 - concave$ operator theory implies uniqueness of the Koopmans operator's fixed point in $\left\langle \theta, U^T\right\rangle \subset \mathrm{B}^{+}(a,b)$. For each $C \in \langle ae, be\rangle$, $U_\infty(C) = U^\infty(C)$. Extension of the Koopmans operator's unique solution to the norm-interior of $\mathrm{B}^{+}(\ell_\infty^{++})$ follows since $\langle ae, be\rangle$ is, from this perspective, arbitrarily chosen.

5. The Koopmans Operator Is $u_0 - concave$ on $\mathrm{B}^{+}(\ell_\infty^{++})$

We prove that the Koopmans equation has at most one solution in the set of φ_γ-bounded functions defined on the set ℓ_∞^{++} by application of Krasnosel'skiĭ and Zabreĭko's [37] $u_0 - concave$ operator theory. Fix the arbitrarily chosen order interval $\langle ae, be\rangle$ and the corresponding function space $\mathrm{B}(a,b)$ together with its positive cone, $\mathrm{B}^{+}(a,b)$. We apply Liang et al's sufficiency theory in this positive cone.

Consider the Koopmans operator's LFP, U_∞ in $\left\langle \theta, U^T\right\rangle \subset \mathrm{B}^{+}$. It is $\gamma-$ norm lower semicontinuous. Moreover, $U_\infty(C) \geq T_W\theta(C) \geq \theta(C) = 0$ for all $C \in \ell_\infty^{+}$ and a strict inequality obtains for some C. The restriction of the LFP to $\langle ae, be\rangle$ must also solve the Koopmans equation on $\mathrm{B}^{+}(a,b)$. The reason is simple: $T_W U_\infty(C) = U_\infty(C) = W(c_1, U_\infty(SC))$ must hold for ALL $C \in \ell_\infty^{+}$, so *a fortiori*, this equality must also hold for each $C \in \langle ae, be\rangle$. This observation makes use of the shift invariance property of $\langle ae,be\rangle$. Note $U_\infty > \theta$ on each $\langle ae,be\rangle$ since $S(ae) = ae$ and $W(a,U_\infty(S(ae))) \geq W(a,0) > 0$.

Thus if we can show there is at most one solution in $\mathrm{B}^+(a,b)$, then it is the uniquely determined solution on that domain. Given the LFP exists on ℓ_∞^+, this implies U_∞ is the unique solution in $\mathrm{B}^+(\ell_\infty^{++})$. *The least and greatest fixed points agree on* ℓ_∞^{++}.

The first step is to prove the Koopmans operator is an order-concave operator mapping from $\mathrm{B}^+(a,b)$ to itself.

Lemma 5.1. $T_W : \mathrm{B}^+(a,b) \to \mathrm{B}^+(a,b)$ *is an order-concave operator on* $B^+(a,b)$.

Proof. That T_W is an order-concave operator follows from the concavity of the aggregator function and convexity of the cone $\mathrm{B}^+(a,b)$. Fix an arbitrary $C \in \langle ae,be\rangle$. Let $U^0, U^1 \in \mathrm{B}^+(a,b)$ and let $U^t = (1-t)U^0 + tU^1$, for $0 \leq t \leq 1$. Evidently $U^t \in \mathrm{B}^+(a,b)$. By W concave in its second argument for each c_1,

$$W\left(c_1, U^t(SC)\right) \geq (1-t)W\left(c_1, U^0(SC)\right) + tW\left(c_1, U^1(SC)\right).$$

The lefthand side of this inequality is $T_W U^t(C)$ and the righthand side is the convex combination $(1-t)T_W U^0(C) + tT_W U^1(C)$. Thus,

$$T_W U^t \geq (1-t)T_W U^0 + tT_W U^1,$$

and T_W is an order-concave operator on $\mathrm{B}^+(a,b)$ as well.[39] □

This proof does not actually depend on the order interval $\langle ae,be\rangle$ and goes through in $\mathrm{B}^+(\ell_\infty^+)$. That is, T_W is, in general, an order-concave operator acting on the convex cone $\mathrm{B}^+(\ell_\infty^+)$.

