EXAMPLES of STOCHASTIC PROCESSES
(Measure Theory and Filtering by Aggoun and Elliott)

Example 1: Let

$$\Omega = \{\omega_1, \omega_2, ...\},$$

and let the time index n be finite $0 \leq n \leq N$. A stochastic process in this setting is a two-dimensional array or matrix such that:

$$X = \begin{bmatrix} X_1(\omega_1) & X_1(\omega_2) & ... \\ X_2(\omega_1) & X_2(\omega_2) & ... \\ ...... & ... & ... \\ X_N(\omega_1) & X_N(\omega_2) & ... \end{bmatrix}$$

Each row represents a random variable and each column is a sample path or realization of the stochastic process X. If the time index is unbounded, each sample path is given by an infinite sequence.

Example 2: Let $N = 4$ in the previous example and suppose that X is given by the following array

$$\begin{bmatrix} 2 & 3 & 5 & 7 & 11 & 3 & 2.3 & 1 \\ -1 & 1 & 7.5 & \sqrt{2} & 3 & 6 & 83 & 19 \\ 11 & 7 & 70 & 3 & 2 & -5 & 2 & 21 \\ 5 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \end{bmatrix}$$

The sample space of $\{X_n\}$ is $\mathbf{R}^4$ and the stochastic process can be thought of as a mapping (in fact a random variable)

$$\omega_i \rightarrow X(\omega_i) = (X_1(\omega_i), X_2(\omega_i), X_3(\omega_i), X_4(\omega_i)) = (x_1^i, x_2^i, x_3^i, x_4^i) \equiv x^i \in \mathbf{R}^4.$$

The random variable X induces a probability measure P_X on the Borel $\sigma - field\ \mathcal{B}(\mathbf{R}^4)$,

$$P_X(B) \equiv P[\omega : X(\omega) \in B] = P(X^{-1}(B)).$$

For instance,

$$B_1 = \{x \in \mathbf{R}^4 : 3 \leq x_1 \leq 5,\ 2 \leq x_2 \leq 7\}$$

contains a single trajectory (column 6 in the table) so that $P_X(B) = P(\omega_6)$.

$$B_2 = \{x \in \mathbf{R}^4 : \max_{1 \leq n \leq 4} x_n \leq 7\}$$

contains three trajectories (column 2, 4,and 6 in the table) so that $P_X(B) = P(\omega_2, \omega_4, \omega_6)$.

Exercise 1: In economics you will only be observing a time series, for instance column 3. Assuming that the process is *i.i.d* try to obtain the probabilities of B_1 and B_2. Later on, these standard assumptions will be substituted by *Ergodocity* and *Stationarity.*

Example 3: Let $\Omega = \{\omega_1, \omega_2, ...\}$ and P a probability measure on $(\Omega, \mathcal{F})$. Suppose the time index set is the set of positive integers. A real valued stochastic process X in this setting is a two-dimensional infinite array such that:

$$X = \begin{bmatrix} X_1(\omega_1) & X_1(\omega_2) & ... \\ X_2(\omega_1) & X_2(\omega_2) & ... \\ ... & ... & ... \\ ... & ... & ... \end{bmatrix}$$

Here the sample space is

$$\mathbf{R}^\infty = \{(x_1, x_2, ...) \in \mathbf{R} \times \mathbf{R} \times ...\}.$$

Here $\mathbf{R}^\infty$ denotes the space consisting of all infinite sequences $(x_1, x_2, ...)$ of real numbers. In $\mathbf{R}^\infty$ and n-dimensional rectangle is a set of the form (sometimes called cylinder sets)

$$\{x \in \mathbf{R}^\infty ; \, x_1 \in I_1, ..., \, x_n \in I_n\},$$

where I_1, ..., I_n are finite or infinite intervals. Take the Borel field $\mathcal{B}(\mathbf{R}^\infty)$ to be the smallest $\sigma - field$ of subsets of $\mathbf{R}^\infty$ containing all finite-dimensional rectangles.

Now think of the stochastic process X as an $\mathbf{R}^\infty$ valued random variable

$$\omega_i \rightarrow X(\omega_i) = (X_1(\omega_i), X_2(\omega_i), ...) = (x_1^i, x_2^i, ...) \equiv x^i \in \mathbf{R}^\infty.$$

The random variable X induces a probability measure P_X on the $\sigma - field$ $\mathcal{B}(\mathbf{R}^\infty)$. For instance, if

$$A = \{x \in \mathbf{R}^\infty : \sup x_n \geq a\} \in \mathcal{B}(\mathbf{R}^\infty),$$

then the set A consists of all sequences with some of their entries larger than a and $P_X(A) \equiv P[\omega : X(\omega) \in A]$.

If all we observe are the values of a process $X_1(\omega), X_2(\omega), ...$the underlying probability space is certainly not uniquely determined. As an example, suppose that in one room a fair coin is being tosses independently, and calls zero or one are being made for tails or heads respectively. In another room a well-balanced die is being cast independently and zero or one called as the resulting face is odd or even. There is, however, no way of discriminating between these two experiments on the basis of the calls.

Denote $X = (X_1, X_2, ...)$. From an observational point of view, the thing that really interests us is not the space $(\Omega, \mathcal{F}, P)$ but the distribution of the values of X. If two processes, X, on $(\Omega, \mathcal{F}, P)$ and X' on $(\Omega', \mathcal{F}', P')$ have the same probability distribution,

$$P_X(B) \equiv P[X(\omega) \in B] = P'[X'(\omega) \in B], \text{ for all } B \in \mathcal{B}(\mathbf{R}^\infty),$$

then there is no way of distinguishing between the processes by observing them (*more formally see definition 4*).

