Solow: It is all about physical capital accumulation

Felix Wellschmied

Universidad Carlos III de Madrid

Growth Theory

Modern economic growth

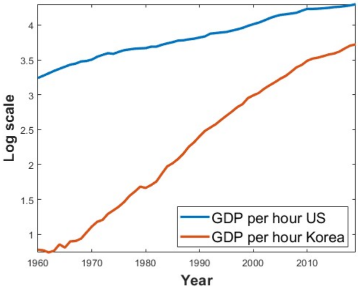
Over the last 150 years:

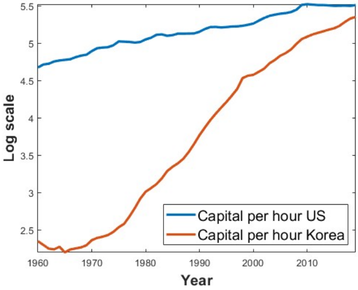
- Economic growth in advanced economies can be described by exponential growth.

- There exist huge cross-country differences in income per person across countries.

- Income differences are not stable. In fact, there are some "growth miracles".
    - Growth miracles are associated with rapid capital accumulation.

Korea, a growth miracle





Modern economic growth

- To understand these phenomena, we need a theory.
- Our theory will be guided by certain data facts.
- These data facts should be "universally" true, i.e., relatively stable over time.

Kaldor facts

Kaldor (1961) summarized six facts about income. We will consider the first five:

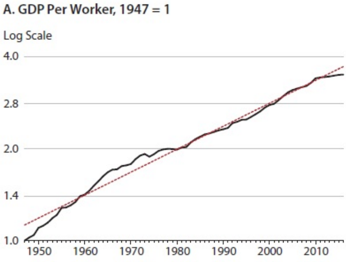
1. Labor productivity grows at a constant rate over time.
2. Capital per worker grows at a constant rate over time.
3. Capital has a constant rate of return over time.
4. The capital to output ratio is constant over time.
5. The share of income going to capital is constant over time.

Kaldor facts II

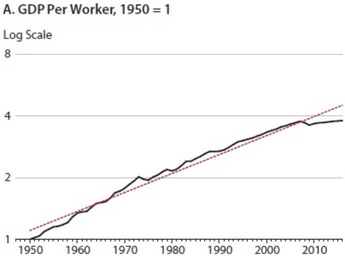
Recently, Herrendorf et al. (2019) consider these facts anew including recent data:

1. Broadly speaking, the Kaldor facts still hold.
2. However, some data moments show some time variation.
3. In this course, we will, nevertheless, use the Kaldor facts as the benchmark.

Constant growth in labor productivity



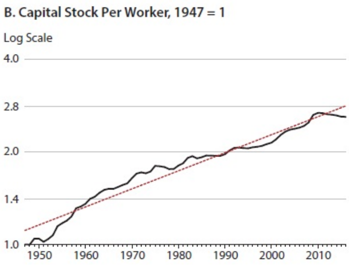
U.S.



UK

- A constant labor productivity growth is a good approximation.
- However, we observe a slow-down after 1970.

Constant growth in capital per worker



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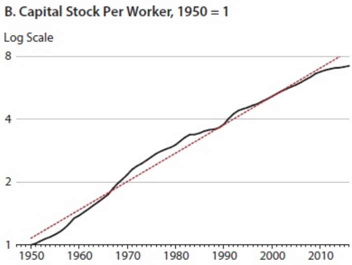

UK

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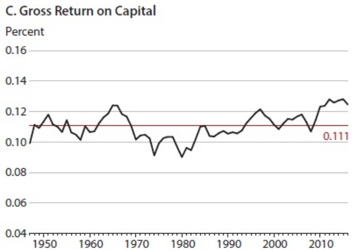
Constant return to capital



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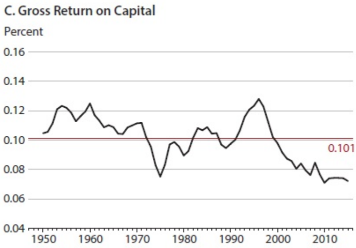
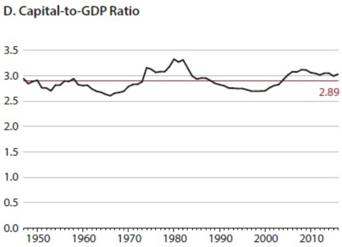


UK

- A constant return to capital is a good approximation.
- Since the 2000s, we observe some time variation that is different across countries.