The Lemma's order-concavity result clearly implies the Koopmans operator is subhomogeneous on $\mathrm{B}^+(a,b)$ as $T_W U \geq T_W\theta \geq \theta$. We note below that $T_W\theta$ is an order unit in $\mathrm{B}^+(a,b)$, so $T_W\theta >> \theta$. This implies, for each $t \in (0,1)$, that $T_W(tU) > tU$ for each $U \in \mathrm{B}^+(a,b)$ and $U \neq \theta$ and $U = U^1$ and $\theta = U^0$ in the order-concavity condition. This strict subhomogeneity property for $\mathrm{B}^+(a,b)$ is stronger than the Koopmans operator's subhomogeneity property derived from its order-concavity on $\mathrm{B}^+(\ell_\infty^+)$. The strict subhomogeneity of the Koopmans operator for concave Thompson aggregators is weaker than the *strong subhomogeneous* condition employed by Marinacci and Montrucchio's [44] Thompson (concave) aggregator uniqueness theorem.

Liang et al's [40] sufficient condition holds for the Koopmans operator on $\mathrm{B}^+(a,b)$. We add additional notation applicable to the restriction of the general commodity space ℓ_∞^+ to an order interval, $\langle ae,be\rangle$. For $U \in \mathrm{B}^+(\ell_\infty^+)$, its restriction to the order interval is denoted by $U[a,b]$. In particular, for $T_W\theta \in \mathrm{B}^+(\ell_\infty^+)$ let its restriction be $T_W\theta[a,b]$. Denote the set of functions comparable to $T_W\theta[a,b] \in \mathrm{B}^+(a,b)$ as the constituent $Q(T_W\theta[a,b])$.

$\mathrm{B}(a,b)$ is a Banach lattice with unit, φ_γ. Its positive cone $\mathrm{B}^+(a,b)$, is a nonempty, $\|\bullet\|_\gamma-$norm-closed, normal and solid convex cone. T_W is clearly a self map on $\mathrm{B}^+(a,b)$.

The next lemma confirms the Koopmans operator's range is $Q(T_W\theta[a,b])$. Its corollary states the Koopmans operator is $T_W\theta - concave$ on $Q(T_W\theta[a,b])$. In Liang et al's [40] notational setup set $P \equiv \mathrm{B}^+(a,b)$ and $u_0 \equiv T_W\theta$ and $Q(T_W\theta[a,b])$ plays the role of P_{u_0} as follows:

$$Q(T_W\theta[a,b]) = \left\{ \begin{array}{c} V \in \mathrm{B}^+(a,b) : \exists\alpha(V), \beta(V) > 0 \\ \text{and } \alpha(V)T_W\theta \leq V \leq \beta(V)T_W\theta \end{array} \right\}.$$

Let $T_W U \in \mathrm{B}^+(a,b)$ when $U \in \mathrm{B}^+(a,b)$. We prove $T_W U \in Q(T_W\theta[a,b])$.

[39]The proof is valid if either $U^0 \geq U^1$ or $U^1 \geq U^0$.

Lemma 5.2. $T_W : \mathrm{B}^+(a,b) \to Q(T_W\theta[a,b])$.

Proof. Since T_W is a monotone operator, we always have $T_WU \geq T_W\theta$ whenever $U \geq \theta$. Set $\alpha(U) = 1$ and write the scalar $\alpha(T_WU) \equiv \alpha(U)$ to simplify notation. This scalar depends on U and its image under T_W.

On the other hand, since U is $\varphi_\gamma - \mathit{bounded}$, there is a number M^U such that $U(C) \leq M^U\varphi_\gamma(C)$ for each $C \in \ell_\infty^+$. Also note that $\varphi_\gamma(C) \leq \varphi_\gamma(be) = (1+b)^{1/\gamma}$ for each $C \in \langle ae, be\rangle$. Thus, by monotonicity of the Koopmans operator we obtain for each $C \in \langle ae, be\rangle$:

$$\begin{aligned} T_WU(C) &\leq T_W(M^U\varphi_\gamma(C)) \leq M^U W\left(c_1, M^U\varphi_\gamma(SC)\right) \\ &\leq W\left(c_1, M^U(1+b)^{1/\gamma}\right). \end{aligned}$$

Moreover, $T_W\theta(C) = W(c_1,0) \geq W(a,0)$ by monotonicity of the aggregator function and the definition of $\langle ae, be\rangle$. Therefore, $T_W\theta[a,b] >> \theta$ and it is an order unit in $\mathrm{B}^+(a,b)$ since $W(a,0) > 0$ by $(T4)$. This shows $Q(T_W\theta[a,b])$ is a well-defined constituent of $\mathrm{B}^+(a,b)$. Evidently $\mathrm{B}^{++}(a,b) = Q(T_W\theta[a,b])$.

