Econometrics II - EXAM

Answer each question in separate sheets in three hours

1. Consider the unobserved effects model for a randomly drawn cross section observation i,

$$y_{it} = \mathbf{x}_{it}'\beta + c_i + u_{it}, \quad t = 1, \ldots, T.$$

Denote $\mathbf{x}_i = (\mathbf{x}_{i1}', \ldots, \mathbf{x}_{iT}')'$ and $\mathbf{u}_i = (u_{i1}, \ldots, u_{iT})'$. Assume that the following conditions hold:

$$\begin{array}{rrcl} (i) & \mathbb{E}[u_{it}|\mathbf{x}_i, c_i] & = & 0, \quad t = 1, \ldots, T \\ (ii) & \mathbb{E}[c_i|\mathbf{x}_i] & = & 0. \end{array}$$

(a) Interpret conditions (i) and (ii). Do they guarantee consistency of OLS estimates when regressing y_{it} on $\mathbf{x}_{it}$ for $i = 1, \ldots, n$, $t = 1, \ldots, T$? Would your conclusions change if you change (i) by $(i*)$?

$$(i*) \quad \mathbb{E}[u_{it}|\mathbf{x}_i] = 0.$$

And if condition (ii) is replaced by $(ii*)$?

$$(ii*) \quad \sum_{i=1}^{n} c_i = 0.$$

And if condition (ii) is replaced by $(ii**)$?

$$(ii**) \quad \mathbb{E}[c_i|\mathbf{x}_i] = \mathbf{x}_i'\delta.$$

(b) Find $\mathrm{E}(\mathbf{v}_i\mathbf{v}_i'|\mathbf{x}_i)$ and $\mathrm{E}(\mathbf{v}_i\mathbf{v}_i')$, $\mathbf{v}_i = (v_{i1}, \ldots, v_{iT})'$,

$$v_{it} = u_{it} + c_i$$

under $(i) - (iv)$, with

$$\begin{array}{rcl} (iii)\ \mathbb{E}[\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i, c_i] & = & \mathbb{E}[\mathbf{u}_i\mathbf{u}_i'] \\ (iv)\ \mathbb{E}\left(c_i^2|\mathbf{x}_i\right) & = & \mathbb{E}\left(c_i^2\right). \end{array}$$

and analyze the asymptotic properties of the corresponding feasible GLS estimate of β based on consistent estimates of $\mathrm{E}(\mathbf{v}_i\mathbf{v}_i')$.

(c) Is the customary Random Effects estimator consistent under $(i)-(iv)$? If so, which estimate is more efficient asymptotically, the Random Effects estimate or the GLS estimate you proposed in (b)?

(d) Could you use usual diagnostics from the Pooled OLS under $(i) - (iv)$? And under $(i) - (v)$?

$$(v) \quad \mathbb{E}[\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i, c_i] = \sigma_u^2\mathbf{I}_T$$

(e) Find $\mathrm{E}(\mathbf{v}_i\mathbf{v}_i'|\mathbf{x}_i)$ and $\mathrm{E}(\mathbf{v}_i\mathbf{v}_i')$ under $(i) - (v)$ and under $(i*), (ii) - (v)$.

2. Consider the nonlinear simultaneous equation model

$$y_1 = \gamma_{12} y_2 + \gamma_{13} y_2^{\gamma_{14}} + \delta_{11} z_1 + \delta_{12} z_2 + u_1 \quad (1)$$
$$y_2 = \gamma_{21} y_1 + \delta_{22} z_2 + u_2. \quad (2)$$

(a) Study the identification of the system when $\gamma_{14} = 2$ and $\delta_{12} = 0$ are known and it is assumed that $\mathbb{E}(u_1|\mathbf{z}) = \mathbb{E}(u_2|\mathbf{z}) = 0$.

(b) Repeat the previous analysis when we do not have information on the value of δ_{12} (but still $\gamma_{14} = 2$ is known).

(c) Consider now the situation where it is known that $\delta_{12} = 0$, but we do not have information on γ_{14} and it has to be estimated along other parameters in the vector

$$\theta = \left(\gamma_{12}, \gamma_{13}, \gamma_{14}, \delta_{11}, \gamma_{21}, \delta_{22}, \sigma_1^2, \sigma_2^2\right)',$$

where $\mathbb{E}(u_1^2) = \sigma_1^2$, $\mathbb{E}(u_2^2) = \sigma_2^2$.