A constant capital to output ratio



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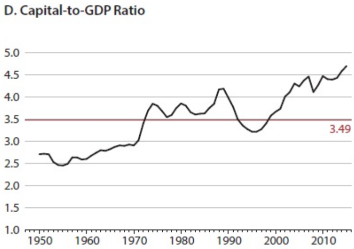
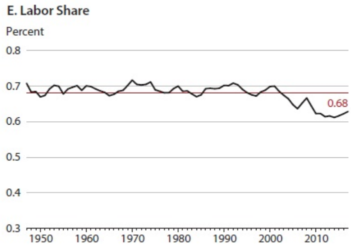


UK

- In the U.S., capital and output grow approximately at the same rate.
- In the UK, capital grows faster than output.

A constant capital share in income



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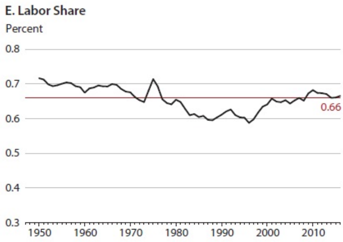


UK

- Instead of the capital share, they consider the easier to measure labor share.
- The labor share started to fall in the U.S. during the 2000s. It was falling in the UK between the 50s and 2000s.

Understanding modern economic growth

A first attempt to understand modern economic growth

- Solow (1956) presents a framework on how to understand the phenomenon of modern economic growth (Kaldor factors). For that work, he won the Nobel price.
- It is a closed economy model where production takes place by labor, capital, and technology.
- Importantly, it takes technological growth as exogenous and puts physical capital accumulation center stage.

How production takes place

The real world

- In a modern economy, production takes place mostly at firms which are often multinational corporations.
- These firms produce thousands of different goods and services relying on thousands on imports and creating thousands of exports.
- For production, they employ
  - labor of different types (education, age, sex...)
  - equipment, structures, roads, land, raw materials...
- A lot of production is also done by the government.
- Most exchange of goods and services as well as factor inputs is conducted in thousands of markets.
- People make decisions about consumption today versus the future.

Abstractions

The world is quite complicated (more so than medieval England) and we will have to make simplifications to make progress in understanding it:

- We assume a closed economy.
- We abstract from the government and treat it just as the private sector.
- There is only one output good.
- Production takes place at the level of firms that rent the factors of production from households.

Abstractions: factors of production

To focus on the right factor inputs, we look at data from national accounts:

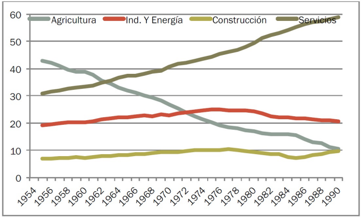
- As we have seen before, labor compensation is around 2/3 of national income, i.e., it is important. To deal with it, we will assume that we can aggregate all different labor inputs into just one input.
- Similarly, we assume that we can aggregate all physical capital inputs into just one.
- This implies we will treat land as physical capital. One may object that land is finite. However, fertilizers and tall buildings suggest otherwise.
- Even for the U.S., a major oil producer, income from natural resources is relatively small. Hence, we will ignore them. One can also interpret them as physical capital recognizing that, so far, their supply did not run out.

Abstractions: technology

We also assume a single measure on how well we use capital and labor to produce output, i.e., technology.

- This may relate to firm organization, e.g., management style.
- This may relate to logistics, e.g., just-in-time delivery.
- This may relate to new products that are better than the old product but not more expensive to produce, e.g., a faster computer algorithm.
- In fact, many of the products we consume today did not exist 50 years ago.
- Hence, we will think of improvements in A as new *ideas*, or better recipes.

More products or new products?



The type of products we produce today looks very different from the type of products we produced in 1955.

Abstractions: markets

To understand markets, we again look at national accounts:

- The profit share of national income is relatively small, around 5%.
- This suggests that product markets and input markets are close to perfectly competitive.

This implies that the factors of production earn their marginal products.

