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Uniform Continuity of the Value of Zero-Sum Games with Differential Information

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We establish uniform continuity of the value for zero-sum games with differential information, when the distance between changing information fields of each player is measured by the Boylan pseudometric. We also show that the optimal strategy correspondence is upper semicontinuous when the information fields of players change (even with the weak topology on players' strategy sets), and is approximately lower semicontinuous.

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1. Introduction. Bayesian games, or games with differential information, describe situations in which there is uncertainty about players' payoffs, and different players have (typically) different private information about the realized state of nature ω that affects the payoffs. Private information of player i is often represented by a partition of the space Ω of all states of nature (in which case i knows to which element of the partition the realized ω belongs), or more generally, by a σ-field $\mathcal{F}^i$ of measurable sets (events) in Ω (in which case i knows, given any event in $\mathcal{F}^i$, whether it has occurred). It was shown by Simon [13] that Bayesian Nash equilibrium (BNE) may fail to exist in games with differential information, as a result of discontinuity of the expected payoff function in Bayesian strategies of all players *simultaneously*. The situation changes, however, when attention is confined to two-person *zero-sum games* with differential information. Indeed, under quite general conditions, the expected payoff function is (weakly) continuous in Bayesian strategies of each player *separately*, and the Sion [14] minimax theorem needs only this form of continuity to guarantee existence of the value and of optimal strategies for each player.

This work concerns the behavior of the value of a zero-sum game with differential information when players' information endowments (fields) undergo small changes, and the distance between informations fields is measured by means of the Boylan [3] pseudometric. It turns out that the value has strong continuity properties. We show that when the payoff function is Lipschitz continuous in strategies at each state of nature,[1] a mild integrability assumption[2] on the state-dependent Lipschitz constant guarantees that the value is a *uniformly* continuous function of players' information fields (see Theorem 1). If, in addition, the state-dependent Lipschitz constant of the payoff function is bounded, then the value is in fact Lipschitz continuous in information fields (see Corollary 1). Moreover, the correspondence describing players' optimal strategies as a function of information is upper semicontinuous, even with respect to the weak convergence topology on each player's set of strategies, and is approximately lower semicontinuous (see Theorem 3).

These continuity properties of the value (and optimal strategies) in zero-sum games stand in contrast to discontinuity of the BNE correspondence in general (nonzero-sum) games with differential information. The

[1] This requirement is satisfied, for example, by games which have a matrix-game form in all states of nature (see the example in §2).

[2] The assumption is that the state-dependent Lipschitz constant is *q-integrable* for $q > 1$. When this constant is merely integrable, the value is still continuous (see Theorem 2), but possibly not uniformly.

BNE correspondence is not lower semicontinuous, that is, BNE strategies/payoffs may not be approachable by BNE (or even ε-BNE) strategies/payoffs in games with slightly modified information endowments. This was shown by Monderer and Samet [10][3] in a setting similar to ours. The BNE are also not upper semicontinuous as was shown by Milgrom and Weber [9] and Cotter [4].

The continuity of the BNE correspondence has been investigated in two different setups. In this paper, we use the basic setup of Monderer and Samet [10], who work with information fields to describe players' varying private information, with fixed common prior belief about the distribution of the states of nature. (This follows a certain tradition of modelling information in economic theory; see, e.g., Allen [1], Cotter [4], Stinchcombe [16], and Van Zandt [17, 18].)) In other words, the underlying uncertainty in the game (represented by the common prior) is fixed, but information endowments of players (represented by information fields) are variable. However, there is a different approach to continuity of NE correspondences, which is with respect to the common prior belief (see, e.g., Milgrom and Weber [9] and Kajii and Morris [7]). In this approach, contrary to ours, the underlying uncertainty (the common prior) is variable, but information endowments are fixed (the space of states of nature is assumed to be the cross product of *fixed* sets of players' types, and each player's private information is given by the knowledge of his type). Perturbing the underlying uncertainty influences the expected payoffs of *all* players, but does not affect their strategy sets. However, our setting emphasizes *differences* in information, allowing information endowments in the game to be perturbed in a way that directly affects only one individual player, or in a way that affects all players differently. Indeed, a change in the private information of both players induces (typically different) changes in players' strategy sets, due to the constraint of measurability of each player's strategies with respect to his information field. Although the impact of these information changes on the structure of the game might appear to be significant, our theorems show that the value and the optimal strategies in zero-sum games are nevertheless well behaved with respect to these changes.

In nonzero-sum games, upper semicontinuity of BNE is obtained at the cost of imposing certain restrictions on information structure in the game. Indeed, in the setup of types, a sufficient *spread* of the common prior distribution on the product of players' types is needed for upper semicontinuity of BNE (see Milgrom and Weber [9]; the common prior is required to be absolutely continuous with respect to the product of its marginal distributions). In the setup of information fields, an analogous condition in Cotter [5] also yields upper semicontinuity with respect to the Boylan topology on information endowments, but only under the assumption that all fields are generated by at most countable partitions of the space of states of nature. Our results show, however, that for the continuity of the value or upper semicontinuity of optimal strategies in zero-sum games, no restrictions on information fields are necessary.

This paper is organized as follows. The setup is described in §2. Our results (Theorems 1, 2, and 3 and Corollaries 1 and 2) are stated and proved in §3; Remarks 1 and 2 appear at the end of this section. Section 4 contains some concluding remarks. The appendix contains the proof of technical Lemma 2.

2. Preliminaries. We consider zero-sum games with two players, $i = 1, 2$. Games are played in an uncertain environment, which affects payoff functions of the players. The underlying uncertainty is described by a probability space $(\Omega, \mathcal{F}, \mu)$, where Ω is a set of states of nature, $\mathcal{F}$ is a countably generated σ-field of subsets of Ω, and μ is a countably additive probability measure on $(\Omega, \mathcal{F})$, which represents the *common prior belief* of the players about the distribution of the realized state of nature. The initial *information endowment* of player i is given by a σ-subfield $\mathcal{F}^i$ of $\mathcal{F}$.