Analyze the identification of the system provided by the four sets of moment conditions given by $\mathbb{E}(u_1 \mathbf{z}) = \mathbb{E}(u_2 \mathbf{z}) = 0$ and $\mathbb{E}(\left(u_1^2 - \sigma_1^2\right) \mathbf{z}) = 0$, $\mathbb{E}(\left(u_2^2 - \sigma_2^2\right) \mathbf{z}) = 0$.

Is any equation identified when $\gamma_{14} = 0$ (but this is unknown).

(d) Repeat the analysis of part (c) when $\gamma_{14} = 1$ (but this is unknown).

(e) Repeat the analysis of part (c) when it is known that $\delta_{11} = \delta_{22}$.

3. Given zero mean (scalar) time series data x_t, $t = 1, \ldots, T$ we wish to test the null hypothesis of first order uncorrelation

$$H_0 : \rho_1 = 0.$$

For that we consider the moment conditions

$$m_t(\beta) = \begin{bmatrix} x_t^2 - \sigma^2 \\ x_t x_{t-1} - \rho_1 \sigma^2 \end{bmatrix}, \quad \beta = \begin{bmatrix} \sigma^2 \\ \rho_1 \end{bmatrix}$$

where σ^2 is the variance of x_t and $\rho_1 \sigma^2$ is the first-order autocovariance.

(a) Investigate the identification of the parameters σ^2 and ρ_1.

(b) Obtain the asymptotic distribution of the GMM estimates of σ^2 and ρ_1 under H_0 when x_t is $N(0, \sigma^2)$. [Hint: $E[z^4] = 2\sigma_z^4$ if $z \sim N(0, \sigma_z^2)$].

(c) Propose and iterative scheme to obtain the GMM estimates of β and a Wald test for H_0.

(d) Consider now the enlarged set of moment conditions

$$M_t(\theta) = \begin{bmatrix} x_t^2 - \sigma^2 \\ x_t x_{t-1} - \rho_1 \sigma^2 \\ x_t x_{t-2} - \rho_2 \sigma^2 \end{bmatrix}$$

where $\theta = (\sigma^2, \rho_1, \rho_2)'$ with ρ_1 and ρ_2 the first and second order autocorrelation coefficients, resp. Consider restricted estimation of θ using $M_t(\theta)$ under H_0^*,

$$H_0^* : \rho_2 = 0$$

and propose a Lagrange Multiplier test for H_0^*.

(e) Study the asymptotic distribution of the GMM estimates of σ^2 and ρ_1 defined by $m_t(\beta)$ in (b) when H_0 does not hold, $\rho_1 \neq 0$, and any additional conditions you may require.

Econometrics II - EXAM Outline Solutions
Answer each question in separate sheets in three hours

1. *Consider the unobserved effects model for a randomly drawn cross section observation i,*

$$y_{it} = \mathbf{x}_{it}'\beta + c_i + u_{it}, \quad t = 1, \ldots, T.$$

Denote $x_i = (\mathbf{x}_{i1}', \ldots, \mathbf{x}_{iT}')'$ and $u_i = (u_{i1}, \ldots, u_{iT})'$. Assume that the following conditions hold

$$\begin{aligned} (i) \quad \mathbb{E}[u_{it}|\mathbf{x}_i, c_i] &= 0, \quad t = 1, \ldots, T \\ (ii) \quad \mathbb{E}[c_i|\mathbf{x}_i] &= 0. \end{aligned}$$

(a) *Interpret conditions (i) and (ii). Do they guarantee consistency of OLS estimates when regressing y_{it} on x_{it} for $i = 1, \ldots, n$, $t = 1, \ldots, T$?*
Yes, consistent as $\mathbb{E}[c_i + u_{it}|\mathbf{x}_i] = 0$ and an appropriate rank condition holds, $\sum_{t=1}^{T} \mathbb{E}[\mathbf{x}_{it}\mathbf{x}_{it}'] > 0$.
Would your conclusions change if you change (i) by $(i*)$?