Abstractions: The aggregate production function

Our assumptions imply that firms operate an aggregate production function that combines a single labor input, L, a single capital input, K, and some technology level, A, into a single output good: $Y = F(K, L, A)$. Our model should be consistent with the Kaldor facts. We now use the fact of constant income shares to figure out how F should look like:

$$\frac{r(t)K(t)}{Y(t)} = \alpha, \tag{1}$$

$$\frac{w(t)L(t)}{Y(t)} = 1 - \alpha. \tag{2}$$

Given the assumption of competitive markets, we have:

$$\frac{\frac{\partial Y(t)}{\partial K(t)} K(t)}{Y(t)} = \alpha, \tag{3}$$

$$\frac{\frac{\partial Y(t)}{\partial L(t)} L(t)}{Y(t)} = 1 - \alpha. \tag{4}$$

Abstractions: The aggregate production function II

This holds for, among others, the Cobb-Douglas production function:

$$Y(t) = K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} \tag{5}$$

Note, the place of $A(t)$ in the production function is not particularly important:

$$Y(t) = K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} = A(t)^{1-\alpha} K(t)^{\alpha} L(t)^{1-\alpha} = E(t) K(t)^{\alpha} L(t)^{1-\alpha}, \tag{6}$$

with $E(t) = A(t)^{1-\alpha}$. The way I have written it above will make the math easier.

Abstractions: Household decisions

- One of the most important questions in modern macroeconomics is how households trade-off consumption today against tomorrow.

- The Solow model abstracts from this and assumes that households save a constant fraction of their income each period.

Solving the model

The distribution of income

Even without thinking of the dynamics of the model, we can already derive the distribution of income. From perfectly competitive markets, we have:

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = (1-\alpha)K(t)^{\alpha}A(t)^{1-\alpha}L(t)^{-\alpha} \tag{7}$$

$$r(t) = \frac{\partial Y(t)}{\partial K(t)} = \alpha K(t)^{\alpha-1}\left(A(t)L(t)\right)^{1-\alpha} \tag{8}$$

Hence, as households own the factors of production, total household income is

$$r(t)K(t) + w(t)L(t) = Y(t). \tag{9}$$

This will be convenient, as we do not need to distinguish between production and household income. For example, aggregate savings are simply $S(t) = sY(t)$.

Capital accumulation

The Solow model assumes that every period a fraction δ of the capital stock depreciates. Working against this, households invest $I(t) = S(t)$:

$$\dot{K}(t) = S(t) - \delta K(t) \tag{10}$$

$$\dot{K}(t) = sY(t) - \delta K(t) \tag{11}$$

$$\dot{K}(t) = sK(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} - \delta K(t). \tag{12}$$

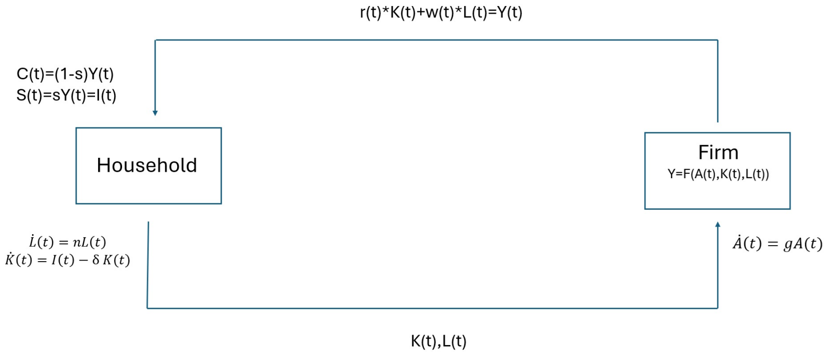
Dynamics of population and technology

The Solow model assumes that the population and technology grow at exogenous rates. To be consistent with the Kaldor facts on labor productivity, it assumes they grow exponentially:

$$L(t) = L(0)\exp(nt) \Rightarrow \frac{\dot{L}(t)}{L(t)} = n \tag{13}$$

$$A(t) = A(0)\exp(gt) \Rightarrow \frac{\dot{A}(t)}{A(t)} = g. \tag{14}$$

Summary of the economy



Steady state

As before, we will start our analysis with the steady state of the model. For this, we need to find a variable that has a steady state. It turns out, in the Solow model, these are output and capital per efficient worker:

$$\tilde{k}(t) = \frac{K(t)}{A(t)L(t)} \tag{15}$$

$$\tilde{y}(t) = \frac{Y(t)}{A(t)L(t)} = \tilde{k}(t)^{\alpha}. \tag{16}$$

Rewriting the capital accumulation equation

$$\dot{K}(t) = sK(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} - \delta K(t) \tag{17}$$

$$\frac{\dot{K}(t)}{K(t)} = s\tilde{k}(t)^{\alpha-1} - \delta \tag{18}$$

We need to rewrite the left-hand-side in terms of efficient workers. For this, we need to find an expression for $\frac{\dot{K}(t)}{K(t)}$.

