Econometrics II - EXAM

Answer each question in separate sheets in three hours

1. Consider the two linear simultaneous equations ($G = 2$) system with two exogenous variables $\mathbf{z} = (z_1, z_2)'$ $(K = 2)$,

$$\begin{aligned} y_1\gamma_{11} + y_2\gamma_{12} + z_1\delta_{11} + z_2\delta_{12} &= u_1 \\ y_1\gamma_{21} + y_2\gamma_{22} + z_1\delta_{21} + z_2\delta_{22} &= u_2 \end{aligned}$$

where, $\mathbf{u} = (u_1, u_2)'$,

$$E[\mathbf{u}\mathbf{u}'] = \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{21} \\ \gamma_{12} & \gamma_{22} \end{bmatrix}.$$

(a) Using the standard normalization in Γ, write the general form of the order and rank conditions for single equation identification and for system identification.

Then, stating the implied restrictions on the system parameters, check the identification of the above system and state your recommended estimation method in the following cases:

(b) A different exogenous variable is omitted from each structural equation.

(c) The variable z_2 does not appear in the system.

(d) Neither z_1 nor z_2 appear in the first equation.

(e) Γ is constrained to be symmetric and the coefficient of z_1 is the same in both equations.

(f) Γ is constrained to be lower triangular (with diagonal elements equal to 1) and Σ is diagonal. In this case explain how you would estimate the structural form parameters from the estimation of the reduced form. Are these estimates efficient in general? And if you further assume the restrictions on (a)?

2. Consider the following model with lagged endogenous variables and correlated error term,

$$y_t = [\mathbf{z}_t' \;\; y_{t-1}]\boldsymbol{\beta} + u_t, \qquad u_t = \alpha u_{t-1} + v_t,$$

where the v_t are IID, zero mean and $E[v_t \mathbf{z}_j] = \mathbf{0}$ all t, j, and set $\boldsymbol{\beta} = \left(\boldsymbol{\beta}_z', \beta_y\right)'$.

(a) Under which conditions is u_t strictly and covariance stationary? Find the autocovariance function of u_t in the latter case and calculate

$$\lim_{T\to\infty} \operatorname{Var}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} u_t\right].$$

(b) Study the consistency properties of the OLS estimate of $\boldsymbol{\beta}$.

(c) Define a GMM estimate of $\boldsymbol{\beta}$ exploiting the restrictions $E[v_t \mathbf{z}_j] = \mathbf{0}$ all t, j, and study its consistency and asymptotic distribution.

(d) Study the consistency and asymptotic distribution of the OLS estimate of $\boldsymbol{\beta}$, using the transformed model

$$y_t^* = [\mathbf{z}_t^{*\prime} \;\; y_{t-1}^*]\boldsymbol{\beta} + u_t^*, \qquad t = 2, \ldots, T,$$

where

$$\begin{aligned} y_t^* &= y_t - \alpha y_{t-1} \\ \mathbf{z}_t^* &= \mathbf{z}_t - \alpha \mathbf{z}_t \\ u_t^* &= u_t - \alpha u_{t-1}. \end{aligned}$$

3. Consider a zero mean (scalar) time series MA(1) model,

$$x_t = \theta_0 \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

where $\varepsilon_t \sim Independent N\left(0,1\right).$

(a) Find the autocovariance sequence for x_t in terms of the parameters θ_0 and θ_1.

(b) Given a sample of x_t, $t = 1, \ldots, T$, define the corresponding (generalized) moment estimates of the parameters $\theta = (\theta_0, \theta_1)'$ in terms of the previous nonzero autocovariances, and investigate the rank condition for identification and the asymptotic properties of the estimates. [Hint: $E\left[z^4\right] = 2\sigma_z^4$ if $z \sim N\left(0, \sigma_z^2\right)$].

(c) Propose an iterative scheme to obtain the GMM estimates of θ and a Wald test for H_0,

$$H_0 : \theta_1 = 0.$$

(d) Consider restricted estimation of θ under H_0, and propose a Lagrange Multiplier test for H_0.