Each player $i = 1, 2$ has a set S^i of *strategies*, which is a convex and compact subset of a Euclidean space $\mathbb{R}^{n_i}$. We will assume, without loss of generality, that $\max_{s\in S^1\cup S^2} \|s\| \le 1$, where $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^{n_1}$ or $\mathbb{R}^{n_2}$. One simple example of such strategy set S^i, to which we return later, is the $(n_i - 1)$-dimensional simplex of i's mixed strategies, provided player i has n_i pure strategies.

There is, in addition, a measurable[4] real-valued *payoff function* u: $\Omega \times S^1 \times S^2 \to \mathbb{R}$, such that $u(\cdot, s^1, s^2)$ is integrable for every $(s^1, s^2) \in S^1 \times S^2$. At every state of nature $\omega \in \Omega$, $u(\omega, s^1, s^2)$ is the payoff received by player 1, and $-u(\omega, s^1, s^2)$ is the payoff of player 2, when each player i chooses to play s^i. We assume that each $u(\omega, \cdot, \cdot)$ is a Lipschitz function with constant $K(\omega)$, that is,

$$|u(\omega, s^1, s^2) - u(\omega, t^1, t^2)| \le K(\omega)(\|s^1 - t^1\| + \| s^2 - t^2\|) \tag{1}$$

[3] In fact, Monderer and Samet [10] (as well as Kajii and Morris [7] in a fixed-types model of differential information) are concerned precisely with the question of what topology on information endowments would lead to approximate lower semicontinuity of BNE. It must be significantly weaker than the Boylan topology.

[4] The measurability is in all coordinates jointly (with respect to the Borel σ-fields in the second and third coordinates).

for every $(s^1, s^2), (t^1, t^2) \in S^1 \times S^2$. We also assume that the state-dependent Lipschitz constant $K(\cdot)$ is $\mathcal{F}$-measurable, and that there exists $q \geq 1$ such that it is q-integrable (and, in particular, integrable):

$$\int_\Omega (K(\omega))^q \, d\mu(\omega) < \infty. \tag{2}$$

The probability space $(\Omega, \mathcal{F}, \mu)$, information endowments $\mathcal{F}^1$ and $\mathcal{F}^2$, strategy sets S^1 and S^2, and the payoff function u fully describe a *zero-sum Bayesian game*. To concentrate on the effects of changes in information endowments, we keep all the attributes of the game fixed, with the exception of $\mathcal{F}^1$ and $\mathcal{F}^2$ that are variable. Thus, we denote the game by $G(\mathcal{F}^1, \mathcal{F}^2)$, to emphasize its changeable characteristics.

A *Bayesian strategy* of player i is an $\mathcal{F}^i$-measurable function $x^i\colon \Omega \to S^i$. The set of all Bayesian strategies of player i will be denoted by $X^i(\mathcal{F}^i)$.

For $1 \leq p \leq \infty$, $L_p^n(\Omega, \mathcal{F}, \mu)$ denotes the Banach space of all $\mathcal{F}$-measurable functions[5] $x\colon \Omega \to \mathbb{R}^n$ such that

$$\|x\|_p \equiv \left(\int_\Omega \|x(\omega)\|^p \, d\mu(\omega) \right)^{1/p} < \infty \tag{3}$$

(recall that $\|\cdot\|$ stands for the Euclidean norm on $\mathbb{R}^n$) if $p < \infty$, and $\|x\|_\infty \equiv$ essential supremum of $\|x(\cdot)\| < \infty$ if $p = \infty$. For every $1 \leq p \leq \infty$, $X^i(\mathcal{F})$ is a subset of $L_p^{n_i}(\Omega, \mathcal{F}, \mu)$. We will call the topology that the $\|\cdot\|_1$-norm on $L_1^{n_i}(\Omega, \mathcal{F}, \mu)$ induces on $X^i(\mathcal{F})$ the *strong topology* on $X^i(\mathcal{F})$. Since functions in $X^i(\mathcal{F})$ are uniformly bounded, the strong topology on $X^i(\mathcal{F})$ coincides[6] with the one induced by the $\|\cdot\|_p$-norm for any $1 < p < \infty$.

For $1 \leq p < \infty$, the weak topology on $L_p^n(\Omega, \mathcal{F}, \mu)$ is the (weakest) one in which the linear functional[7] $\varphi_y(x) \equiv \int_\Omega x(\omega) \cdot y(\omega) \, d\mu(\omega)$ is continuous for any given $y \in L_q^n(\Omega, \mathcal{F}, \mu)$, where $q = p/(p-1)$ if $p > 1$ and $q = \infty$ if $p = 1$. The *weak topology* on $X^i(\mathcal{F})$ will be defined as the one induced on it by the weak topology of $L_1^{n_i}(\Omega, \mathcal{F}, \mu)$. The weak topology on $X^i(\mathcal{F})$ coincides[8] with the one that the weak topology of $L_p^{n_i}(\Omega, \mathcal{F}, \mu)$ induces on it, for any $1 < p < \infty$, since functions in $X^i(\mathcal{F})$ are uniformly bounded.

For $1 < p < \infty$, the unit ball in $L_p^{n_i}(\Omega, \mathcal{F}, \mu)$ is metrizable in the weak topology of $L_p^{n_i}(\Omega, \mathcal{F}, \mu)$ (because its dual $L_q^{n_i}(\Omega, \mathcal{F}, \mu)$ is separable due to our assumption on $\mathcal{F}$), and it is also compact. Note that $X^i(\mathcal{F})$ is a weakly closed subset of the unit ball, and thus it is metrizable and compact in its weak topology. The weakly closed subset $X^i(\mathcal{F}^i)$ of $X^i(\mathcal{F})$ is also metrizable and compact in the weak topology.

The expected payoff of player 1 (and the expected loss of player 2) when $x^i \in X^i\ (\mathcal{F}^i)$ is chosen by i is

$$U(x^1, x^2) \equiv E(u(\cdot, x^1(\cdot), x^2(\cdot))) = \int_\Omega u(\omega, x^1(w), x^2(w)) \, d\mu(\omega)$$

(the integral is well defined due to integrability of each $u(\cdot, s^1, s^2)$, assumption (1), and integrability of $K(\cdot)$). This also defines U for all $(x^1, x^2) \in X^1(\mathcal{F}) \times X^2(\mathcal{F})$.