Rewriting the capital accumulation equation II

Given the definition:

$$\tilde{k}(t) = \frac{K(t)}{A(t)L(t)} \tag{19}$$

$$\ln \tilde{k}(t) = \ln K(t) - \ln A(t) - \ln L(t). \tag{20}$$

Now take the derivative with respect to time and use the fact that the derivative of a variable in logs with respect to time is the growth rate of that variable:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \tag{21}$$

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - g - n \tag{22}$$

Rewriting the capital accumulation equation III

Combining the equations yields:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} + n + g = s\tilde{k}(t)^{\alpha-1} - \delta \tag{23}$$

$$\dot{\tilde{k}}(t) = s\tilde{k}(t)^{\alpha} - (n + g + \delta)\tilde{k}(t). \tag{24}$$

Capital per efficient worker grows over time because of savings per efficient worker, $s\tilde{k}(t)^{\alpha}$. It shrinks because of population growth, technological progress, and capital depreciation.

Solving for the steady state

Conjecture that in steady state, $\dot{\tilde{k}}(t) = 0$:

$$0 = s(\tilde{k}^*)^\alpha - (n + g + \delta)\tilde{k}^* \tag{25}$$

$$\tilde{k}^* = \left(\frac{s}{n + g + \delta}\right)^{\frac{1}{1-\alpha}}. \tag{26}$$

Note, we have found indeed a steady state. Our variable, $\tilde{k}^*$, depends only on time-invariant parameters. The steady state capital per efficient worker increases in the savings rate and decreases in the population growth rate, technological progress, and capital depreciation rate.

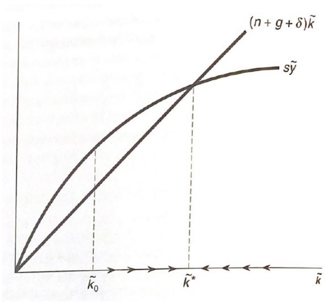
Solving for the steady state II

Once we know $\tilde{k}^*$, it is straight forward to compute the other endogenous variables in steady state:

$$\tilde{y}^* = \frac{K(t)^\alpha \left(A(t)L(t)\right)^{1-\alpha}}{A(t)L(t)} = (\tilde{k}^*)^\alpha \tag{27}$$

$$\tilde{c}^* = (1-s)\tilde{y}^* = (1-s)(\tilde{k}^*)^\alpha. \tag{28}$$

The steady state graphically



- Note, there exist one steady state with $\tilde{k}^* > 0$.
- Key for this is that $s(\tilde{k}^*)^\alpha = s\tilde{y}^*$ is concave. For this, we require diminishing marginal returns to capital.

Output per capita in steady state

$$\tilde{y}^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{29}$$

$$\left(\frac{Y(t)}{L(t)}\right)^* = A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{30}$$

Different from the Malthus model, the Solow model can explain long-run differences in income per capita:

- A higher technology level increases output per capita.
- A higher savings rate increases output per capita.
- A higher population growth rate or capital depreciation rate decreases output per capita.

Factor payments in steady state

The rental price of capital is given by

$$r(t) = \frac{\partial Y(t)}{\partial K(t)} = \alpha K(t)^{\alpha-1} \left(A(t)L(t)\right)^{1-\alpha} = \alpha \tilde{k}(t)^{\alpha-1}, \tag{31}$$

which is a constant in the long run. Hence, the Solow model is consistent with the Kaldor fact on constant returns to capital.

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = (1-\alpha)K(t)^{\alpha} A(t) \left(A(t)L(t)\right)^{-\alpha} = (1-\alpha)A(t)\tilde{k}(t)^{\alpha}, \tag{32}$$

which is growing with technology. Kaldor did not study wages but this fact is also born out by the data.

Growth in steady state

In steady state, by definition, capital (and output) per efficient worker, $\tilde{k}(t)$, is constant. This does not mean, however, that capital or capital per capita, $k(t) = \frac{K(t)}{L(t)}$, are constant. In fact, we already know that:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \tag{33}$$

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^* = n + g \tag{34}$$

$$\left(\frac{\dot{k}(t)}{k(t)}\right)^* = g. \tag{35}$$

That is, in steady state, capital per capita grows at the rate of technological progress. A constant growth rate of capital per capita is one of the Kaldor facts.