Econometrics II - EXAM Outline Solutions
Answer each question in separate sheets in three hours

1. *Consider the two linear simultaneous equations* $(G=2)$ *system with two exogenous variables* $z=(z_1,z_2)'$ $(K=2)$,

$$\begin{aligned} y_1\gamma_{11}+y_2\gamma_{12}+z_1\delta_{11}+z_2\delta_{12} &= u_1 \\ y_1\gamma_{21}+y_2\gamma_{22}+z_1\delta_{21}+z_2\delta_{22} &= u_2 \end{aligned}$$

where, $u=(u_1,u_2)'$,

$$E[\mathbf{uu}'] = \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{21} \\ \gamma_{12} & \gamma_{22} \end{bmatrix}.$$

(a) *Using the standard normalization in* Γ, *write the general form of the order and rank conditions for single equation identification and for system identification.*

Imposing the normalization $\gamma_{11}=\gamma_{22}=1$, these are:

Single equation: rank$[\mathbf{R}_1\mathbf{B}] = G-1$, with $\mathbf{R}_1\boldsymbol{\beta}_1=\mathbf{0}$. Order condition: rank$[\mathbf{R}_1] \ \geq \ G-1$.

System: rank$[\mathbf{R}(\mathbf{I}_G\otimes\mathbf{B})] = G(G-1)$, with restrictions $\mathbf{R}\boldsymbol{\beta}=\mathbf{0}$, $\boldsymbol{\beta}=\text{vec}[\mathbf{B}]$. Order condition: rank$[\mathbf{R}] \ \geq \ G(G-1)$.

Then, stating the implied restrictions on the system parameters, check the identification of the above system and state your recommended estimation method in the following cases:

(b) *A different exogenous variable is omitted from each structural equation.*

For example, $\delta_{12}=\delta_{21}=0$. Each equation (just) identified if these variables are in the other equations, i.e. if $\delta_{22}\neq 0$ (for eq. 1) and $\delta_{11}\neq 0$ (for eq.2).

(c) *The variable* z_2 *does not appear in the system.*

$\delta_{12}=\delta_{22}=0$. Rank conditions fail: no single equation identified.

(d) *Neither* z_1 *nor* z_2 *appear in the first equation.*

$\delta_{11}=\delta_{12}=0$. First equation overidentified if $\delta_{21}\delta_{22}\neq 0$ (just identified if $\delta_{21}\neq 0$ or and $\delta_{22}\neq 0$). Second equation not identified.

(e) Γ *is constrained to be symmetric and the coefficient of* z_1 *is the same in both equations.*

$\gamma_{12}=\gamma_{21}$, $\delta_{11}=\delta_{21}$. This corresponds to

$$\mathbf{R} = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

so, with $\boldsymbol{\beta}=(\gamma_{11},\gamma_{12},\delta_{11},\delta_{12},\gamma_{21},\gamma_{22},\delta_{21},\delta_{22})'$, we obtain that

$$\mathbf{R}(\mathbf{I_G}\otimes\mathbf{B}) = \begin{pmatrix} \gamma_{12} & \gamma_{22} & -\gamma_{11} & -\gamma_{21} \\ \delta_{11} & \delta_{21} & -\delta_{11} & -\delta_{21} \end{pmatrix} = \begin{pmatrix} \gamma_{12} & -1 & 1 & -\gamma_{12} \\ \delta_{11} & \delta_{11} & -\delta_{11} & -\delta_{11} \end{pmatrix}$$

which is of rank $2(2-1)=2$, if any two vectors are linearly independent, ie. if $\delta_{11}\neq 0$ and $\gamma_{12}\neq -1$, so the system would be (just) identified.