$$(i*) \quad \mathbb{E}[u_{it}|\mathbf{x}_i] = 0.$$

No, $\mathbb{E}[c_i + u_{it}|\mathbf{x}_i] = 0$ still holds.
And if condition (ii) is replaced by $(ii*)$?

$$(ii*) \quad \sum_{i=1}^{n} c_i = 0.$$

Yes, because we can not establish $\mathbb{E}[c_i + u_{it}|\mathbf{x}_i] = 0$.
And if condition (ii) is replaced by $(ii**)$?

$$(ii**) \quad \mathbb{E}[c_i|\mathbf{x}_i] = \mathbf{x}_i'\delta.$$

Yes, now POLS is inconsistent.

(b) *Find $E(\mathbf{v}_i\mathbf{v}_i'|\mathbf{x}_i)$ and $E(\mathbf{v}_i\mathbf{v}_i')$, $v_i = (v_{i1}, \ldots, v_{iT})'$,*

$$v_{it} = u_{it} + c_i$$

under $(i) - (iv)$, with

$$\begin{aligned} (iii) \ \mathbb{E}[\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i, c_i] &= \mathbb{E}[\mathbf{u}_i\mathbf{u}_i'] \\ (iv) \ \mathbb{E}\left(c_i^2|\mathbf{x}_i\right) &= \mathbb{E}\left(c_i^2\right) \end{aligned}$$

and analyze the asymptotic properties of the corresponding feasible GLS estimate of β based on consistent estimates of $E(\mathbf{v}_i\mathbf{v}_i')$.

$$\begin{aligned} E(\mathbf{v}_i\mathbf{v}_i'|\mathbf{x}_i) &= E(\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i) + E\left(c_i^2|\mathbf{x}_i\right) j_T j_T' + 2E(c_i\mathbf{u}_i|\mathbf{x}_i) j_T' \\ &= E(\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i) + E\left(c_i^2|\mathbf{x}_i\right) j_T j_T' \\ &= E(\mathbf{u}_i\mathbf{u}_i') + E\left(c_i^2\right) j_T j_T' \\ &= E(\mathbf{v}_i\mathbf{v}_i') \end{aligned}$$

by (i). GLS is as usual, under $\mathbb{E}\left[\mathbf{X}\boldsymbol{\Omega}^{-1}\mathbf{X}'\right] > 0$, with unrestricted $E(\mathbf{v}_i\mathbf{v}_i') = \Omega$, where we can check that $E\left[\mathbf{X}\Omega^{-1}\mathbf{v}\right] = 0$ by (i) and (ii) (assuming $E\left[\mathbf{X}\Omega^{-1}\mathbf{X}'\right] > 0$). Estimates of $\Omega > 0$ can be obtained from POLS residuals.

(c) *Is the customary Random Effects estimator consistent under* $(i)-(iv)$?

Yes, because $E\left[\mathbf{X}\mathbf{A}^{-1}\mathbf{v}\right]=0$ for any $\mathbf{A}>0$.

If so, which estimate is more efficient asymptotically, the Random Effects estimate or the GLS estimate you proposed in (b)*?*

In this case there is no guarantee that the RE is using the right weighting, while the feasible GLS is, so this one should be more efficient.

(d) *Could you use usual diagnostics and s.e.'s from the Pooled OLS under* $(i)-(iv)$*?*

No, because (iv) does not guarantee time uncorrelation of u_{it}.

And under $(i)-(v)$?

$$(v)\ \ \mathbb{E}\left[\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i,c_i\right]=\sigma_u^2\mathbf{I}_T$$

Still no, because the individual effects always induce serial dependence in v_{it}.

(e) *Find* $E(\mathbf{v}_i\mathbf{v}_i'|\mathbf{x}_i)$ *and* $E(\mathbf{v}_i\mathbf{v}_i')$ *under* $(i)-(v)$

This produces the usual RE variance,

$$E(\mathbf{v}_i\mathbf{v}_i'|\mathbf{x}_i)=\sigma_u^2I_t+E\left(c_i^2\right)j_Tj_T'.$$

and under $(i*),(ii)-(v)$.

In this case $E(c_i\mathbf{u}_i|\mathbf{x}_i)$ is not necessarily zero, which is implied by (i), so additional terms show up in all elements of Ω.