Growth in steady state II

Similarly for output,

$$\frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)} = \frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \tag{36}$$

$$\left(\frac{\dot{Y}(t)}{Y(t)}\right)^{*} = n + g \tag{37}$$

$$\left(\frac{\dot{y}(t)}{y(t)}\right)^{*} = g. \tag{38}$$

Hence, output per capita in steady state also growth at the rate of technological progress. Moreover, as Y and K both grow at rate $n + g$, there ratio is constant which completes the Kaldor facts.

Growth in steady state III

Finally, for consumption,

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{\dot{C}(t)}{C(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \tag{39}$$

$$\left(\frac{\dot{C}(t)}{C(t)}\right)^* = n + g \tag{40}$$

$$\left(\frac{\dot{c}(t)}{c(t)}\right)^* = g. \tag{41}$$

Hence, consumption per capita in steady state also growth at the rate of technological progress. A steady state in which all endogenous variables grow at the same rate is referred to as a *balanced growth path*.

A detour: an alternative to solve for the steady state dynamics

The fact that the capital to output ratio is constant in steady state provides an alternative to solve for the steady state dynamics that does not require us to define $\tilde{k}(t)$. In some models, this will prove useful. Start with the production function:

$$Y(t) = K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} \tag{42}$$

$$\frac{Y(t)}{Y(t)^{\alpha}} = \left(\frac{K(t)}{Y(t)}\right)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} \tag{43}$$

$$Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)L(t). \tag{44}$$

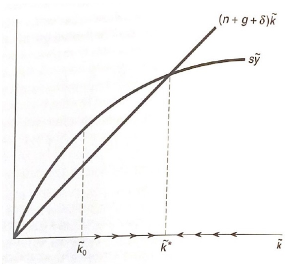
As the first term is constant in steady state, we directly see that output growth must be $n + g$.

What happens outside the steady state?

Hence, we have a model that is consistent with the Kaldor facts once the economy is in steady state. However, we would also like to understand how the economy behaves outside steady state:

- Nothing guarantees that a particular economy is in steady state in a particular year.
- It allows us to study how the economy moves from one steady state to another if parameters change.
- In fact, we will see that outside steady state dynamics allow us to understand growth miracles.

Do we convergence to steady state?



Recall the capital per efficient worker accumulation equation:

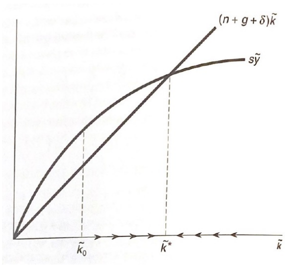
$$\dot{\tilde{k}}(t) = s\tilde{k}(t)^{\alpha} - (n+g+\delta)\tilde{k}(t). \tag{45}$$

$$s\tilde{k}(t)^{\alpha} > (n+g+\delta)\tilde{k}(t) \text{ if } \tilde{k}(t) < \tilde{k}^* \tag{46}$$

$$s\tilde{k}(t)^{\alpha} < (n+g+\delta)\tilde{k}(t) \text{ if } \tilde{k}(t) > \tilde{k}^*. \tag{47}$$

Hence, we converge to steady state from any starting point $\tilde{k}(0) > 0$.

Intuition for convergence



$$s\tilde{k}(t)^{\alpha-1} = \frac{s}{\alpha}MPK(t) > (n+g+\delta) \text{ if } \tilde{k}(t) < \tilde{k}^* \tag{48}$$

$$s\tilde{k}(t)^{\alpha-1} = \frac{s}{\alpha}MPK(t) < (n+g+\delta) \text{ if } \tilde{k}(t) > \tilde{k}^*. \tag{49}$$

With too little capital, the marginal product of capital is high and savings exceed effective depreciation. The reverse is true when capital is too high.