(f) Γ *is constrained to be lower triangular (with diagonal elements equal to 1) and* Σ *is diagonal. In this case explain how you would estimate the structural form parameters from the estimation of the reduced form. Are these estimates efficient in general? And if you further assume the restrictions on (a)?*

$\gamma_{12} = 0$: this is a triangular system: (just) identified (since y_1 is exogenous in the second equation).

We can estimate the reduced form

$$\mathbf{y} = \mathbf{\Pi}'\mathbf{x} + \mathbf{v},$$

where $\mathbf{\Pi} = \mathbf{\Delta}\mathbf{\Gamma}^{-1}$, by OLS and the covariance matrix $\mathbb{E}\left[\mathbf{v}\mathbf{v}'\right] = \mathbf{\Lambda} = \mathbf{\Gamma}'^{-1}\mathbf{\Sigma}\mathbf{\Gamma}^{-1}$ using OLS residuals. In this case we have that

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{21} \\ \gamma_{12} & \gamma_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \gamma_{12} & 1 \end{bmatrix}, \quad \text{so} \quad \Gamma^{-1} = \begin{bmatrix} 1 & 0 \\ -\gamma_{12} & 1 \end{bmatrix},$$

and

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

and therefore from $\mathbf{\Pi} = \mathbf{\Delta}\mathbf{\Gamma}^{-1}$ we obtain 4 equations and from $\mathbf{\Lambda} = \mathbf{\Gamma}'^{-1}\mathbf{\Sigma}\mathbf{\Gamma}^{-1}$ we obtain another 3 equations (because of symmetry), and we have 7 unknowns (1 element in $\mathbf{\Gamma}$, 4 in $\mathbf{\Delta}$. and 2 in $\mathbf{\Sigma}$). In particular note that

$$\begin{aligned} \mathbf{\Gamma}'^{-1}\mathbf{\Sigma}\mathbf{\Gamma}^{-1} &= \begin{bmatrix} 1 & -\gamma_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\gamma_{12} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1^2 & -\gamma_{12}\sigma_2^2 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\gamma_{12} & 1 \end{bmatrix} = \begin{bmatrix} \sigma_1^2\left(1-\gamma_{12}^2\right) & -\gamma_{12}\sigma_2^2 \\ -\gamma_{12}\sigma_2^2 & \sigma_2^2 \end{bmatrix}, \end{aligned}$$

we obtain three equations for three unknowns $\left(\sigma_1^2, \gamma_{12}, \sigma_2^2\right)$.

Regarding the estimation method, when the system is identified, and if we can not assume some form of conditional homoskedasticity, i.e. $E\left[\mathbf{u}\ \mathbf{u}'|\mathbf{x}\right] = E\left[\mathbf{u}\ \mathbf{u}'\right] = \Sigma$, then we should rely on system efficient system GMM (chi-square) estimates using $\hat{W}_n = \hat{E}\left[\mathbf{X}\mathbf{u}\ \mathbf{u}'\mathbf{X}'\right]^{-1}$ with $\mathbf{X} = (I_2 \otimes \mathbf{x})$ and the corresponding restrictions. Otherwise we could also use 3SLS.

In case (d) we can only use efficient single equation GMM for the first one.

In the just identified cases we have that system GMM is equal to System IV (and 3SLS to 2SLS).

2. *Consider the following model with lagged endogenous variables and correlated error term,*

$$y_t = [\mathbf{z}_t' \ \ y_{t-1}]\boldsymbol{\beta} + u_t, \qquad u_t = \alpha u_{t-1} + v_t,$$

where the v_t are IID, zero mean and $E[v_t z_j] = 0$ all t, j, and set $\beta = \left(\boldsymbol{\beta}_z', \beta_y\right)'$.