The distribution of a process contains all the information which is relevant to probality theory. All the theorems in this probabilistic introduction depend on the distribution of the process, and hence hold for all the processes having that distribution. Among all the processes having a given distribution P on $\mathcal{B}(\mathbf{R}^\infty)$, there is one which has some claim to being the simplest.

Definition 1 *1. For any given distribution P define the random variables $X_1, X_2, ...$, on $(\mathbf{R}^\infty,\ \mathcal{B}(\mathbf{R}^\infty),\ P)$ by*

$$X_n(x_1, x_2, ...) = x_n.$$

This process is called the coordinate representation process and has the same distribution as the original process.

This representation will be used when we discuss stationarity, ergodicity, etc.

Now we complicate things a bit more. Let X_t be a continuous time stochastic process. That is, the time index belongs to some interval of the real line, say, $t \in [0, \infty)$. If we are interested in the behavior of X_t during an interval of time $[t_0 \leq t \leq t_1]$ it is necessary to consider simultaneously an uncountable family of $X_t\prime s$ $\{X_t,\ t_0 \leq t \leq t_1\}$. This results in a technical problem because of the uncountability of the index parameter t . Recall that $\sigma - fields$ are, by definition, closed under countable operations only and that statements like $\{X_t \geq x,\ t_0 \leq t \leq t_1\} = \bigcap_{t_0 \leq t \leq t_1} \{X_t \geq x\}$ are not events!!! However, for most practical situations this difficulty is bypassed by replacing uncontable index sets by countable dense subsets without losing any significant information. In general, these arguments are based on the separability of a continuous time stochastic process. This is possible, for example, if the stochastic process X is almost surely continuous (see next definition).

Playing with stochastic processes: Let $X = \{X_t : t \geq 0\}$ and $Y = \{Y_t : t \geq 0\}$ be two stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, P)$. Because of the presence of ω the functions $X_t(\omega)$ and $Y_t(\omega)$can be compared in different ways:

Definition 2 *2. X and Y are called indistinguishable if*

$$P(\{\omega : X_t(\omega) = Y_t(\omega),\ t \geq 0\}) = 1.$$

Definition 3 *3. Y is a modification of X if for every $t \geq 0$, we have*

$$P(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1.$$

Definition 4 *4. X and Y have the same law or probability distribution iff all their finite dimensional probability distributions coincide, that is, iff for any sequence of times $0 \leq t_1 \leq .... \leq t_n$ the joint probability distributions of $(X_{t_1}, ..., X_{t_n})$ and $(Y_{t_1}, ..., Y_{t_n})$ coincide.*

Note that the first property is much stronger than the other two. The null sets in the second and thirs property may depend on t .

Recall that there are different definitions of limit for sequences of random variables. To each definition corresponds a type of continuity of real valued time index process. For instance:

Definition 5 *5. $\{X_t\}$ is continuous in probability if for every t and $\varepsilon > 0$,*

$$lim_{h \to 0} P(|X_{t+h} - X_t| > \varepsilon) = 0.$$

....almost sure, in L^p, etc............

However, none of the above notions is strong enough to differentiate, for instance, between a process for which almost all the sample paths are continuous for every t , and a process for which almost all sample paths have a countable number of discontinuities, when the two processes have the same finite dimensional distributions. A much stronger criterion for continuity is *sample paths continuity* that requires continuity for all ts simultaneously!!!!!!!!!!! In other words for almost all ω the function $X_{(.)t}(\omega)$ is continuous in the usual sense. Unfortunately, the definition of a stochastic process in terms of its finite dimensional distributions does not help here since we are faced with whole intervals containing uncountable numbers of ts. Fortunately for most useful processes in applications, continuous versions (sample paths continuous) or right-continuous versions, can be constructed.

If a stochastic process with index set $[0, \infty)$ is continuous its sample space can be identified with $\mathbf{C}[0, \infty)$, the space of all real valued continuous functions, with a corresponding metric ρ.

Let $\mathcal{B}(\mathbf{C})$ the smallest $\sigma - field$ containing the open sets of the topology induced by ρ on $\mathbf{C}[0, \infty)$, the borel $\sigma - field$. Then same $\sigma - field$ $\mathcal{B}(\mathbf{C})$ is generated by the cylinder sets of $\mathbf{C}[0, \infty)$ which have the form

$$\{x \in \mathbf{C}[0, \infty) : x_{t_1} \in I_1,\, x_{t_2} \in I_2,\, ...,\, x_{t_n} \in I_n\},$$

where each I_i is an interval of the form $(a_i,\, b_i]$. In other words, a cylinder set is a set of functions with restrictions put on a finite number of coordinates (is the set of functions that, at times $t_1, ..., t_n$ get through the windows I_1, I_2, ..., I_n and at other times have arbitrary values.

An example of a Borel set from $\mathcal{B}(\mathbf{C})$ is

$$A = \{x : \sup x_t \geq a,\, t \geq 0\}.$$

Note that the set given by A depends on the behavior of functions on an uncountable set of points and would not be in the $\sigma - field$ $\mathcal{B}(\mathbf{C})$ if $\mathbf{C}[0, \infty)$ were replaced by the much larger space $\mathbf{R}^{[0,\, \infty)}$. In this latter space every Borel set is determined by restrictions imposed on the functions x , on an at most countable set of points $t_1, ..., t_n$.