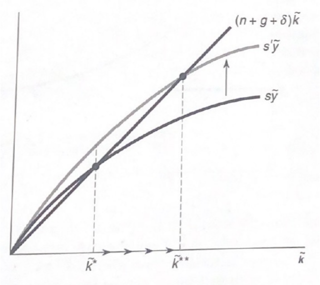
Convergence to steady state

We will see that we can solve for the convergence path explicitly in terms of the capital to output ratio:

$$\underbrace{\tilde{k}(t)^{1-\alpha}}_{\frac{K(t)}{Y(t)}=\frac{\alpha}{MPK(t)}} = \frac{s}{n+g+\delta} - \left[\underbrace{\frac{s}{n+g+\delta}}_{\left(\frac{\tilde{k}}{\tilde{y}}\right)^{*}=\left(\frac{K}{Y}\right)^{*}} - \tilde{k}(0)^{1-\alpha}\right]\exp(-\beta t). \tag{50}$$

- $\tilde{k}(t)^{1-\alpha} - \frac{s}{n+g+\delta}$ converges to zero at rate $\beta = (1-\alpha)(n+g+\delta)$.
- In words: The absolute gap between the capital to output ratio and its steady state vanishes at rate β.
- Hence, the (absolute) growth rate is higher the further the economy is away from steady state (the more different is $MPK(t)$).

Comparative statics: An increase in the savings rate



One way to obtain a higher steady state is a higher savings rate. For any level of $\tilde{k}(t)$, $s\tilde{y}(t) = s\tilde{k}(t)^\alpha$ increases. The new steady state is associated with a higher $\tilde{k}^*$ and, hence, a higher $\tilde{y}^*$. It directly follows that output per capita is also higher in the new steady state.

Comparative statics: An increase in the savings rate II

Note, in the old and new steady state, $\tilde{y}^*$ are constant and, hence, output per capita grows in each case at rate g. That is, the savings rate changes the level of output per capita in steady state but not its growth rate. However, on the transition path to the new steady state, $\tilde{y}$ is not constant. To see this, note

$$\frac{\dot{y}(t)}{y(t)} = g + \frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)} = g + \alpha \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)}. \tag{51}$$

Now, consider again the capital accumulation equation

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = s\tilde{k}(t)^{\alpha-1} - (n + g + \delta). \tag{52}$$

During the transition, $s\tilde{k}(t)^{\alpha-1} > n + g + \delta$ and, hence, capital per efficient worker grows.

The transition path: An increase in the savings rate

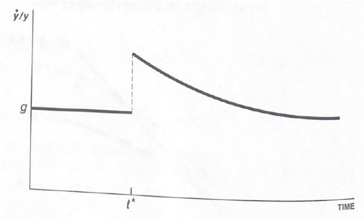
Given our solution for $\tilde{k}(t)^{1-\alpha}$, we can compute the entire transition path:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = s\tilde{k}(t)^{\alpha-1} - (n+g+\delta) \tag{53}$$

$$= \frac{s}{\frac{s}{n+g+\delta} - \left[\frac{s}{n+g+\delta} - \tilde{k}(0)^{1-\alpha}\right]\exp(-\beta t)} - (n+g+\delta). \tag{54}$$

As time passes, the distance $\frac{s}{n+g+\delta} - \tilde{k}(t)^{1-\alpha}$ becomes smaller. That is, the denominator becomes larger and $\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)}$ slows down.

Output per worker growth outside steady state

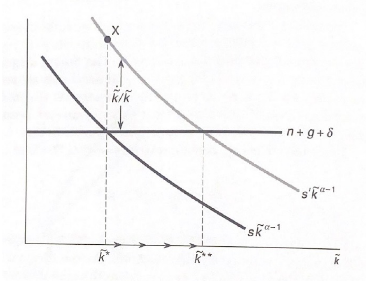


Consider an economy changing to a higher steady state. It accumulates capital rapidly initially, and output per capita is growing rapidly initially:

$$\frac{\dot{y}(t)}{y(t)} = g + \frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)} = g + \alpha\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)}. \tag{55}$$

As capital accumulation slows down, so does output per capita growth.

The transition path: A graphical representation



Note, $\tilde{k}(t)^{\alpha-1}$ is a downward-sloping convex function. Hence, the distance between $s\tilde{k}(t)^{\alpha-1}$ and $(n+g+\delta)$ is largest in the first period of the adjustment. This is a result from the diminishing marginal returns to capital.

Output per worker growth and the capital to output ratio

We can use again our rewriting of the production function to highlight again the key role that the capital to output ratio plays in the Solow model:

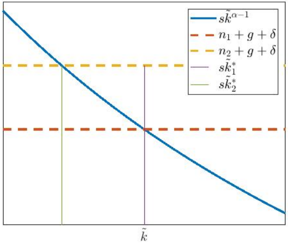
$$Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)L(t) \tag{56}$$

$$\frac{Y(t)}{L(t)} = y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t) \tag{57}$$

$$\frac{\dot{y}(t)}{y(t)} = g + \frac{\alpha}{1-\alpha} g_{\frac{K(t)}{Y(t)}}. \tag{58}$$

That is, growth of output per worker in excess of g must result from a growing capital to output ratio.