(a) *Under which conditions is u_t strictly and covariance stationary? Find the autocovariance function of u_t in the latter case and calculate*

$$\lim_{T\to\infty} \operatorname{Var}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} u_t\right].$$

u_t is strictly stationary if $|\alpha| < 1$ and and cov. stationary if it further $E\left[v_t^2\right] < \infty$. u_t is an $AR(1)$ process, so that the ACF is given by $\rho_v(j) = \alpha^j$. Then

$$\begin{aligned}
\lim_{T\to\infty} \operatorname{Var}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} u_t\right] &= \sigma_u^2 \sum_{-\infty}^{\infty} \rho_v(j) \\
&= \frac{\sigma_v^2}{1-\alpha^2}\left(2\sum_{j=0}^{\infty}\alpha^j - 1\right) \\
&= \frac{\sigma_v^2}{1-\alpha^2}\left(\frac{2}{1-\alpha} - 1\right) = \frac{\sigma_v^2}{1-\alpha^2}\left(\frac{2-1+\alpha}{1-\alpha}\right) \\
&= \frac{\sigma_v^2}{(1-\alpha)(1+\alpha)}\left(\frac{1+\alpha}{1-\alpha}\right) = \frac{\sigma_v^2}{(1-\alpha)^2}.
\end{aligned}$$

(b) *Study the consistency properties of the OLS estimate of β.*

The OLS is inconsistent when $\alpha \neq 0$ (or $\sigma_u^2 \neq 0$) because

$$\begin{aligned}
E(y_{t-1}u_t) &= E\left(\left([\mathbf{z}_{t-1}' \ \ y_{t-2}]\boldsymbol{\beta} + u_{t-1}\right)(\alpha u_{t-1} + v_t)\right) \\
&= E\left(\left(y_{t-2}\boldsymbol{\beta}_y + u_{t-1}\right)(\alpha u_{t-1} + v_t)\right) \\
&= \alpha\boldsymbol{\beta}_y E(y_{t-2}u_{t-1}) + E\left(u_{t-1}(\alpha u_{t-1} + v_t)\right) \\
&= E(y_{t-1}u_t) + \alpha\sigma_u^2 \\
&= \frac{\alpha\sigma_u^2}{1-\alpha\boldsymbol{\beta}_y}.
\end{aligned}$$

(c) *Define a GMM estimate of β exploiting the restrictions $E[v_t z_j] = 0$ all t, j, and study its consistency and asymptotic distribution.*

A GMM estimate can be defined for instruments $\mathbf{x}_t := \left(\mathbf{z}_t', \mathbf{z}_{t-1}^{\dagger\prime}\right)'$, where $\mathbf{z}_{t-1}^{\dagger}$ is not including the intercept (which guarantees that $\mathbb{E}[v_t] = 0$, all t), since $E[\mathbf{x}_t u_t] = 0$, by means of

$$\hat{\boldsymbol{\beta}}_{T,GMM} = \hat{\boldsymbol{\beta}}_T\left(\hat{\mathbf{W}}_T\right) = \left(\sum_{t=1}^{T}\left[\begin{array}{c}\mathbf{z}_t \\ y_{t-1}\end{array}\right]\mathbf{z}_t\mathbf{x}_t'\hat{\mathbf{W}}_T\sum_{t=1}^{T}\mathbf{x}_t[\mathbf{z}_t' \ \ y_{t-1}]\right)^{-1}\sum_{t=1}^{T}\left[\begin{array}{c}\mathbf{z}_t \\ y_{t-1}\end{array}\right]\mathbf{x}_t'\hat{\mathbf{W}}_T\sum_{t=1}^{T}\mathbf{x}_t y_t$$

for a particular weighting matrix $\hat{\mathbf{W}}_T \to \mathbf{W} > \mathbf{0}$.

If the $\mathbf{z}_t$ are strictly exogenous we could use an enlarged set of IV's for y_{t-1}, with leads and lags of $\mathbf{z}_t$, and also we could include some lags of y_t, such as $y_{t-2}, y_{t-3}, \ldots$

The complication in the asymptotics resides in the fact that the sequence $\mathbf{x}_t u_t$ need not be uncorrelated, so Newey-West type of asymptotic variances show up.