If

$$\inf_{x^2 \in X^2(\mathcal{F}^2)} \sup_{x^1 \in X^1(\mathcal{F}^1)} U(x^1, x^2) = \sup_{x^1 \in X^1(\mathcal{F}^1)} \inf_{x^2 \in X^2(\mathcal{F}^2)} U(x^1, x^2), \tag{4}$$

then the common value $v = v(\mathcal{F}^1, \mathcal{F}^2)$ of the two expressions in (4) is called the *value* of the zero-sum Bayesian game $G(\mathcal{F}^1, \mathcal{F}^2)$. Given $\varepsilon \geq 0$, $x^1 \in X^1(\mathcal{F}^1)$ is called *ε-optimal* for player 1 in $G(\mathcal{F}^1, \mathcal{F}^2)$ if

$$U(x^1, y^2) \geq v(\mathcal{F}^1, \mathcal{F}^2) - \varepsilon$$

for every $y^2 \in X^2(\mathcal{F}^2)$. Similarly, $x^2 \in X^2(\mathcal{F}^2)$ is called *ε-optimal* for player 2 in $G(\mathcal{F}^1, \mathcal{F}^2)$ if

$$U(y^1, x^2) \leq v(\mathcal{F}^1, \mathcal{F}^2) + \varepsilon$$

for every $y^1 \in X^1(\mathcal{F}^1)$. If a strategy x^i is 0-optimal for player i, it is called *optimal* for i.

We shall assume that each player's payoff is concave in his own strategy; that is, the state-dependent payoff function $u(\omega, \cdot, \cdot)$ is concave in $s^1 \in S^1$ for a fixed $s^2 \in S^2$, and convex in $s^2 \in S^2$ for a fixed $s^1 \in S^1$. This will guarantee the existence of the value and optimal strategies in $G(\mathcal{F}^1, \mathcal{F}^2)$:

Proposition. *Under the above assumption,* (a) *each player's expected payoff is concave and upper semicontinuous in his own Bayesian strategy. That is, the expected payoff function U is weakly upper semicontinuous*

[5] Or, to be precise, their equivalence classes, where any two functions which are equal μ-almost everywhere are identified. This identification applies to Bayesian strategies as well.

[6] It is also the same as the topology of convergence in measure, and as the Mackey topology if $X^i(\mathcal{F})$ is viewed as a subset of $L_\infty^{n_i}(\Omega, \mathcal{F}, \mu)$.

[7] The dot stands for the inner product in R^n.

[8] This topology is also the same as the weak* topology if $X^i(\mathcal{F})$ is viewed as a subset of $L_\infty^{n_i}(\Omega, \mathcal{F}, \mu)$.

and concave in $x^1 \in X^1(\mathcal{F})$ for a fixed $x^2 \in X^2(\mathcal{F})$, and weakly lower semicontinuous and convex in $x^2 \in X^2(\mathcal{F})$ for a fixed $x^1 \in X^1(\mathcal{F})$; and (b) *the game $G(\mathcal{F}^1, \mathcal{F}^2)$ possesses a value and both players have optimal strategies.*

Proof. (a) Since $u(\omega, \cdot, s^2)$ is a continuous and concave function of s^1, and its maximum $\psi(\omega, s^2) \equiv \max_{s^1 \in S^1} u(\omega, s^1, s^2)$ is integrable in ω due to the integrability of the Lipschitz constant $K(\omega)$, Theorem 2.8 of Balder and Yannelis [2] can be applied[9] to deduce weak upper semicontinuity of U in $x^1 \in X^1(\mathcal{F})$. The concavity of U in $x^1 \in X^1(\mathcal{F})$ is obvious. A mirror argument shows lower semicontinuity and convexity of U in $x^2 \in X^2(\mathcal{F})$.

(b) Properties of U shown in (a) guarantee existence of the value and optimal strategies by the Sion minimax theorem (see, e.g., Theorem A.7 in Sorin [15]) because $X^1(\mathcal{F}^1) \times X^2(\mathcal{F}^2)$ is weakly compact. □

Example. The most prevalent payoff function that gives rise to such U comes from the usual matrix game. In a matrix game, each player i has n_i pure strategies, and S^i is the $(n_i - 1)$-dimensional simplex of i's mixed strategies. In each $\omega \in \Omega$, the payoff function is

$$u(\omega, s^1, s^2) = s^1 A(\omega) s^2, \tag{5}$$

where strategy $s^1 \in S^1$ is regarded as a row vector, $s^2 \in S^2$ as a column vector, and $A(\omega)$ is an $n_1 \times n_2$-matrix, with $A(\omega)_{j,k}$ being the payoff of player 1 when he chooses pure strategy j and 2 chooses pure strategy k. Conditions (1) and (2) are guaranteed if $a(\omega) = \max_{j,k} |A_{j,k}(\omega)|$ is integrable.

Finally, we define convergence of players' information endowments by means of the Boylan pseudometric (introduced in Boylan [3]) on the family $\mathcal{F}^*$ of σ-subfields of $\mathcal{F}$:[10]

$$d(\mathcal{F}_1, \mathcal{F}_2) = \sup_{A \in \mathcal{F}_1} \inf_{B \in \mathcal{F}_2} \mu(A \triangle B) + \sup_{B \in \mathcal{F}_2} \inf_{A \in \mathcal{F}_1} \mu(A \triangle B), \tag{6}$$

where $A \triangle B = (A \backslash B) \cup (B \backslash A)$ is the "symmetric difference" of A and B.

If $x^i \in X^i\ (\mathcal{F})$ and $\mathcal{F}' \in \mathcal{F}^*$, denote by $E(x^i \mid \mathcal{F}') \in X^i\ (\mathcal{F}')$ the conditional expectation of x^i with respect to the field $\mathcal{F}'$. The conditional expectation $E(x^i \mid \mathcal{F}')$ is well behaved with respect to small changes in $\mathcal{F}'$, as is shown in the following lemma, based on the result of Rogge [12]. Inequality (7) established in the lemma will be used in the proofs of our results in the next section, and it will be of crucial importance in the proof of Theorem 1.