2. *Consider the simultaneous equation model*

$$y_1 = \gamma_{12}y_2+\gamma_{13}y_2^{\gamma_{14}}+\delta_{11}z_1+\delta_{12}z_2+u_1 \quad (3)$$
$$y_2 = \gamma_{21}y_1+\delta_{22}z_2+u_2. \quad (4)$$

(a) *Study the identification of the system when* $\gamma_{14}=2$ *and* $\delta_{12}=0$ *are known and it is assumed that* $E(u_1|\mathbf{z})=E(u_2|\mathbf{z})=0$.

See lecture notes and Wooldridge.

The second equation is always identified as far as $\delta_{12}\neq 0$, while the first one is identified for any value of γ_{13} as far as $\delta_{22}\neq 0$, since z_2 is a valid instrument for y_2 always, and z_1^2, z_2^2 and z_1z_2 are instruments for y_2^2.

(b) *Repeat the previous analysis when we do not have information on the value of* δ_{12} *(but still* $\gamma_{14}=2$ *is known).*

See lectures notes and Wooldridge.

The second equation is always identified as far as $\delta_{12}\neq 0$, while the first one is identified only for $\gamma_{13}\neq 0$ (and $\delta_{22}\neq 0$) since when the model is linear there are no valid instruments for y_2.

(c) *Consider now the situation where it is known that* $\delta_{12}=0$, *but we do not have information on* γ_{14} *and it has to be estimated along other parameters in the vector*

$$\theta=\left(\gamma_{12},\gamma_{13},\gamma_{14},\delta_{11},\gamma_{21},\delta_{22},\sigma_1^2,\sigma_2^2\right)',$$

where $E(u_1^2)=\sigma_1^2,\ E(u_2^2)=\sigma_2^2$.

Analyze the identification of the system provided by the four moment conditions given by $E(u_1z)=E(u_2z)=0$ *and* $E(\left(u_1^2-\sigma_1^2\right)z)=0,\ E(\left(u_2^2-\sigma_2^2\right)z)=0,$ *when* $\gamma_{14}=0$ *(but this is unknown).*

We have that the general moment condition for the whole system (the analysis could be done equation by equation since there are no cross-equations restrictions),

$$E\left[M\left(\theta\right)\right]:=E\left[\left(\begin{array}{c}\mathbf{z}u_1\left(\theta\right)\\ \mathbf{z}u_2\left(\theta\right)\\ \mathbf{z}\left[u_1^2\left(\theta\right)-\sigma_1^2\right]\\ \mathbf{z}\left[u_2^2\left(\theta\right)-\sigma_2^2\right]\end{array}\right)\right]=E\left[\left(\begin{array}{c}\mathbf{z}\left(y_1-\gamma_{12}y_2-\gamma_{13}y_2^{\gamma_{14}}-\delta_{11}z_1\right)\\ \mathbf{z}\left(y_2-\gamma_{21}y_1-\delta_{22}z_2\right)\\ \mathbf{z}\left[\left(y_1-\gamma_{12}y_2-\gamma_{13}y_2^{\gamma_{14}}-\delta_{11}z_1\right)^2-\sigma_1^2\right]\\ \mathbf{z}\left[\left(y_2-\gamma_{21}y_1-\delta_{22}z_2\right)^2-\sigma_2^2\right]\end{array}\right)\right]=0$$

Now we can consider the 8×8 matrix

$$\frac{\partial}{\partial\theta'}M\left(\theta\right)=\left(\begin{array}{cccccccc}-y_2 & -y_2^{\gamma_{14}} & -\gamma_{13}y_2^{\gamma_{14}}\log y_2 & -z_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -y_1 & -z_2 & 0 & 0\\ 2u_1\left(\theta\right)\left(\,-y_2\right. & -y_2^{\gamma_{14}} & -\gamma_{13}y_2^{\gamma_{14}}\log y_2 & -z_1 & 0 & \left.0\,\right) & -1 & 0\\ 2u_2\left(\theta\right)\left(\,0\right. & 0 & 0 & 0 & -y_1 & \left.-z_2\,\right) & 0 & -1\end{array}\right)\otimes\mathbf{z}$$