(d) *Study the consistency and asymptotic distribution of the OLS estimate of* β, *using the transformed model*

$$y_t^* = [\mathbf{z}_t^{*\prime} \;\; y_{t-1}^*]\boldsymbol{\beta} + u_t^*, \qquad t = 2, \ldots, T,$$

where

$$\begin{aligned} y_t^* &= y_t - \alpha y_{t-1} \\ \mathbf{z}_t^* &= \mathbf{z}_t - \alpha \mathbf{z}_t \\ u_t^* &= u_t - \alpha u_{t-1}. \end{aligned}$$

This model is equivalent to the model

$$y_t^* = [\mathbf{z}_t^{*\prime} \;\; y_{t-1}^*]\boldsymbol{\beta} + v_t, \qquad t = 2, \ldots, T,$$

where the error term v_t is IID and independent of the regressors, so OLS is consistent under the appropriate rank condition on the second moment matrix of $[\mathbf{z}_t^{*\prime} \;\; y_{t-1}^*]'$, $E\left([\mathbf{z}_t^{*\prime} \;\; y_{t-1}^*]'[\mathbf{z}_t^{*\prime} \;\; y_{t-1}^*]\right)$
In this case the asymptotic covariance matrix of the OLS estimates depend on $E\left([\mathbf{z}_t^{*\prime} \;\; y_{t-1}^*]'[\mathbf{z}_t^{*\prime} \;\; y_{t-1}^*]v_t^2\right)$.

3. *Consider a zero mean (scalar) time series MA(1) model,*

$$x_t = \theta_0 \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

where $\varepsilon_t \sim IndependentN\,(0,1)$.

(a) *Find the autocovariance sequence for* x_t *in terms of the parameters* θ_0 *and* θ_1.

x_t is a MA(1) process, so

$$\begin{aligned} \gamma_x(0) &= \theta_0^2 + \theta_1^2 \\ \gamma_x(1) &= \theta_0 \theta_1 \end{aligned}$$

and $\gamma_x(j) = 0$, $|j| > 1$.

(b) *Given a sample of* x_t, $t = 1, \ldots, T$, *define the corresponding (generalized) moment estimates of the parameters* $\theta = (\theta_0, \theta_1)'$ *in terms of the previous nonzero autocovariances, and investigate the rank condition for identification and the asymptotic properties of the estimates. [Hint:* $E\left[z^4\right] = 2\sigma_z^4$ *if* $z \sim N\left(0, \sigma_z^2\right)$*].*

The GMM estimate is defined in terms of $E\left[m_t\left(\theta^0\right)\right] = 0$ for the true θ^0, where

$$m_t(\theta) = \begin{bmatrix} x_t^2 - \left(\theta_0^2 + \theta_1^2\right) \\ x_t x_{t-1} - \theta_0 \theta_1 \end{bmatrix}.$$

Then we find that

$$\Xi(\theta) = E\left[\frac{\partial}{\partial \theta'} m_t(\theta)\right] = \begin{bmatrix} -2\theta_0 & -2\theta_1 \\ -\theta_1 & -\theta_0 \end{bmatrix},$$

which is of rank 2 as far as $\theta_1 \neq \theta_0$ (in which case the MA polynomial has a unit root).

The GMM estimates minimize

$$Q_T(b) = \left\{\frac{1}{T}\sum_{t=1}^{T} m_t(b)\right\}' \left\{\frac{1}{T}\sum_{t=1}^{T} m_t(b)\right\}$$

(note that the model is just identified, so we can set weighting $W_T = I_2$ wlog) so we need to consider the distribution of

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T} m_t\left(\theta^0\right) = \frac{1}{T^{1/2}}\sum_{t=1}^{T} \begin{bmatrix} x_t^2 - \left(\theta_0^2 + \theta_1^2\right) \\ x_t x_{t-1} - \theta_0 \theta_1 \end{bmatrix} := \frac{1}{T^{1/2}}\sum_{t=1}^{T} \mathbf{v}_t,$$

where $\mathbf{v}_t$ is no independent nor Gaussian, but zero mean and with finite dependence because x_t is $MA(1)$, so independent at lags larger than 1. Then we need a general CLT for $T^{-1/2}\sum_{t=1}^{T} m_t(\beta_0)$ and its AVar $\mathbf{V}$ would involve all the autocorrelations of $\mathbf{v}_t$.

