Karush-Kuhn-Tucker Review

Math Review-II

Universidad Carlos III de Madrid

Monetary and Financial Macroeconomics

Karush-Kuhn-Tucker Method

- KKT is a generalized version of Lagrange's method.
- It is used when a constraint may not hold with equality at the optimum (unlike the previous example).
- We will make a quick review of KKT method by solving an example in order to solve maximization with inequality constraints.

KKT Example

$$\begin{aligned} \max_{x_1, x_2} \quad & \ln x_1 + x_2 \\ \text{s. to} \quad & 2x_1 + x_2 \leq 10 \\ & x_2 \geq 0 \end{aligned}$$

KKT conditions

$$L(x_1, x_2, \lambda, \mu) = \ln x_1 + x_2 + \lambda(10 - 2x_1 - x_2) + \mu(x_2)$$

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial \lambda} \geq 0, \quad \frac{\partial L}{\partial \mu} \geq 0$$

$$\lambda(10 - 2x_1 - x_2) = 0, \quad \mu x_2 = 0$$

$$\lambda \geq 0, \quad \mu \geq 0$$

KKT conditions

Remember we have

$$L(x_1, x_2, \lambda, \mu) = \ln x_1 + x_2 + \lambda(10 - 2x_1 - x_2) + \mu(x_2)$$

Then,

$$\frac{\partial L}{\partial x_1} = 0 \implies \frac{1}{x_1} = 2\lambda$$
$$\frac{\partial L}{\partial x_2} = 0 \implies 1 + \mu = \lambda$$

KKT conditions

Remember we have

$$L(x_1, x_2, \lambda, \mu) = \ln x_1 + x_2 + \lambda(10 - 2x_1 - x_2) + \mu(x_2)$$

Then,

$$\frac{\partial L}{\partial \lambda} \geq 0 \implies 10 - 2x_1 - x_2 \geq 0$$
$$\frac{\partial L}{\partial \mu} \geq 0 \implies x_2 \geq 0$$

KKT conditions

Let's combine all the conditions

$$\frac{1}{x_1} = 2\lambda \tag{1}$$

$$1 + \mu = \lambda \tag{2}$$

$$10 - 2x_1 - x_2 \geq 0 \tag{3}$$

$$x_2 \geq 0 \tag{4}$$

$$\lambda(10 - 2x_1 - x_2) = 0 \tag{5}$$

$$\mu x_2 = 0 \tag{6}$$

$$\lambda \geq 0, \quad \mu_2 \geq 0 \tag{7}$$

Possible Maxima

There is a unique solution we can drive from KKT conditions are

$$(x_1, x_2, \lambda, \mu) = \left(\frac{1}{2}, 9, 1, 0\right)$$

which makes objective

$$\ln(0.5) + 9 \cong 8.31$$

and we can say the unique maximum is given by

$$(x_1^*, x_2^*, \lambda^*, \mu^*) = \left(\frac{1}{2}, 9, 1, 0\right)$$