Lemma 1. *If $x^i \in X^i\ (\mathcal{F})$ and $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}^*$, then*

$$\|E(x^i \mid \mathcal{F}_1) - E(x^i \mid \mathcal{F}_2)\|_1 \le 16 n_i d(\mathcal{F}_1, \mathcal{F}_2). \tag{7}$$

Proof. If $n_i = 1$ (that is, if $S^i \subset [-1, 1]$), (7) was established in Rogge [12]. (See, e.g., Rogge [12] and Landers and Rogge [8], who show that $\|E(f \mid \mathcal{F}_1) - E(f \mid \mathcal{F}_2)\|_1 \le 8 d(\mathcal{F}_1, \mathcal{F}_2)$ for all $\mathcal{F}$-measurable functions f with values in $[0, 1]$.) When $n_i > 1$,

$$\begin{aligned}
\|E(x^i \mid \mathcal{F}_1) - E(x^i \mid \mathcal{F}_2)\|_1 &= \int_\Omega \|E(x^i \mid \mathcal{F}_1) - E(x^i \mid \mathcal{F}_2)\| \, d\mu(\omega) \\
&\le \int_\Omega \sum_{j=1}^{n_i} |E(x^i_j \mid \mathcal{F}_1) - E(x^i_j \mid \mathcal{F}_2)| \, d\mu(\omega) \\
&= \sum_{j=1}^{n_i} \|E(x^i_j \mid \mathcal{F}_1) - E(x^i_j \mid \mathcal{F}_2)\|_1 \\
&\le 16 n_i d(\mathcal{F}_1, \mathcal{F}_2). \quad \square
\end{aligned}$$

[9] Theorem 2.8 of Balder and Yannelis [2] is a little too heavy for our purpose (it aims to show weaker upper semicontinuity of U by assuming $u(\omega, \cdot)$ to be only upper semicontinuous), but it is a convenient reference.

[10] If $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}^*$ contain the same sets of positive measure, then $d(\mathcal{F}_1, \mathcal{F}_2) = 0$. For this reason, d is indeed a pseudometric rather than a metric. It would have become a metric if we passed to work with equivalence sets of σ-subfields, dropping the distinction between any such $\mathcal{F}_1$, $\mathcal{F}_2$.

3. Results. Given two pairs of fields in F^*, (F_1^1, F_1^2) and (F_2^1, F_2^2) (where F_j^i is the information endowment of player $i = 1, 2$ in pair $j = 1, 2$), the distance between them will be measured by the following pseudometric:

$$\bar{d}((F_1^1, F_1^2), (F_2^1, F_2^2)) \equiv \max[d(F_1^1, F_2^1), d(F_1^2, F_2^2)],$$

where d is the Boylan pseudometric, defined in (6).

Theorem 1. *If the state-dependent Lipschitz constant $K(\cdot)$ of the payoff function is q-integrable for some $q > 1$, the value $v(F^1, F^2)$ is a uniformly-continuous (in fact, Hölder-continuous) function of $(F^1, F^2) \in F^* \times F^*$ with respect to the pseudometric $\bar{d}$. Specifically, for any two $(F_1^1, F_1^2), (F_2^1, F_2^2) \in F^* \times F^*$,*

$$|v(F_1^1, F_1^2) - v(F_2^1, F_2^2)| \leq C[\bar{d}((F_1^1, F_1^2), (F_2^1, F_2^2))]^{(q-1)/q}, \tag{8}$$

where $C > 0$ is a constant given by

$$C \equiv 4(8\max(n_1, n_2))^{(q-1)/q} \|K\|_q. \tag{9}$$

Proof. For any two given $(F_1^1, F_1^2), (F_2^1, F_2^2) \in F^* \times F^*$, let $x^1 \in X^1(F_1^1)$ be an optimal strategy of player 1 in the game $G(F_1^1, F_1^2)$, and pick $y^2 \in X^2(F_2^2)$. Now denote $x_2^1 \equiv E(x^1 \mid F_2^1) \in X^1(F_2^1)$ and $y_1^2 \equiv E(y^2 \mid F_1^2) \in X^2(F_1^2)$. The optimality of x^1 in $G(F_1^1, F_1^2)$ implies

$$U(x^1, y_1^2) \geq v(F_1^1, F_1^2). \tag{10}$$

Note that

$$|U(x^1, y_1^2) - U(x_2^1, y^2)|$$

(by (1))

$$\leq \int_\Omega K(\omega)\|x^1(\omega) - x_2^1(\omega)\| \, d\mu(\omega) + \int_\Omega K(\omega)\|y_1^2(\omega) - y^2(\omega)\| \, d\mu(\omega)$$

(by the Hölder inequality, for $p = q/(q-1)$)

$$\leq \|K\|_q(\|x^1 - x_2^1\|_p + \|y_1^2 - y^2\|_p)$$

(since $\|x^1(\omega) - x_2^1(\omega)\|$, $\|y_1^2(\omega) - y^2(\omega)\| \leq 2$ for μ-almost every $\omega \in \Omega$)

$$\leq 2^{(p-1)/p}\|K\|_q\left(\left(\int_\Omega \|x^1(\omega) - x_2^1(\omega)\| \, d\mu(\omega)\right)^{1/p} + \left(\int_\Omega \|y_1^2(\omega) - y^2(\omega)\| \, d\mu(\omega)\right)^{1/p}\right)$$

$$= 2^{(p-1)/p}\|K\|_q\left(\left\|x^1 - x_2^1\right\|_1^{1/p} + \left\|y_1^2 - y^2\right\|_1^{1/p}\right)$$

$$= 2^{(p-1)/p}\|K\|_q\left(\left\|E(x^1 \mid F_1^1) - E(x^1 \mid F_2^1)\right\|_1^{1/p} + \left\|E(y^2 \mid F_1^2) - E(y^2 \mid F_2^2)\right\|_1^{1/p}\right)$$