Then, noting that

$$\Xi\left(\theta^0\right) = -\begin{bmatrix} 2\theta_0^0 & 2\theta_1^0 \\ \theta_1^0 & \theta_0^0 \end{bmatrix}$$

under $H_0, T^{1/2}\left(\hat{\beta} - \beta_0\right) \to_d N\left(0, \text{Avar}\left(\hat{\theta}\right)\right)$where $\text{Avar}\left(\hat{\theta}\right)$ is

$$\left(\begin{bmatrix} 2\theta_0^0 & 2\theta_1^0 \\ \theta_1^0 & \theta_0^0 \end{bmatrix}' \begin{bmatrix} 2\theta_0^0 & 2\theta_1^0 \\ \theta_1^0 & \theta_0^0 \end{bmatrix}\right)^{-1} \begin{bmatrix} 2\theta_0^0 & 2\theta_1^0 \\ \theta_1^0 & \theta_0^0 \end{bmatrix}' \mathbf{V} \begin{bmatrix} 2\theta_0^0 & 2\theta_1^0 \\ \theta_1^0 & \theta_0^0 \end{bmatrix} \left(\begin{bmatrix} 2\theta_0^0 & 2\theta_1^0 \\ \theta_1^0 & \theta_0^0 \end{bmatrix}' \begin{bmatrix} 2\theta_0^0 & 2\theta_1^0 \\ \theta_1^0 & \theta_0^0 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 2\theta_0^0 & 2\theta_1^0 \\ \theta_1^0 & \theta_0^0 \end{bmatrix} \mathbf{V} \begin{bmatrix} 2\theta_0^0 & 2\theta_1^0 \\ \theta_1^0 & \theta_0^0 \end{bmatrix}'.$$

(c) *Propose an iterative scheme to obtain the GMM estimates of θ and a Wald test for H_0,*

$$H_0 : \theta_1 = 0.$$

GMM numerical approximation:

$$\hat{\theta}_i = \hat{\theta}_{i-1} - \left(\sum_t \Xi_t \left(\hat{\theta}_{i-1}\right)' \sum_t \Xi_t \left(\hat{\theta}_{i-1}\right) \right)^{-1} \sum_t \Xi_t \left(\hat{\theta}_{i-1}\right)' \sum_{t=1}^{T} m_t \left(\hat{\theta}_{i-1}\right)$$

where

$$\Xi_t (\theta) = \frac{\partial}{\partial \theta'} m_t (\theta) = \begin{bmatrix} -2\theta_0 & -2\theta_1 \\ -\theta_1 & -\theta_0 \end{bmatrix},$$

so that

$$\hat{\theta}_i = \hat{\theta}_{i-1} - \left(\begin{bmatrix} -2\hat{\theta}_{0i-1} & -2\hat{\theta}_{1i-1} \\ -\hat{\theta}_{1i-1} & -\hat{\theta}_{0i-1} \end{bmatrix} \right)^{-1} \frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} -2\hat{\theta}_{0i-1} & -2\hat{\theta}_{1i-1} \\ -\hat{\theta}_{1i-1} & -\hat{\theta}_{0i-1} \end{bmatrix}' \begin{bmatrix} x_t^2 - \left(\hat{\theta}_{0i-1}^2 + \theta_{1i-1}^2\right) \\ x_t x_{t-1} - \hat{\theta}_{0i-1}\hat{\theta}_{1i-1} \end{bmatrix}$$

Since under the null $T_T^{1/2}\hat{\theta}_{1T} \to_d N(0, V_{11})$ we have that

$$Wald = T\hat{\theta}_{1T}^2 \hat{V}_{11}^{-1} \to_d \chi_1^2$$

for a consistent $\hat{V}_{11}$.