We need the expectation of this matrix at θ_0 to be full column rank to have local identification, but in the general case this depends on the cross moments between the variables of the system. If $\gamma_{14}=0$

$$E\left[\frac{\partial}{\partial\theta'}M\left(\theta_0\right)\right]=E\left(\begin{array}{cccccccc}-y_2 & -1 & -\gamma_{13}\log y_2 & -z_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -y_1 & -z_2 & 0 & 0\\ 2u_1(\,-y_2 & -1 & -\gamma_{13}\log y_2 & -z_1 & 0 & 0\,) & -1 & 0\\ 2u_2(\,0 & 0 & 0 & 0 & -y_1 & -z_2\,) & 0 & -1\end{array}\right)\otimes\mathbf{z}.$$

If $\gamma_{14}=0$ then the model is linear and

$$y_1=\gamma_{12}y_2+\gamma_{13}+\delta_{11}z_1+u_1.$$

From this expression and the corresponding (linear) reduced forms we cuold obtain all moments involving the variables y_i, u_i and z_i, and we can check whether the matrix $E\left[\frac{\partial}{\partial\theta'}M\left(\theta_0\right)\right]$ is of full column rank. A necessary condition is that $\gamma_{13}\neq 0$, because otherwise the matrix has a column of zeros and θ_0 would not be identified by the moment conditions.

(d) *Repeat the analysis of part (c) but only changing that $\gamma_{14}=1$ (but this is unknown). Discuss also the identification of each of the equations on its own.*

If $\gamma_{14}=1$, then the first equation has colinear regressors,

$$y_1=\gamma_{12}y_2+\gamma_{13}y_2+\delta_{11}z_1+u_1$$

and cannot be identified either. We can check this by

$$E\left[\frac{\partial}{\partial\theta'}M\left(\theta_0\right)\right]=E\left(\begin{array}{cccccccc}-y_2 & -y_2 & -\gamma_{13}y_2\log y_2 & -z_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -y_1 & -z_2 & 0 & 0\\ 2u_1\left(\theta\right)\left(\,-y_2\right. & -y_2 & -\gamma_{13}y_2\log y_2 & -z_1 & 0 & \left.0\,\right) & -1 & 0\\ 2u_2\left(\theta\right)\left(\,0\right. & 0 & 0 & 0 & -y_1 & \left.-z_2\,\right) & 0 & -1\end{array}\right)\otimes\mathbf{z}$$

and we can observe that the first two columns are the same, so rank is at most 7.

All these problems affect the first equation, the second one is always identified as far as $\delta_{12}\neq 0$ (o if $\gamma_{13}\neq 0$ and z_2 no constant wp1 in case (c)).

(e) *Repeat the analysis of part (c) when it is known that $\delta_{11}=\delta_{22}$.*

Now we can consider a reparametrization in terms of

$$\theta^*=\left(\gamma_{12},\gamma_{13},\gamma_{14},\delta_{11},\gamma_{21},\sigma_1^2,\sigma_2^2\right)',$$

eliminating δ_{22} from the system. Now the analysis can not be done equation by equation since there are cross-equations restrictions: the parameter δ_{11} shows up in two equations,

$$E[M(\theta)] := E\left[\begin{pmatrix} \mathbf{z}u_1(\theta) \\ \mathbf{z}u_2(\theta) \\ \mathbf{z}\left[u_1^2(\theta)-\sigma_1^2\right] \\ \mathbf{z}\left[u_2^2(\theta)-\sigma_2^2\right] \end{pmatrix}\right] = E\left[\begin{pmatrix} \mathbf{z}\left(y_1-\gamma_{12}y_2-\gamma_{13}y_2^{\gamma_{14}}-\delta_{11}z_1\right) \\ \mathbf{z}\left(y_2-\gamma_{21}y_1-\delta_{11}z_2\right) \\ \mathbf{z}\left[\left(y_1-\gamma_{12}y_2-\gamma_{13}y_2^{\gamma_{14}}-\delta_{11}z_1\right)^2-\sigma_1^2\right] \\ \mathbf{z}\left[\left(y_2-\gamma_{21}y_1-\delta_{11}z_2\right)^2-\sigma_2^2\right] \end{pmatrix}\right].$$