Next, choose $\beta(U)$ sufficiently large so that

$$W\left(c_1, M^U(1+b)^{1/\gamma}\right) \leq \beta(U)W(a,0).$$

Then note that $\beta(U)W(a,0) \leq \beta(U)T_W\theta(C)$ and the previous inequality yields for each $C \in \langle ae, be\rangle$:

$$T_WU(C) \leq \beta(U)T_W\theta(C).$$

Thus, for $\alpha(U) = 1$, and this choice of $\beta(U)$, we find

$$T_W\theta \leq T_WU \leq \beta(U)T_W\theta. \tag{5.1}$$

This proves $T_WU \in Q(T_W\theta[a,b])$. □

Inequality (5.1) is readily rearranged in the form of inequality (3.5): for $\mu(U) = 1/\beta(U) > 0$:

$$\mu(U)T_WU \leq T_W\theta. \tag{5.2}$$

Combining Lemmas 10 and 11 yields:

Corollary 5.3. *$T_W : \mathrm{B}^+(a,b) \to \mathrm{B}^+(a,b)$ is a $T_W\theta - concave$ operator for each $0 < a < b < \infty$.*

Proof. Lemmas 10 and 11 imply the Koopmans operator satisfies the hypotheses of Liang et al's Theorem. That is, (5.2) obtains. Hence T_W is a $T_W\theta - \mathit{concave}$ operator acting on each cone $\mathrm{B}^+(a,b)$. □

Finally, we invoke the *Krasnosel'skiĭ and Zabreĭko Theorem* when the Koopmans operator acts on $\mathrm{B}^+(a,b)$.

Proposition 5.4. *For each given $0 < a < b < \infty$, the Koopmans equation, $T_WU = U$, has at most one non-zero solution in the cone $B^+(a,b)$.*

The nonzero principal fixed point $U_\infty \in \mathrm{B}^+(\ell_\infty^+)$ remains a nonzero solution to the Koopmans equation on the domain $\mathrm{B}^+(a,b)$. Hence, Proposition 13 implies

$$T_WU_\infty[a,b] = U_\infty[a,b]$$

for each $0 < a < b < \infty$ is the unique solution in $\mathrm{B}^+[a,b]$. The order interval $\langle ae, be\rangle$ is arbitrary. Hence U_∞ is the unique solution among all the functions in $\mathrm{B}^+(\ell_\infty^{++})$.

Uniqueness on ℓ_∞^{++} implies $U^\infty = U_\infty$ on that domain. U_∞ is sup norm lower semicontinuous and U^∞ is sup norm upper semicontinuous. Hence, the operator's principal solution is also a sup norm continuous function on ℓ_∞^{++}.

We sum up our findings:

Theorem 5.5. *Let W be a concave Thompson aggregator. Then, there is a unique nonzero norm continuous utility function $U_\infty \in B^+(\ell_\infty^{++})$ that solves the Koopmans equation. That is, for each $C \in \ell_\infty^{++}$:*

$$T_W U_\infty(C) = U_\infty(C) = W(c_1, U_\infty(SC)).$$

6. Uniform Approximation and *a posteriori* Error Estimation

Order theoretic iterative construction of the LFP and GFP as found in our paper [14] does *not* imply the sequence of successive approximations initiated at $U_0 = \theta$, $\{T_W^N \theta\}$ converges *uniformly* to the unique fixed point of the Koopmans operator established in Section 5. More broadly, the same can be said for iterations initiated at a arbitrary positive $U_0 \in \mathrm{B}^+(a,b)$. The purpose of this section is to note uniform convergence holds for a $u_0 -$ *concave* operator acting on $\mathrm{B}^+(a,b)$. However, the uniform convergence does not come with an *a posteriori* error estimate yielding information about the rate of convergence to the unique solution, U_∞, found in Theorem 14. We draw on the monographs by Guo and Lakshmikantham [27], Guo et al [28], and Balbus's [10] article in order to resolve these issues.

An *a posteriori error estimate*, denoted $E(N)$, is a function that depends on N and the approximate solution at iterate stage N.[40] The particular error estimate depends on the choice of U_0 that initiates the successive approximations iterations. We are interested in the case where the sequence is $\{T_W^{N+1}\theta\}$, so $T_W^{N+1}\theta$ is the approximate solution in that step. Therefore, we set $U_0 = T_W\theta$ to compute an error estimate in terms of the parameters a, b defining $\mathrm{B}^+(a,b)$ and functional form, W. Formally, an *a posteriori* error estimate yields the inequality $\left\|T_W^{N+1}\theta - U_\infty\right\|_\gamma \leq E(N)$ with $\lim_{N\to\infty} E(N) \to 0$. Computing this error bound is the main result in this section since uniform convergence of $\{T_W^{N+1}\theta\}$ to U_∞ follows.