(by (7) in Lemma 1)

$$\leq 2^{(p-1)/p}(16\max(n_1, n_2))^{1/p}\|K\|_q([d(F_1^1, F_2^1)]^{1/p} + [d(F_1^2, F_2^2)]^{1/p})$$

$$\leq 4(8\max(n_1, n_2))^{1/p}\|K\|_q[\bar{d}((F_1^1, F_1^2), (F_2^1, F_2^2))]^{1/p}.$$

To summarize, we have shown that

$$|U(x^1, y_1^2) - U(x_2^1, y^2)| \leq C[\bar{d}((F_1^1, F_1^2), (F_2^1, F_2^2))]^{(q-1)/q}. \tag{11}$$

Together with (10), (11) implies that

$$U(x_2^1, y^2) \geq v(F_1^1, F_1^2) - C[\bar{d}((F_1^1, F_1^2), (F_2^1, F_2^2))]^{(q-1)/q}.$$

This holds for every $y^2 \in X^2(F_2^2)$, and hence it follows that[11]

$$v(F_2^1, F_2^2) = \max_{y^1 \in X^1(F_2^1)} \min_{y^2 \in X^2(F_2^2)} U(y^1, y^2) \tag{12}$$

$$\geq \min_{y^2 \in X^2(F_2^2)} U(x_2^1, y^2) \geq v(F_1^1, F_1^2) - C[\bar{d}((F_1^1, F_1^2), (F_2^1, F_2^2))]^{(q-1)/q}. \tag{13}$$

[11] Since optimal strategies exist for both players, and the function U is weakly lower semicontinuous in x^2 on the weakly compact $X^2(F^2)$, we can replace inf by min, and sup by max, in the right-hand side of definition (4) of the value v.

Using similar arguments (when we start from an optimal strategy $x^2 \in X^2(\mathcal{F}_1^2)$ of player 2 in the game $G(\mathcal{F}_1^1, \mathcal{F}_1^2)$), we can show that for $x_2^2 = E(x^2 \mid \mathcal{F}_2^2) \in X^2(\mathcal{F}_2^2)$, the inequality

$$U(y^1, x_2^2) \le v(\mathcal{F}_1^1, \mathcal{F}_1^2) + C[\bar{d}((\mathcal{F}_1^1, \mathcal{F}_1^2), (\mathcal{F}_2^1, \mathcal{F}_2^2))]^{(q-1)/q}$$

holds for every $y^1 \in X^1(\mathcal{F}_2^1)$. This leads to

$$v(\mathcal{F}_2^1, \mathcal{F}_2^2) = \min_{y^2 \in X^2(\mathcal{F}_2^2)} \max_{y^1 \in X^1(\mathcal{F}_2^1)} U(y^1, y^2) \tag{14}$$

$$\le \max_{y^1 \in X^1(\mathcal{F}_2^1)} U(y^1, x_2^2) \le v(\mathcal{F}_1^1, \mathcal{F}_1^2) + C[\bar{d}((\mathcal{F}_1^1, \mathcal{F}_1^2), (\mathcal{F}_2^1, \mathcal{F}_2^2))]^{(q-1)/q}. \tag{15}$$

The combination of (12)–(13) and (14)–(15) now implies (8). □

The continuity of the value as a function of $(\mathcal{F}^1, \mathcal{F}^2)$ is, of course, an immediate implication of Theorem 1:

Corollary 1. *Suppose that $\{\mathcal{F}_k^i\}_{k=1}^\infty \subset \mathcal{F}^*$ is a sequence such that $\lim_{k\to\infty} \mathcal{F}_k^i = \mathcal{F}^i$ in the Boylan pseudometric for $i = 1, 2$, and that the condition of Theorem 1 holds. Then, $\lim_{k\to\infty} v(\mathcal{F}_k^1, \mathcal{F}_k^2) = v(\mathcal{F}^1, \mathcal{F}^2)$.*

If $K(\cdot)$ is a bounded function, it is obvious that (2) holds for every $q > 1$, and thus q can be chosen to be arbitrarily high. The constant $C = C(q)$, defined in (9), converges to the limit

$$32 \max(n_1, n_2) \|K\|_\infty$$

when q approaches infinity. Inequality (8) of Theorem 1 thus provides us with the following corollary:

Corollary 2. *If $K(\cdot)$ is a bounded function, the value $v(\mathcal{F}^1, \mathcal{F}^2)$ is a Lipschitz function of $(\mathcal{F}^1, \mathcal{F}^2) \in \mathcal{F}^* \times \mathcal{F}^*$, with respect to the pseudometric $\bar{d}$.*

It is natural to ask whether the value is continuous when $K(\cdot)$ is merely *integrable*. Our next theorem shows that the continuity still obtains under this more general assumption. However, it does not follow from Theorem 1 (because we do not have uniform continuity in this case) and has to be established directly (using similar techniques).

Theorem 2. *If the state-dependent Lipschitz constant $K(\cdot)$ is integrable, and if $\{\mathcal{F}_k^i\}_{k=1}^\infty \subset \mathcal{F}^*$ is a sequence such that $\lim_{k\to\infty} \mathcal{F}_k^i = \mathcal{F}^i$ in the Boylan pseudometric for $i = 1, 2$, then $\lim_{k\to\infty} v(\mathcal{F}_k^1, \mathcal{F}_k^2) = v(\mathcal{F}^1, \mathcal{F}^2)$.*

Proof. We will show that the limit v' of any convergent subsequence of $\{v(\mathcal{F}_k^1, \mathcal{F}_k^2)\}_{k=1}^\infty$ (which we assume, w.l.o.g., to be the sequence itself) is equal to $v(\mathcal{F}^1, \mathcal{F}^2)$. Let x_k^1 be an optimal strategy of player 1 in the game $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$ for every $k \ge 1$. The set $X^1(\mathcal{F})$ is metrizable and compact, and therefore there is a subsequence of $\{x_k^1\}_{k=1}^\infty$ (which we again let, w.l.o.g., to be the sequence itself) that converges weakly to some $x^1 \in X^1(\mathcal{F})$. By Lemma 2 in the appendix, x^1 is $\mathcal{F}^1$-measurable, which implies that $x^1 \in X^1(\mathcal{F}^1)$.