Now we can consider the 8×8 matrix

$$\frac{\partial}{\partial\theta'}M(\theta) = \begin{pmatrix} -y_2 & -y_2^{\gamma_{14}} & -\gamma_{13}y_2^{\gamma_{14}}\log y_2 & -z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z_2 & -y_1 & 0 & 0 \\ 2u_1(\theta)(\,-y_2 & -y_2^{\gamma_{14}} & -\gamma_{13}y_2^{\gamma_{14}}\log y_2 & -z_1 & 0\,) & -1 & 0 \\ 2u_2(\theta)(\,0 & 0 & 0 & -z_2 & -y_1\,) & 0 & -1 \end{pmatrix}\otimes\mathbf{z}$$

so when $\gamma_{14}=0$

$$E\left[\frac{\partial}{\partial\theta'}M(\theta_0)\right] = E\begin{pmatrix} -y_2 & -1 & -\gamma_{13}\log y_2 & -z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z_2 & -y_1 & 0 & 0 \\ 2u_1(\,-y_2 & -1 & -\gamma_{13}\log y_2 & -z_1 & 0\,) & -1 & 0 \\ 2u_2(\,0 & 0 & 0 & -z_2 & -y_1\,) & 0 & -1 \end{pmatrix}\otimes\mathbf{z},$$

which can be of rank 7 if $\gamma_{13}\neq 0$, so identification is possible because θ^* has dimension 7 now.

3. *Given zero mean (scalar) time series data x_t, $t=1,\ldots,T$ we wish to test the null hypothesis of first order uncorrelation*

$$H_0:\rho=0.$$

For that we consider the moment conditions

$$m_t(\beta)=\begin{bmatrix} x_t^2-\sigma^2 \\ x_tx_{t-1}-\rho\sigma^2 \end{bmatrix},\quad \beta=\begin{bmatrix}\sigma^2\\ \rho\end{bmatrix}$$

where σ^2 is the variance of x_t and $\rho\sigma^2$ is the first-order autocovariance.

(a) *Investigate the identification of the parameters σ^2 and ρ.*

For that we can consider the derivatives of the moment condition

$$\Xi(\beta)=E\left[\frac{\partial}{\partial\beta'}m_t(\beta)\right]=\begin{bmatrix}-1 & 0\\ -\rho & -\sigma^2\end{bmatrix}$$

so $\Xi(\beta_0)$ which is full rank if $\sigma_0^2>0$, with no restrictions on ρ_0 or on the distribution of x_t, so β is locally identified if $\sigma_0^2>0$.

(b) *Obtain the asymptotic distribution of the GMM estimates of σ^2 and ρ under H_0 with weighting $W_T=I_2$ when x_t is $N(0,\sigma^2)$. [Hint: $E[z^4]=2\sigma_z^4$ if $z\sim N(0,\sigma_z^2)$].*

The GMM estimates minimize

$$Q_T(b)=\left\{\frac{1}{T}\sum_{t=1}^T m_t(b)\right\}'\left\{\frac{1}{T}\sum_{t=1}^T m_t(b)\right\}$$

so we need to consider the distribution of

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T} m_t(\beta_0) = \frac{1}{T^{1/2}}\sum_{t=1}^{T}\left[\begin{array}{c} x_t^2-\sigma_0^2 \\ x_t x_{t-1}\end{array}\right].$$

Note that the model is just identified, so we can set weighting $W_T = I_2$ wlog.

Since x_t is Gaussian and under the null $\rho = 0$, we could assume that all autocorrelations $\rho(j) = 0$, $j \neq 0$, so that $x_t \sim iid\left(0,\sigma_0^2\right)$, $x_t^2-\sigma_0^2 \sim iid(0,\mu_4)$ where $\mu_4 = E\left[\left(x_t^2-\sigma_0^2\right)^2\right] = E\left[x_t^4\right]-\sigma_0^2 = 2\sigma_0^4$, $x_t x_{t-1} \sim iid\left(0,\sigma_0^4\right)$. Then

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T} m_t(\beta_0) \rightarrow_d N\left(0,\left[\begin{array}{cc} 2\sigma_0^2 & 0 \\ 0 & \sigma_0^4\end{array}\right]\right)$$

because $E\left[\left(x_t^2-\sigma_0^2\right)x_t x_{t-1}\right] = E\left[x_t^3 x_{t-1}\right] - \sigma_0^2 E\left[x_t x_{t-1}\right] = E\left[x_t^3\right]E\left[x_{t-1}\right] - 0 = 0$. Otherwise, if higher order $\rho(j) \neq 0$ for $j \neq 0$, then we need a more general CLT for $T^{-1/2}\sum_{t=1}^{T} m_t(\beta_0)$ and its AVar would involve all these autocorrelations.