(d) *Consider restricted estimation of θ under H_0, and propose a Lagrange Multiplier test for H_0.*

The restricted estimation fixes $\tilde{\theta}_1 = 0$, so that we consider the following GMM objective function and estimate

$$\begin{aligned} \tilde{\theta}_0 &= \arg\min_b Q_T((b,0)) = \arg\min_b \frac{1}{T}\sum_{t=1}^{T} m_t((b,0))' \frac{1}{T}\sum_{t=1}^{T} m_t((b,0)) \\ &= \arg\min_b \left\{ \frac{1}{T}\sum_{t=1}^{T} \left(x_t^2 - \theta_0^2\right) \right\}^2. \end{aligned}$$

Then the LM test is

$$LM_T = TQ_{T,\theta_1}\left(\left(\tilde{\theta}_0, 0\right)\right)' \widehat{AVar}\left(Q_{T,\theta_1}\left(\left(\tilde{\theta}_0, 0\right)\right)\right) Q_{T,\theta_1}\left(\left(\tilde{\theta}_0, 0\right)\right),$$

where

$$\begin{aligned} Q_{T,\theta_1}\left(\left(\tilde{\theta}_0, 0\right)\right) &= \frac{\partial}{\partial \theta_1} Q_T(\theta)_{\theta = \left(\tilde{\theta}_0, 0\right)'} \\ &= \frac{2}{T}\sum_{t=1}^{T} \begin{bmatrix} x_t^2 - \left(\theta_0^2 + \theta_1^2\right) \\ x_t x_{t-1} - \theta_0\theta_1 \end{bmatrix}'_{\theta = \left(\tilde{\theta}_0, 0\right)'} \frac{1}{T}\sum_{t=1}^{T} \begin{bmatrix} -2\theta_1 \\ -\theta_0 \end{bmatrix}_{\theta = \left(\tilde{\theta}_0, 0\right)'} \\ &= \frac{2}{T}\sum_{t=1}^{T} \begin{bmatrix} x_t^2 - \tilde{\theta}_0 \\ x_t x_{t-1} \end{bmatrix}' \frac{1}{T}\sum_{t=1}^{T} \begin{bmatrix} 0 \\ -\tilde{\theta}_0 \end{bmatrix} \\ &= -\frac{2\tilde{\theta}_0}{T}\sum_{t=1}^{T} x_t x_{t-1} \\ &= -2\tilde{\theta}_0 \hat{\gamma}_T(1) \end{aligned}$$

and $AVar\left(Q_{T,\theta_1}\left(\left(\tilde{\theta}_0,0\right)\right)\right)$ is given, under $H_0:\theta_1=0$ (so that x_t is Gaussian White Noise) and $\tilde{\theta}_0:\widehat{Var}\left[x_t\right]\to_p\theta_0=Var\left[x_t\right]$ under H_0, by

$$\begin{aligned} 4\theta_0^2 Var\left[\frac{1}{T^{1/2}}\sum_{t=1}^T x_t x_{t-1}\right] &= 4\theta_0^2\sum_{j=-\infty}^{\infty} Cov\left(x_t x_{t-1}, x_{t-j}x_{t-1-j}\right) \\ &= 4\theta_0^4. \end{aligned}$$

Then, using $\widehat{AVar}=4\tilde{\theta}_0^4$,

$$LM_T = T\tilde{\theta}_0^{-2}\left(\frac{1}{T}\sum_{t=1}^T x_t x_{t-1}\right)^2=\left(\frac{1}{T^{1/2}}\sum_{t=1}^T\frac{x_t x_{t-1}}{\tilde{\theta}_0}\right)^2=T\hat{\rho}_T\left(1\right)^2\to_d\chi_1^2,$$

under H_0, which is the first standardized autocorrelation coefficient squared.