6.1. Successive Approximations Uniformly Converge to the LFP.

Guo and Lakshmikantham ([27], Theorem 2.2.4 and Corollary 2.2.1) show iteration of a $u_0 -$ *concave* operator has a uniform convergence property. We specialize their result for the Banach lattice with unit, $\mathrm{B}(X)$. Recall order unit norm and uniform convergence are equivalent in this situation. Here u_0 is an order unit (and may be different from e).

Theorem 6.1. *Suppose $A : B^+(X) \to B^+(X)$, is monotone and $u_0 -$ concave. Assume there is $x^* > \theta$ with $Ax^* = x^*$. Then the successive approximation sequence $\{x_N\}$ with $x_N = Ax_{N-1}$ for each $N \in \mathbb{N}$ for any initial $x_0 > \theta$ uniformly converges to x^*. That is,*

$$\|x_N - x^*\|_\infty \to 0 \text{ as } N \to \infty.$$

[40] Linz ([41], p.27) describes *a posteriori* error estimates. It is also addressed in Krasnosel'skii et al ([36], pp. 167-168) in the context of nonlinear operator theory. Krasnosel'skii and Zabreĭko ([37], pp. 318-319) note the importance of uniform convergence in actual computations owing to roundoff errors.

Apply this result to the Koopmans operator acting as a self-map on $\mathrm{B}^{+}(a,b)$ to obtain for each $U_0 \in \mathrm{B}^{+}(a,b)$ that $\left\{T_W^N U_0\right\}$ converges uniformly to the unique fixed point, U^*. If $U_0 = T_W\theta$, then T_W is $T_W\theta - concave$, the sequence $\left\{T_W^{N+1}\theta\right\}$ is increasing, converges uniformly by Theorem 15, and order converges to $U^* = U_\infty = \sup_N \left\{T_W^{N+1}\theta\right\}$ by Lemma 1.

This convergence result does not come with an error estimate, so we cannot say anything about the *rate* that $\{T_W^{N+1}\theta\}$ converges to U_∞. However, this problem can be resolved with some additional structure, as first noted by Balbus [10].

6.2. ***A Posteriori*** **Error Estimate for an** $r - concave$ **Koopmans Operator in** $\mathbf{B}^{++}(a,b)$**.** The $T_W\theta - concavity$ of the Koopmans operator for Thompson aggregators can be further specialized for some special Thompson aggregator cases, such as the CES aggregator. Balbus [10] introduces an additional condition (Assumption 3, p. 558), called *(T7)* here. This property, together with *(T1)-(T6),* and concavity of the aggregator, implies the Koopmans operator is also an *r-concave operator*. Liang et al ([40], Lemma 2) observe an $r - concave$ operator is also $u_0 - concave$. As such, it is reasonable to expect the additional structure of $r - concavity$ to yield a stronger solution theory for the Koopmans operator equation than $u_0 - concavity$ alone delivers. Indeed, there are two features: there exist a unique fixed point in $\mathrm{B}^{++}(a,b)$ and there is an *a posteriori* estimate of the rate of convergence obtained by iterating the operator for a given initial input. This estimate arises here as a by-product of proving the unique fixed point of the given operator has a global attracting property as defined below.

The additional aggregator condition sufficient for the Koopmans operator to be $r - concave$ is:

(T7) W is increasing in (x,y)*, jointly continuous, and* $W(0,0) = 0$*. Furthermore, there is* $r \in (0,1)$ *such that for each* $x \geq 0$*,* $y > 0$*, and each* $t \in (0,1)$

$$W(x,ty) \geq t^r W(x,y). \tag{6.1}$$

We say the *aggregator* is $r - concave$ **in** y **for each** x (shortening the expression: $r - concave$ **of order** r **in** y **for each** x).

Balbus [10] demonstrates by an example that *(T7)* alone does NOT imply *(T3)*. In particular, he supplies an example where W is $r - concave$ but is not concave at the origin as required by Marinacci and Montrucchio ([43], [44]). We combine *(T7)* and the previous six Thompson aggregator conditions. This yields information on the rate at which the iterates $T_W^{N+1}\theta$ uniformly converge to the operator's unique fixed point.