Comparative statics: An increase in the population growth rate



The new steady state is associated with a lower $\tilde{k}^*$ and, hence, a lower $\tilde{y}^*$. It directly follows that output per capita is also lower in the new steady state.

Comparative statics: An increase in the population growth rate II

Consider again the capital accumulation equation

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = s\tilde{k}(t)^{\alpha-1} - (n + g + \delta). \quad (59)$$

During the transition, $s\tilde{k}(t)^{\alpha-1} < n + g + \delta$ and, hence, capital per efficient worker shrinks. As a consequence, output per capita is growing at a rate less than g during the transition phase.

Introducing human capital

Why human capital

So far, we assume all workers are equally productive across time and countries. However,

- the number of average years of schooling varies substantially within a country over time and across countries.
- the quality of schooling varies substantially within a country over time and across countries.
- within a country at a point in time, income differences across education groups are large suggesting that education matters for productivity.

Production

To introduce human capital, we make a small change to the production function:

$$Y(t) = K(t)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} \tag{60}$$

$$H(t) = \exp(\psi\, u)L(t), \tag{61}$$

where $L(t)$ is the amount of labor, and $H(t)$ is the amount of total human capital. Total human capital not only depends on the amount of labor but also in the time invested in learning, u.

Quality of learning

Note that

$$\frac{\partial H(t)}{\partial u} = \psi \exp(\psi\, u) L(t) = \psi H(t) \tag{62}$$

$$\frac{\partial \ln H(t)}{\partial u} = \psi. \tag{63}$$

That is, a change in u translates into ψ percent more human capital. One way to interpret ψ is to think about the quality of the education system for a fixed time spend in it.

Steady state

We need to define again a variable that has a steady state. We will define again:

$$\tilde{k}(t) = \frac{K(t)}{A(t)L(t)} \tag{64}$$

$$\tilde{y}(t) = \frac{Y(t)}{A(t)L(t)} = \tilde{k}(t)^{\alpha} \left(\exp(\psi u)\right)^{1-\alpha} \tag{65}$$

Rewriting the capital accumulation equation

$$\dot{K}(t) = sK(t)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} - \delta K(t) \tag{66}$$

$$\frac{\dot{K}(t)}{K(t)} = s\tilde{k}(t)^{\alpha-1} \left(\exp(\psi u)\right)^{1-\alpha} - \delta \tag{67}$$

We need to find again an expression for $\frac{\dot{K}(t)}{K(t)}$.

Rewriting the capital accumulation equation II

Note that

$$\tilde{k} = \frac{K(t)}{A(t)L(t)} \tag{68}$$

$$\ln \tilde{k} = \ln K(t) - \ln A(t) - \ln L(t) \tag{69}$$

Now take the derivative with respect to time:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \tag{70}$$

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - g - n \tag{71}$$

Rewriting the capital accumulation equation III

Combining the equations yields:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} + n + g = s\tilde{k}(t)^{\alpha-1} \left(\exp(\psi u)\right)^{1-\alpha} - \delta \tag{72}$$

$$\dot{\tilde{k}}(t) = s\tilde{k}(t)^{\alpha} \left(\exp(\psi u)\right)^{1-\alpha} - (n+g+\delta)\tilde{k}(t). \tag{73}$$

Note, this is almost the same dynamic system as in the model without human capital. The only difference is the additional education term.

Solving for the steady state:

$$\tilde{k}^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} \exp(\psi u) \tag{74}$$

$$\tilde{y}^* = \tilde{k}(t)^{\alpha} \left(\exp(\psi u)\right)^{1-\alpha} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \exp(\psi u) \tag{75}$$

$$\tilde{c}^* = (1-s)\tilde{y}^* = (1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \exp(\psi u). \tag{76}$$

Output per capita in steady state

Hence, output per worker in steady state is:

$$\left(\frac{Y(t)}{L(t)}\right)^* = \left(\underbrace{\frac{s}{n+g+\delta}}_{\left(\frac{K}{Y}\right)^*}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi\, u) \tag{77}$$

Output per capita is increasing in the amount of education. A more educated workforce is more productive and, thereby, allows each worker to produce more.

