Introduction and mathematical primer

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Growth Theory

Practical Information

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Grading

Final exam 50%.

2 midterms 40-50%.

Participation during reducidos 0-10%.

- Random calls on students during reducidos about past magistral material.
- The probability to be called decreases in number of past calls.
- You obtain a 0 if you are not there when called.
- If (and only if) you are never called, the weight goes to the midterms.
- Presenting exercises or finishing computer exercises fast also counts as a participation.

Basics

To study the theory of economic growth, we need some mathematical background.

We will not have time to cover all the basics. For those missing some of those, I recommend reviewing the following topics from Sydsæter and Hammond (2021) and Larson and Edwards (2016) which you can find in our library:

- Working with exponential functions (Chapter 4.9 (SH), 5.4 (LE))
- Working with logarithmic functions (Chapter 4.10 (SH), 5.1 (LE))
- Inverse functions (Chapter 5.3 (SH)+(LE))
- Differentiation (Chapter 6 (SH), 2 (LE))
- Optimization (Chapter 9 (SH), 3 (LE))
- Integration (Chapter 10.1-10.3 (SH), 4 (LE))

Introduction to continuous time

As we are interested in variables over time, we will often write explicitly the time dependence. For example, we write output, Y, in period t as $Y(t)$.

This raises the issue on how to deal with changes over time and what length a period takes. Mathematically, it will be convenient to think about a time period which length approaches zero. Therefore, the change of a variable over time (denoted by a dot) is simply the derivative with respect to time:

$$\dot{Y}(t) = \frac{dY(t)}{dt} = \lim_{\Delta t \to 0} \frac{Y(t) - Y(t - \Delta t)}{\Delta t}. \tag{1}$$

Exponential growth

We will see that in continuous time, a constant percentage growth rate implies exponential growth of the variable. Starting with the percentage growth rate, we have

$$\frac{Y(t) - Y(t - \Delta t)}{Y(t)} = \frac{\dot{Y}(t)}{Y(t)} \tag{2}$$

We have that $Y(t)$ has a constant growth rate when

$$\frac{\dot{Y}(t)}{Y(t)} = g_Y. \tag{3}$$

Exponential growth

This equation is a first-order differential equation:

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\frac{dY(t)}{dt}}{Y(t)} = g_Y, \tag{4}$$

which we can solve:

$$\frac{1}{Y(t)} \frac{dY(t)}{dt} = g_Y \tag{5}$$

$$\int \frac{1}{Y(t)} dY(t) = \int g_Y dt \tag{6}$$

$$\ln Y(t) = tg_Y + k; \qquad \ln Y(0) = k \tag{7}$$

$$Y(t) = Y(0) \exp(tg_Y). \tag{8}$$

Hence, $Y(t)$ having a constant growth rate implies that $Y(t)$ grows exponentially at rate g_Y.

Linear growth

Continuous time makes also the comparison between exponential and linear growth very transparent. Linear growth means

$$Y(t) = Y(0) + tg_L \tag{9}$$

$$\dot{Y}(t) = g_L \tag{10}$$

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{g_L}{Y(t)} \tag{11}$$

That is, an economy experiencing linear growth has a growth rate that converges to zero as time progresses.

Geometric growth

Continuous time makes also the comparison between exponential and geometric growth very transparent. Geometric growth means

$$Y(t) = Y(0)(1+g_g)^t \tag{12}$$

$$\dot{Y}(t) = Y(0)(1+g_g)^t \ln(1+g_g) \tag{13}$$

$$\frac{\dot{Y}(t)}{Y(t)} = \ln(1+g_g) < g_g \tag{14}$$

That is, an economy experiencing geometric growth is growing at a constant rate but the rate is less than with exponential growth.

Computing growth rates in continuous time

For our analysis, the following property of continuous time will often prove useful. Assume we have

$$y(t) = \ln x(t). \tag{15}$$

Then:

$$\frac{dy(t)}{dt} = \frac{dy(t)}{dx(t)}\frac{dx(t)}{dt} = \frac{1}{x(t)}\dot{x}(t) = \frac{\dot{x}(t)}{x(t)} = g_x, \tag{16}$$

as $\frac{dy(t)}{dx(t)} = 1/x(t)$ and $\frac{dx(t)}{dt} = \dot{x}(t)$. That is, the derivative of a variable in logs with respect to time is the growth rate of that variable, i.e.,

$$\frac{d \ln Y(t)}{dt} = \frac{\dot{Y}(t)}{Y(t)}. \tag{17}$$

Examples

A simple example with exponential growth:

$$Y(t) = Y(0)\exp(tg_Y) \tag{18}$$
$$\ln Y(t) = \ln Y(0) + tg_Y \tag{19}$$
$$\frac{d\ln Y(t)}{dt} = \frac{\dot{Y}(t)}{Y(t)} = g_Y. \tag{20}$$

A simple example with linear growth:

$$Y(t) = Y(0) + tg_L \tag{21}$$
$$\ln Y(t) = \ln(Y(0) + tg_L) \tag{22}$$
$$\frac{d\ln Y(t)}{dt} = \frac{\dot{Y}(t)}{Y(t)} = \frac{g_L}{Y_0 + tg_L}. \tag{23}$$

Examples II

A simple example with geometric growth:

$$Y(t) = Y(0)(1+g_g)^t \tag{24}$$
$$\ln Y(t) = \ln Y(0) + t \ln(1+g_g) \tag{25}$$
$$\frac{d \ln Y(t)}{dt} = \frac{\dot{Y}(t)}{Y(t)} = \ln(1+g_g). \tag{26}$$

References

LARSON, R. AND B. EDWARDS (2016): *Calculus*, Cengage Learning, 10ª ed. ed.

SYDSÆTER, K. AND P. J. HAMMOND (2021): *Essential mathematics for economic analysis*, Pearson Education.