Definition 6.2. Let $(E, \|\bullet\|)$ be an ordered Banach space. Let E_0 be a nonempty subset of E. Suppose that an operator $A : E_0 \to E_0$. Assume that this operator has a fixed point, $x^* \in E_0$. We say the fixed point x^* has the **global attracting property on** E_0 if, for each initial input $x_0 \in E_0$, $x_0 > \theta$, $\lim_{N\to\infty} \|x_N - x^*\| = 0$, where $x_N = Ax_{N-1}$.

In practice E is an AM-space with unit and $E_0 = E^{++}$. Our case arises for $E = \mathrm{B}(a,b)$ with $E_0 = \mathrm{B}^{++}(a,b)$. The following Proposition appears in Balbus [10]. It adapts a result in Gou, et al ([28], Theorem 3.1.7) to the case of a Thompson aggregator that is $r - concave$ **in** y for each x. The essential point is that if we iterate the Koopmans operator acting on $\mathrm{B}^{++}(a,b)$ from an initial seed, say $U_0 = T_W\theta >> \theta$, then the γ-norm difference between the $N - th$ iterate of the Koopmans operator, $T_W^{N+1}\theta$, and U_∞ is bounded above by a term that converges to zero as $N \to \infty$. We state the proposition for the special case where the solid positive cone is $\mathrm{B}^{+}(X)$ for some set X and the normality constant is $\mathscr{N}=1$. This result does not exclude the possibility of a second solution in $\partial_0\mathrm{B}^{+}(X)$.[41] The operator's domain in this setting differs from the domain assumed in section 3.4. There, the operator mapped functions in $\mathrm{B}^{+}(X)$ to ones in $Q(u)=\mathrm{B}^{++}(X)$. The next proposition's operator is a selfmap on $Q(u)$.

Proposition 6.3. *If $A:\mathrm{B}^{++}(X)\to\mathrm{B}^{++}(X)$ is an $r-concave$ operator, then A has a unique fixed point $x^*\in\mathrm{B}^{++}(X)$. Moreover, this fixed point is globally attracting with a posteriori error estimate*

$$\|x_N-x^*\|\leq M\left(1-\tau^{r^N}\right) \text{ for each } N,$$

with $x_N=Ax_{N-1}$, and $x_0>>\theta$. The constants $M=2\|x_0\|_\infty, \tau=(t_0/s_0)$, and the numbers t_0 and s_0 are chosen to satisfy the following comparability inequality:

$$0<t_0<1<s_0 \text{ and } t_0^{1-r}x_0\leq Ax_0\leq s_0^{1-r}x_0. \tag{6.2}$$

Inequality (6.2) states $Ax_0\sim x_0$ obtains for a particular choice of the positive scalars t_0 and s_0. Both x_0 and Ax_0 are linked to the order unit, u. The proposition pins down the particular choice of scalars in the linking condition (6.2). Krasnosel'skiĭ and Zabreĭko ([37], p. 319) prove $x^*\sim u$, each $x_N\sim x^*$, and $\{x_N\}$ converges pointwise to x^*. The error estimate $E(N)\to 0$ as $N\to\infty$, due to the $r-concavity$ property, is given by

$$E(N)=M\left(1-\tau^{r^N}\right) \text{ for each } N.$$

The proposition's application to Thompson aggregators requires verification its hypotheses obtain when the Koopmans operator satisfies the following three conditions when W satisfies *(T1)-(T7)*.

(1) T_W is a self-map on $\mathrm{B}^{++}(a,b)$;
(2) T_W is an $r-concave$ operator;
(3) the parameters (t_0,s_0), as well as the initial seed, U_0, can be appropriately chosen in terms of (a,b) so that (6.2) obtains.

It turns out that this last property holds when $U_0=T_W\theta$ and we use the link property with $T_W^2\theta$. This is the special case of interest in our uniqueness theory.

Lemma 6.4. *Suppose W is a Thompson aggregator satisfying (T5) and (T6). Then $T_W:\mathrm{B}^{++}(a,b)\to\mathrm{B}^{++}(a,b)$.*

Proof. Let $U\in\mathrm{B}^{++}(a,b)$. Then

$$T_WU(C)=W(c_1,U(SC))\geq W(a,0)>0$$

for each $C\in\langle ae,be\rangle$. Set $V=T_WU$. Then $V\in\mathrm{B}^{++}(a,b)$ with $V>>\theta$, $\|V\|_\gamma>0$ and $V\in\mathrm{B}^{++}(a,b)$. ☐

In particular, $T_W\theta\in\mathrm{B}^{++}(a,b)$ so $T_W^2\theta\in\mathrm{B}^{++}(a,b)$ and so on. Hence, $T_W\theta$ may be chosen as the initial condition yielding the sequence $\{T_W^{N+1}\theta\}$.