Growth in steady state

We can ask again about the growth rate in steady state. As education is assumed to be constant, nothing really changes:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \tag{78}$$

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^* = n + g \tag{79}$$

$$\left(\frac{\dot{k}(t)}{k(t)}\right)^* = g. \tag{80}$$

That is, capital per capita (and output/consumption per capita) grows at the rate of technological progress.

Transition dynamics

We have already seen that changes in population growth rates and saving rates can have rich transition dynamics for output per worker. We can now also analyze changes in education. In general, output per worker is:

$$y(t) = \frac{Y(t)}{L(t)} = \tilde{k}(t)^{\alpha} A(t) \left(\exp(\psi\, u)\right)^{1-\alpha} \tag{81}$$

Increasing the time spend in education, u, or the quality of education, ψ, has a an initial level impact on output per worker of $(\exp(\psi\, u))^{1-\alpha}$.

Transition dynamics II

The growth rate of output per worker outside the steady state is:

$$\frac{\dot{y}(t)}{y(t)} = \alpha \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} + g. \tag{82}$$

From the growth of capital per efficient worker, we know that increasing the marginal product of capital leads to additional capital accumulation:

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = s\tilde{k}(t)^{\alpha-1} \left(\exp(\psi u)\right)^{1-\alpha} - (n+g+\delta) > 0. \tag{83}$$

By the same logic as before, capital (output) per worker grows particularly fast initially. Hence, increasing education leads to a period of rapid capital accumulation and output growth.

How much should we save

So far, we take the savings rate as given. One may ask whether there is an optimal savings rate. One possibility to define optimal is the savings rate that maximizes long-run consumption per worker:

$$\left(\frac{C(t)}{L(t)}\right)^* = (1-s)\left(\frac{Y(t)}{L(t)}\right)^* = (1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi\, u) \tag{84}$$

As productivity and education follow exogenous processes, this is equivalent to maximize consumption per efficient worker:

$$\tilde{c}^* = (1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \exp(\psi u). \tag{85}$$

The resulting savings rate is referred to as the golden rule s_{Gold}.

The golden rule

Taking the first order condition of (84) yields

$$s_{Gold} = \alpha. \tag{86}$$

Intuition: The more important is capital in the production function the more we should save.

The economics behind the golden rule

In steady state, an alternative to write the problem is:

$$\tilde{c}^* = \tilde{y}^* - (n + g + \delta)\tilde{k}^*. \tag{87}$$

Now take the derivative with respect to the steady state capital stock per efficient worker:

$$\alpha(\tilde{k}^*)^{\alpha-1} (\exp(\psi u))^{1-\alpha} = n + g + \delta \tag{88}$$

$$MPK = n + g + \delta. \tag{89}$$

The marginal gain of savings need to equal its marginal cost (the effective depreciation rate).

Are we saving enough?

To assess whether Spain has a savings rate consistent with the golden rule, consider the following data facts

1. The capital output ratio is 2.75: $k = 2.75y$.
2. Capital depreciation is 10 percent of yearly output: $\delta k = 0.1y$.
3. The capital share of income is 30%
4. Output growth is 3%.

Combining 1 and 2 tells us that the depreciation rate is 3.6%.

According to our model, 3 implies $MPK^*k = 0.3y$. Combining with 1 we have $MPK = 0.11$.

According to our model, 4 implies $n + g = 0.03$.

Hence, $MPK > n + g + \delta$, i.e., we save to little.

Can we rationalize a high MPK

We can draw two possible conclusions from a high MPK:

1. Either we need to change savings incentives. Reforming the pension system is one aspect economists have advocated.
2. Or optimizing long-run consumption per worker is not optimal. If we discount future consumption relative to today's consumption, the golden rule is not optimal. A yearly discount rate of over 4% would be needed to explain the high capital returns.

Back to our three big questions

1. Why are we so rich and they so poor?
    - Different saving rates, population growth rates, education levels, and technology levels.
2. Why are there growth miracles?
    - Rapid accumulation of physical capital or increases in human capital.
3. What are the engines of long-run economic growth?
    - Technological progress.

References

Herrendorf, B., R. Rogerson, and A. Valentinyi (2019): "Growth and the kaldor facts," *Federal Reserve Bank of St. Louis Review*.

Kaldor, N. (1961): "Capital accumulation and economic growth," in *The theory of capital*, Springer, 177–222.

Solow, R. M. (1956): "A contribution to the theory of economic growth," *The quarterly journal of economics*, 70, 65–94.