Now fix $y^2 \in X^2(\mathcal{F}^2)$, and, for every $k \ge 1$, let $y_k^2 \equiv E(y^2 \mid \mathcal{F}_k^2) \in X^2(\mathcal{F}_k^2)$. Since x_k^1 is an optimal strategy of 1 in $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$,

$$U(x_k^1, y_k^2) \ge v(\mathcal{F}_k^1, \mathcal{F}_k^2). \tag{16}$$

Because $\lim_{k\to\infty} y_k^2 = y^2$ in the strong topology by (7), there is a subsequence of $\{y_k^2\}_{k=1}^\infty$ that converges pointwise to y^2 μ-almost everywhere; w.l.o.g., the sequence itself converges pointwise. Note that

$$U(x_k^1, y_k^2) - U(x^1, y^2) = [U(x_k^1, y_k^2) - U(x_k^1, y^2)] + [U(x_k^1, y^2) - U(x^1, y^2)]$$

(by (1))

$$\le \int_\Omega K(\omega) \|y_k^2(\omega) - y^2(\omega)\| \, d\mu(\omega) + [U(x_k^1, y^2) - U(x^1, y^2)].$$

The first term in the above expression converges to zero as $k \to \infty$ by the bounded convergence theorem. As for the second term,

$$\limsup_{k\to\infty} [U(x_k^1, y^2) - U(x^1, y^2)] \le 0$$

since U is weakly upper semicontinuous in the first coordinate by the Proposition. Thus,

$$\limsup_{k\to\infty} U(x_k^1, y_k^2) \le U(x^1, y^2), \tag{17}$$

and together with (16), this implies

$$U(x^1, y^2) \geq \lim_{k\to\infty} v(\mathcal{F}^1_k, \mathcal{F}^2_k) = v'; \tag{18}$$

this inequality holds for every $y^2 \in X^2(\mathcal{F}^2)$. Thus,

$$v(\mathcal{F}^1, \mathcal{F}^2) = \max_{y^1\in X^1(\mathcal{F}^1)} \min_{y^2\in X^2(\mathcal{F}^2)} U(y^1, y^2) \tag{19}$$

$$\geq \min_{y^2\in X^2(\mathcal{F}^2)} U(x^1, y^2) \geq v'. \tag{20}$$

Using similar arguments (when we start from finding a limit point x^2 of a sequence $\{x^2_k\}_{k=1}^\infty$ of optimal strategies of player 2 in games $G(\mathcal{F}^1_k, \mathcal{F}^2_k)$), we can show that

$$U(y^1, x^2) \leq \lim_{k\to\infty} v(\mathcal{F}^1_k, \mathcal{F}^2_k) = v' \tag{21}$$

for every $y^1 \in X^1(\mathcal{F}^1)$. This leads to

$$v(\mathcal{F}^1, \mathcal{F}^2) = \min_{y^2\in X^2(\mathcal{F}^2)} \max_{y^1\in X^1(\mathcal{F}^1)} U(y^1, y^2) \tag{22}$$

$$\leq \max_{y^1\in X^1(\mathcal{F}^1)} U(y^1, x^2) \leq v'. \tag{23}$$

The combination of (19)–(20) and (22)–(23) now implies $v' = v(\mathcal{F}^1, \mathcal{F}^2)$. This establishes $\lim_{k\to\infty} v(\mathcal{F}^1_k, \mathcal{F}^2_k) = v(\mathcal{F}^1, \mathcal{F}^2)$. □

Remark 1 (Monotonic Convergence of Information Fields, or Convergence in Cotter [4] Topology). Theorem 2 also applies in the important case when information fields of players converge *monotonically* (i.e., for each player i, $\{\mathcal{F}^i_k\}_{k=1}^\infty \subset \mathcal{F}^*$ is a sequence of fields such that either $\mathcal{F}^i_1 \subset \mathcal{F}^i_2 \subset \cdots \subset \mathcal{F}^i$ and $\mathcal{F}^i$ is generated by $\bigcup_{k=1}^\infty \mathcal{F}^i_k$, or $\mathcal{F}^i_1 \supset \mathcal{F}^i_2 \supset \cdots \supset \mathcal{F}^i$ and $\mathcal{F}^i = \bigcap_{k=1}^\infty \mathcal{F}^i_k$). Although monotonic convergence of information fields does not necessarily imply convergence in the Boylan pseudometric (as remarked in Boylan [3]), the proof of Theorem 2 remains valid for the monotonic convergence. The only change is in the argument showing strong convergence of $y^2_k \equiv E(y^2 \mid \mathcal{F}^2_k) \in X^2(\mathcal{F}^2_k)$ to y^2: instead of appealing to (7), one has to use the martingale convergence theorem (for increasing or decreasing sequences of σ-fields; see, e.g., Theorems 2 and 3 in §2 of Parry [11]). Similarly, our next Theorem 3 also applies to monotonically converging information fields.

More generally, Theorem 2 holds when $\lim_{k\to\infty} \mathcal{F}^i_k = \mathcal{F}^i$ in the pointwise convergence topology of Cotter [4]. (It is the minimal topology on $\mathcal{F}^*$ in which the mapping $\mathcal{F} \to E(f \mid \mathcal{F})$ is continuous for all $f \in L^1_1(\Omega, \mathcal{F}, \mu)$ with respect to the strong topology on $L^1_1(\Omega, \mathcal{F}, \mu)$.) This is because now the strong convergence of $y^2_k \equiv E(y^2 \mid \mathcal{F}^2_k) \in X^2(\mathcal{F}^2_k)$ to y^2, needed for the proof, is implied directly by $\lim_{k\to\infty} \mathcal{F}^i_k = \mathcal{F}^i$ and the definition of Cotter [4] topology. However, it is only the convergence of information fields in *Boylan* topology that guarantees the *uniform* continuity of the value, as in Theorem 1. □

The following theorem follows quite easily from the proof of Theorem 2.

Theorem 3. *Suppose that* $\{\mathcal{F}^i_k\}_{k=1}^\infty \subset \mathcal{F}^*$ *is a sequence such that* $\lim_{k\to\infty} \mathcal{F}^i_k = \mathcal{F}^i$ *in the Boylan pseudometric for* $i = 1, 2$.