Lemma 6.5. *If W is a concave Thompson aggregator satisfying (T5)-(T7), then T_W is an $r-concave$ operator.*

[41] An obvious example is $\sqrt{x}$ for $x\geq 0$ where both $x=0$ and $x=1$ are fixed points.

Proof. Let $U \in$B$^{++}(a,b)$ and $C \in \langle ae, be\rangle$. Then, for each $0 < t < 1$:

$$T_W(tU)(C) = W(c_1, tU(SC)) \text{ by definition of } T_{W.}$$

Assumption *(T7)* implies inequality (6.1) holds and

$$\begin{aligned} W(c_1, tU(SC)) &\geq t^r W(c_1), U(SC)) \\ &= t^r (T_W U)(C). \end{aligned}$$

Therefore, $T_W(tU) \geq t^r T_W U$ holds pointwise and the Koopmans operator is $r-concave$. □

Proposition 17 implies that there is a unique fixed point for the Koopmans operator in the set B$^{++}(a,b)$. We can relate this immediately to our existence theorem [14].

Corollary 6.6. *If W is a concave Thompson aggregator satisfying* (T5)-(T7)*, then $U_\infty = U^\infty$ is the unique fixed point of the Koopmans operator in $B^{++}(a,b)$. Moreover, this fixed point is a $\varphi_\gamma - bounded, \|\bullet\|_\infty - norm$ continuous real-valued function defined on $\langle ae, be\rangle$.*

The final point is to verify an *a posteriori* estimate holds for the Koopmans operator with an initial seed in B$^{++}(a,b)$. Indeed, we verify this estimate when the initial input is $T_W\theta >> \theta$. It is the natural choice given the Tarski-Kantorovich fixed point theory underpinning our general existence theory and the prominence of the LFP in our related theoretical results. This follows by showing we can choose the numbers (t_0, s_0) to satisfy the pointwise inequalities in (6.2).

Maintain the assumption that W is a concave Thompson aggregator satisfying *(T5)-(T7)*. As $T_W\theta \in$ B$^{++}(a,b)$, $T_W^2\theta \in$ B$^{++}(a,b)$, then $T_W\theta \sim T_w^2\theta$. Hence, there are positive scalars α and β, depending on $T_W^2\theta$ such that

$$\alpha T_W\theta \leq T_W^2\theta \leq \beta T_W\theta. \tag{6.3}$$

There is no loss in generality in taking $\alpha < 1 < \beta$ as the previous inequality still obtains. Here $U_0 = T_W\theta \in$ B$^{++}(a,b)$ and $T_W^2\theta(C) = W(c_1, W(c_2, 0))$ for each $C \in \langle ae, be\rangle$. Moreover, $T_W^2\theta(C) \leq W(b, W(b,0)) = T_W^2\theta(be)$; likewise, $W(a,0) = T_W\theta(ae) \leq T_W\theta(C)$.

Set $M = 2\|T_w\theta\|_\gamma$. In fact, we can estimate M for the given aggregator in terms of the constants a and b by observing

$$\|T_W\theta\|_\gamma = \sup_{C\in\langle ae,be\rangle} \frac{W(c_1,0)}{\varphi_\gamma(C)} \leq \frac{W(b,0)}{(1+a)^{1/\gamma}}.$$

To see this, recall $(1+a)^{1/\gamma} \leq \varphi_\gamma(C) \leq (1+b)^{1/\gamma}$ and $T_W\theta(C) = W(c_1, 0)$ with $a \leq c_1 \leq b$.[42]

Similarly *(T4)* implies

$$\|T_W\theta\|_\gamma \geq \inf_{C\in\langle ae,be\rangle} \frac{W(a,0)}{(1+b)^{1/\gamma}} > 0.$$

Use the upper bound on $\|T_w\theta\|_\gamma$ above to set

$$M = 2\frac{W(b,0)}{(1+a)^{1/\gamma}}.$$

M clearly depends the order interval $\langle ae, be\rangle$, which is well-defined given the parameters (a,b) and γ (from *(T5)*.

[42]The comparative ease of this computation is one rationale for choosing $T_W\theta$ for the initial seed instead of U^T or even $T_W U^T$ as the initial seed.

To complete the verification the link parameters α and β must be found in terms of the aggregator and the pair (a,b). Set

$$\begin{aligned} 0 &< \alpha = \frac{W(a,0)}{W(a,W(a,0))} < 1; \\ 1 &< \beta = \frac{W(b,W(b,0))}{W(b,0)}. \end{aligned} \tag{6.4}$$

It is clear the LHS (6.3) automatically holds as $T_W\theta \le T_W^2\theta$ for each C implies $\alpha T_W\theta \le T_W^2\theta$ for each $\alpha < 1$.

The RHS (6.3) also obtains:

$$T_W^2\theta(C) \le \beta T_W\theta(C) \text{ for some } \beta > 1$$

since $T_W\theta \sim T_w^2\theta$. For each C this inequality implies:

$$T_W^2\theta(C) \le \beta T_W\theta(C) \le \beta W(b,0) = W(b,W(b,0))$$

using the chosen value of β. Hence, the relation $T_W\theta \sim T_w^2\theta$ for this particular choice of β. This inequality string implies $T_W^2\theta(C) \le \beta T_W\theta(C)$ obtains. It is equivalent to the inequality $T_W^2\theta(C) \le T_W^2\theta(be)$.

With the above values, the link parameters may be chose so that (6.2) holds with

$$\begin{aligned} \alpha &= t_0^{1-r}; \\ \beta &= s_0^{1-r}. \end{aligned}$$

Invert each preceding equation to solve for t_0 and s_0, in terms of W and the parameters (a,b). Then $\alpha < 1 < \beta$, and $0 < r < 1$ imply

$$\begin{aligned} 0 &< t_0 = [\alpha]^{1/(1-r)} < 1; \\ 1 &< s_0 = [\beta]^{1/(1-r)}. \end{aligned}$$

The scalars t_0 and s_0 depend on the parameters (a,b). Clearly $\tau \equiv (t_0/s_0) \in (0,1)$ and (6.2) holds with these values that are determined once the values of a and b are set. Using these choices of the parameters M and (t_0,s_0) together with the initial seed, $U_0 = T_W\theta$, we conclude there is a unique $U^* = T_W U^*$ such that

$$\left\| T_W^{N+1}\theta - U^* \right\|_\gamma \le M\left(1 - \tau^{r^N}\right) \text{ for each } N \in \mathbb{N}. \tag{6.5}$$

The existence of a unique solution promised by the next result does not yet insure it coincides with either the LFP or GFP in fix(T_W). However, the Banach lattice properties of the space $\mathrm{B}(a,b)$ yield a connection to the LFP by way of Lemma 1: norm convergence of $\left\{T_W^{N+1}\theta\right\}$ with $\left\{T_W^{N+1}\theta\right\} \nearrow U^*$ implies $U^* = U_\infty = \sup_N \left\{T_W^{N+1}\theta\right\}$.

The importance of establishing a link between norm and order convergence is the latter convergence mode links to the information partial order implicit in the successive approximation sequence $\left\{T_W^{N+1}\theta\right\}$ and our argument suggesting the LFP, U_∞, is a reasonable selection criterion for defining the *principal solution* of the Koopmans equation when it has multiple solutions. Becker and Rincon-Zapatero [14] offers a detailed defense of this selection criterion. However, a dissenting voice favoring the GFP is argued by Bloise, et al [19].

Theorem 6.7. *If W is a concave Thompson aggregator satisfying (T5)-(T7), then $U^* = T_W U^*$ is the unique fixed point of the Koopmans operator in $B^{++}(a,b)$. Furthermore, U^* is globally attracting for each initial seed, $U \in \mathrm{B}^{++}(a,b)$. In particular, we have (6.5) for $U_0 = T_W\theta \in B^{++}(a,b)$ that*

$$\left\|T_W^{N+1}\theta - U^*\right\|_\gamma \leq M\left(1-\tau^{r^N}\right) \text{ for each } N.$$

Hence,

$$\left\|T_W^{N+1}\theta - U^*\right\|_\gamma \to 0 \text{ as } N \to \infty.$$

Moreover, the constants M, t_0, and s_0 satisfying (6.2) are given by

$$\begin{aligned} M &= 2\left\|T_W\theta\right\|_\gamma \text{ with } 0 < \left\|T_W\theta\right\|_\gamma \leq \frac{W(b,0)}{(1+a)^{1/\gamma}}; \\ 0 &< t_0 = [\alpha]^{1/(1-r)} < 1; \\ 1 &< s_0 = [\beta]^{1/r}, \end{aligned}$$

where $\tau = (t_0/s_0) \in (0,1)$ given the choices in (6.4) and $r \in (0,1)$ is specified by (T7).