(1) *The optimal strategy correspondence is weakly upper semicontinuous for each player. That is, if* $i \in \{1, 2\}$, *and* $\{x^i_k\}_{k=1}^\infty \in \prod_{k=1}^\infty X^i(\mathcal{F}^i_k)$ *is a sequence such that for every* k, x^i_k *is an optimal strategy of* i *in the game* $G(\mathcal{F}^1_k, \mathcal{F}^2_k)$, *and* $\lim_{k\to\infty} x^i_k = x^i$ *weakly, then* x^i *is an optimal strategy of* i *in* $G(\mathcal{F}^1, \mathcal{F}^2)$.

(2) *The optimal strategy correspondence is strongly approximately lower semicontinuous for each player. That is, if* $i \in \{1, 2\}$ *and* x^i *is an optimal strategy of* i *in* $G(\mathcal{F}^1, \mathcal{F}^2)$, *then there exist sequences* $\{\varepsilon_k\}_{k=1}^\infty \subset [0, \infty)$ *and* $\{x^i_k\}_{k=1}^\infty \in \prod_{k=1}^\infty X^i(\mathcal{F}^i_k)$ *such that for every* k, x^i_k *is an* ε_k*-optimal strategy of* i *in* $G(\mathcal{F}^1_k, \mathcal{F}^2_k)$, $\lim_{k\to\infty} \varepsilon_k = 0$, *and* $\lim_{k\to\infty} x^i_k = x^i$ *in the strong topology.*

Proof. We will establish both assertions of the theorem for $i = 1$ only because the case of $i = 2$ requires entirely analogous arguments. We therefore fix $i = 1$ for the rest of the proof.

(1) Since $\lim_{k\to\infty} x^1_k = x^1$ weakly, the entire first part of the proof of Theorem 2 (leading to (18)) can be utilized to show that $U(x^1, y^2) \geq \lim_{k\to\infty} v(\mathcal{F}^1_k, \mathcal{F}^2_k)$ for every $y^2 \in X^2(\mathcal{F}^2)$. However, by Theorem 2, $\lim_{k\to\infty} v(\mathcal{F}^1_k, \mathcal{F}^2_k) = v(\mathcal{F}^1, \mathcal{F}^2)$, and so x^1 is indeed an optimal strategy of 1 in $G(\mathcal{F}^1, \mathcal{F}^2)$.

(2) Denote

$$x_k^1 \equiv E(x^1 \mid \mathcal{F}_k^1) \in X^1(\mathcal{F}_k^1) \qquad \text{and} \qquad \varepsilon_k \equiv \sup_{y^2 \in X^2(\mathcal{F}_k^2)} (v(\mathcal{F}_k^1, \mathcal{F}_k^2) - U(x_k^1, y^2)) \geq 0$$

for every k. Thus, x_k^1 is an ε_k-optimal strategy of 1 in $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$. By (7), $\lim_{k\to\infty} x_k^1 = x^1$ in the strong topology. We will now show that $\lim_{k\to\infty} \varepsilon_k = 0$. Indeed, suppose by the way of contradiction that this is not so. Then, there exists an increasing subsequence $\{k_l\}_{l=1}^{\infty}$ of indices such that

$$\liminf_{l\to\infty} U(x_{k_l}^1, y_{k_l}^2) < \lim_{k\to\infty} v(\mathcal{F}_k^1, \mathcal{F}_k^2) = v(\mathcal{F}^1, \mathcal{F}^2) \tag{24}$$

for some $\{y_{k_l}^2\}_{l=1}^{\infty} \subset \prod_{l=1}^{\infty} X^i(\mathcal{F}_{k_l}^2)$. By metrizability and compactness of $X^2(\mathcal{F})$, there is a subsequence of $\{y_{k_l}^2\}_{l=1}^{\infty}$ which converges weakly to some $y^2 \in X^2(\mathcal{F})$ (w.l.o.g., the sequence itself converges to y^2). By Lemma 2 in the appendix, $y^2 \in X^2(\mathcal{F}^2)$. Since $\lim_{l\to\infty} x_{k_l}^1 = x^1$ strongly and $\lim_{l\to\infty} y_{k_l}^2 = y^2$ weakly, it can be shown as in the proof of (17) that $\liminf_{l\to\infty} U(x_{k_l}^1, y_{k_l}^2) \geq U(x^1, y^2)$. But $U(x^1, y^2) \geq v(\mathcal{F}^1, \mathcal{F}^2)$, and therefore (24) is contradicted. We conclude that $\lim_{k\to\infty} \varepsilon_k = 0$. □

Remark 2 (Optimal Strategies Are Not Lower Semicontinuous). Part 2 of Theorem 3 cannot be strengthened because the optimal strategy correspondence is not *lower* semicontinuous in general. That is, it may be the case that $\lim_{k\to\infty} \mathcal{F}_k^i = \mathcal{F}^i$ in the Boylan pseudometric for $i = 1, 2$, and x^i is an optimal strategy of i in $G(\mathcal{F}^1, \mathcal{F}^2)$, but there is no sequence $\{x_k^i\}_{k=1}^{\infty}$ of optimal strategies of i in $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$ that converges to x^i in the strong or even in the weak topology. Indeed, consider the example where $\Omega = [-1, 1]$, $\mathcal{F}$ is the σ-field of Borel sets in Ω, μ is the normalized Lebesgue measure on Ω, $S^1 = [0, 1]$, $S^2 = \{0\}$, and $u(\omega, s^1, s^2) = \omega s^1$. Now, for each $k = 1, 2, 3, \ldots$, let $\mathcal{F}_k^1 = \mathcal{F}_k^2$ be the finite σ-field generated by the intervals $[-1, -1 + 1/k]$ and $(-1 + 1/k, 1]$, and let $\mathcal{F}^1 = \mathcal{F}^2 = \{\varnothing, \Omega\}$. Then, clearly $\lim_{k\to\infty} \mathcal{F}_k^i = \mathcal{F}^i$ for $i = 1, 2$. However, consider a pair $(x^1, x^2) \equiv (0, 0)$ of optimal strategies in the game $G(\mathcal{F}^1, \mathcal{F}^2)$. Since the optimal strategy x_k^1 of 1 in the game $G(\mathcal{F}_k^1, \mathcal{F}_k^2)$ satisfies $x_k^1(\omega) = 1$ for every $\omega \in (-1 + 1/k, 1]$, there exists no sequence of optimal strategies of one in $\{G(\mathcal{F}_k^1, \mathcal{F}_k^2)\}_{k=1}^{\infty}$ that converges to x^1 in either the strong or the weak topology. □

4. Concluding remarks. We have shown that the value $v(\mathcal{F}^1, \mathcal{F}^2)$ of a zero-sum game with differential information responds continuously to changes in players' information endowments $\mathcal{F}^1$ and $\mathcal{F}^2$, under the relatively mild assumption of Lipschitz continuity of the state-dependent payoff function. Moreover, the continuity of the value tends to be *uniform*. Now we would like to mention an additional perspective from which our framework and results can be viewed.

First, as was said in the introduction, changing the information field of a player amounts to a change in his strategy set. Formally, denote by $\mathcal{X}^i$ the set of all nonempty, convex, and weakly closed subsets of $X^i(\mathcal{F})$, and recall that $\mathcal{F}^*$ is the set of all σ-subfields of $\mathcal{F}$. Then, to each player i there corresponds a mapping $X^i\colon \mathcal{F}^* \to \mathcal{X}^i$ that maps player i's information field $\mathcal{F}^i$ into the "constraint set" $X^i(\mathcal{F}^i)$ of i's feasible choices.

Second, although the game $G(\mathcal{F}^1, \mathcal{F}^2)$ is defined for $X^1(\mathcal{F}^1)$ and $X^2(\mathcal{F}^2)$ as players' strategy sets, its definition can be easily modified to allow *general* strategy sets, $Y^1 \in \mathcal{X}^1$ and $Y^2 \in \mathcal{X}^2$. This generalized game possesses a value as can be established using the proof of Proposition; denote it by $V(Y^1, Y^2)$. Note that

$$v(\mathcal{F}^1, \mathcal{F}^2) = V(X^1(\mathcal{F}^1), X^2(\mathcal{F}^2)). \tag{25}$$

Thus, the value mapping $v\colon \mathcal{F}^* \times \mathcal{F}^* \to \mathbb{R}$ that lies at the heart of this work is, in fact, a composition of two components: the mapping $X^1 \times X^2\colon \mathcal{F}^* \times \mathcal{F}^* \to \mathcal{X}^1 \times \mathcal{X}^2$ that determines the players' constraint sets of strategies given their information, and the mapping $V\colon \mathcal{X}^1 \times \mathcal{X}^2 \to \mathbb{R}$ that associates with each pair of players' constraint sets the value of the generalized game with these strategy sets.

We have conducted our analysis of the continuity of value by focusing directly on the composed mapping v. An alternative way to proceed could have been to find conditions guaranteeing continuity of V with respect to constraint sets $Y^1 \in \mathcal{X}^1$ and $Y^2 \in \mathcal{X}^2$ (with the Hausdorff metric on each $\mathcal{X}^i$), and thus deduce the continuity of v using (25) and the fact that each X^i is a Lipschitz function (Van Zandt [17]). We have not chosen this path, believing that studying continuity of V with respect to Y^1 and Y^2 does not have sufficient added value over what we already showed directly: $V(Y^1, Y^2)$ is continuous when each Y^i has the form $X^i(\mathcal{F}^i)$, as follows from our results via (25). However, it might be of interest to look at the continuity of V when it is defined over abstract constraint sets Y^i that are not subsets of $X^i(\mathcal{F}^i)$, but of more general topological spaces. This exceeds the scope of the present work, but could merit future research.

Appendix.

LEMMA 2. *Let $\{\mathcal{F}_k\}_{k=0}^{\infty} \subset \mathcal{F}^*$ be a sequence such that $\lim_{k\to\infty} \mathcal{F}_k = \mathcal{F}_0$ in the Boylan pseudometric. If $\{x_k\}_{k=1}^{\infty} \subset \prod_{k=1}^{\infty} X^i(\mathcal{F}_k)$ is a sequence of functions that converges weakly to $x \in X^i(\mathcal{F})$, then x is $\mathcal{F}_0$-measurable (that is, $x \in X^i(\mathcal{F}_0)$).*

PROOF. Without loss of generality, assume that

$$\sum_{k=1}^{\infty} d(\mathcal{F}_k, \mathcal{F}_0) < \infty \tag{26}$$

(otherwise consider instead some subsequence $\{\mathcal{F}_{k_l}\}_{l=1}^{\infty}$ with $\sum_{l=1}^{\infty} d(\mathcal{F}_{k_l}, \mathcal{F}_0) < \infty$). For every k, denote by $\mathcal{G}_k$ the σ-field $\bigvee_{n=k}^{\infty} \mathcal{F}_k$, that is, the minimal σ-subfield of $\mathcal{F}$ which contains each one of $\{\mathcal{F}_n\}_{n=k}^{\infty}$. It follows from (26) by Corollary 2 of Van Zandt [17] that $\lim_{k\to\infty} \mathcal{G}_k = \mathcal{F}_0$.

By applying the Banach-Saks theorem for the sequence $\{x_n\}_{n=k}^{\infty}$ that converges weakly to x, for every $k \geq 1$, one can find a sequence $\{\bar{x}_k\}_{k=1}^{\infty}$ such that: (a) $\bar{x}_k$ is a convex combination of $\{x_n\}_{n=k}^{\infty}$ and therefore $\bar{x}_k \in X^i(\mathcal{G}_k)$; and (b) $\{\bar{x}_k\}_{k=1}^{\infty}$ converges to x strongly (that is, in the $\|\cdot\|_p$ norm for some $p \geq 1$). By Lemma 1 in Einy et al. [6], the strong limit of $\{\bar{x}_k\}_{k=1}^{\infty}$ is measurable with respect to $\lim_{k\to\infty} \mathcal{G}_k = \mathcal{F}_0$. We conclude that $x \in X^i(\mathcal{F}_0)$. □

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