Corollary 6.8. $U^* = U_\infty = U^\infty$.

Proof. The sequence $\left\{T_W^{N+1}\theta\right\}$ is increasing and uniformly converges to U^*. Lemma 1 implies $\sup_N\left\{T_W^{N+1}\theta\right\} = U_\infty = U^*$ and $\left\{T_W^{N+1}\theta\right\}$ order converges to U_∞. Uniqueness of U^* also implies $U_\infty = U^\infty$. □

The *a posteriori* error estimate (6.5) shows the N^{th} approximate solution, $T_W^{N+1}\theta$, belongs to the ball centered at U_∞ with radius $M\left(1-\tau^{r^N}\right)$. This ball's radius decreases with N and the approximation $T_W^N(T_W\theta) = T_W^{N+1}\theta$ can be made arbitrarily close to the true solution, U^∞, by choosing N sufficiently large.[43] The specific knowledge that the approximate solution, $T_W^{N+1}\theta$, lies in a ball whose radius converges to zero as $N \to \infty$ is new information that does not follow from the $u_0 - $*concave* Koopmans operator's uniform approximation feature assuming *(T1)-(T6)* hold. In that case, $\left\{T_W^{N+1}\theta\right\}$ order converges to U_∞ by our constructive existence theorem. Alternatively, application of Theorem 15 also yields $\{T_W^{N+1}\theta\}$ converges uniformly and in order to U_∞.

The CES aggregator $W(x,y) = (1-\beta)x^\rho + \beta y^\rho$ for $0 < \beta, \rho < 1$ satisfies Assumption *(T7)*. This follows by checking Balbus's sufficient condition ([10], pp. 560-561) for *(T7)*: for each $x \geq 0$ and $y > 0$.

$$\frac{W_2(x,y) * y}{W(x,y)} < 1.$$

Here, W_2 denotes the partial derivative of W with respect to y. A routine computation shows

$$0 < \frac{W_2(x,y) * y}{W(x,y)} = \rho\frac{\beta y^\rho}{(1-\beta)x^\rho + \beta y^\rho} \leq \rho < 1.$$

According to Balbus [10] the parameter r found in Assumption *(T7)* equals ρ in this CES example. He also shows the KDW aggregator satisfies *(T7)*.

[43]Our existence theorem also implies that each iteration of the Koopmans operator starting with $U_0 = T_W\theta$ is, in fact, a $\varphi_\gamma-$ bounded continuous function on (ae, be) (endowed with the sup norm topology inherited from ℓ_∞).

7. Concluding Comments

A recursive utility function is uniquely determined by a Thompson aggregator at each consumption sequence in ℓ_∞^{++}. Null consumption sequences as well as all other consumption sequences in $\partial_0\ell_\infty^+$ may be associated with points where the LFP is strictly smaller than the GFP. These restrictions agree with ones imposed in Martins-da-Rocha and Vailakis [45], Bloise and Vailakis [18] and Marinacci and Montrucchio ([43],[44]). Restricting the commodity space in this manner removes potential sources for multiple solutions to the Koopmans equation by eliminating boundary fixed points.

We strengthen Marinacci and Montrucchio's [43] Thompson aggregator definition to include joint continuity and concavity of the aggregator on its domain. The joint concavity assumption is stronger than their assumption that the aggregator is concave at zero in its second argument. We strengthen their conditions in order to prove the Koopmans operator is an order-concave operator and verify it is also a $u_0 - concave$ operator. This approach differs from the methods used in the existing literature. This technique is an alternative for demonstrating uniqueness results compared to the contraction operator theorems based on the Thompson metric employed by Marinacci and Montrucchio ([43], [44]) or the 0-local contractions introduced by Rincón-Zapatero and Rodriguez-Palmero ([51], [52]) and Martins-da-Rocha and Vailakis ([45], [46]).

The uniqueness contribution by Marinacci and Montrucchio [43] accommodates some endogenous growth models; we are not able to do so. It would be of some interest to adapt the $u_0 - concavity$ approach for an order-concave operator uniqueness theory for the weighted commodity space underlying sustainable or endogenous growth models. That is, find a uniqueness theory compatible with the general existence theory for solutions to the Koopmans equation when the economic model includes a variety of endogenous growth theories.

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