Then, noting that

$$\Xi(\beta_0) = -\left[\begin{array}{cc} 1 & 0 \\ 0 & \sigma_0^2\end{array}\right]$$

under H_0,

$$T^{1/2}\left(\hat{\beta}-\beta_0\right) \rightarrow_d N\left(0, \text{Avar}\left(\hat{\beta}\right)\right)$$

where $\text{Avar}\left(\hat{\beta}\right)$ is

$$\left(\left[\begin{array}{cc} 1 & 0 \\ 0 & \sigma_0^2\end{array}\right]'\left[\begin{array}{cc} 1 & 0 \\ 0 & \sigma_0^2\end{array}\right]\right)^{-1}\left[\begin{array}{cc} 1 & 0 \\ 0 & \sigma_0^2\end{array}\right]'\left[\begin{array}{cc} 2\sigma_0^4 & 0 \\ 0 & \sigma_0^4\end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & \sigma_0^2\end{array}\right]\left(\left[\begin{array}{cc} 1 & 0 \\ 0 & \sigma_0^2\end{array}\right]'\left[\begin{array}{cc} 1 & 0 \\ 0 & \sigma_0^2\end{array}\right]\right)^{-1}$$

$$= \left[\begin{array}{cc} 1 & 0 \\ 0 & \sigma_0^{-4}\end{array}\right]\left[\begin{array}{cc} 2\sigma_0^4 & 0 \\ 0 & \sigma_0^8\end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & \sigma_0^{-4}\end{array}\right] = \left[\begin{array}{cc} 2\sigma_0^4 & 0 \\ 0 & 1\end{array}\right].$$

(c) *Propose an iterative scheme to obtain the GMM estimates of β and a Wald test for H_0.*

GMM numerical approximation:

$$\hat{\beta}_i = \hat{\beta}_{i-1} - \left(\sum_t \Xi_t\left(\hat{\beta}_{i-1}\right)'\sum_t \Xi_t\left(\hat{\beta}_{i-1}\right)\right)^{-1}\sum_t \Xi_t\left(\hat{\beta}_{i-1}\right)'\sum_{t=1}^{T} m_t\left(\hat{\beta}_{i-1}\right)$$

where

$$\Xi_t(\beta) = \frac{\partial}{\partial\beta'}m_t(\beta) = \left[\begin{array}{cc} -1 & 0 \\ -\rho & -\sigma^2\end{array}\right],$$

so that

$$\begin{aligned}\hat{\beta}_i &= \hat{\beta}_{i-1} - \left(\left[\begin{array}{cc} -1 & 0 \\ -\hat{\rho}_{i.1} & -\hat{\sigma}_{i-1}^2\end{array}\right]'\left[\begin{array}{cc} -1 & 0 \\ -\hat{\rho}_{i.1} & -\hat{\sigma}_{i-1}^2\end{array}\right]\right)^{-1}\left[\begin{array}{cc} -1 & 0 \\ -\hat{\rho}_{i-1} & -\hat{\sigma}_{i-1}^2\end{array}\right]'\frac{1}{T}\sum_{t=1}^{T}\left[\begin{array}{c} x_t^2-\hat{\sigma}_{i-1}^2 \\ x_t x_{t-1}-\hat{\rho}_{i-1}\hat{\sigma}_{i-1}^2\end{array}\right] \\ &= \hat{\beta}_{i-1} + \left(\left[\begin{array}{cc} 1+\hat{\rho}_{i.1}^2 & 0 \\ \hat{\sigma}_{i-1}^2\hat{\rho}_{i.1} & \hat{\sigma}_{i-1}^4\end{array}\right]\right)^{-1}\frac{1}{T}\sum_{t=1}^{T}\left[\begin{array}{c} x_t^2-\hat{\sigma}_{i-1}^2+\hat{\rho}_{i-1}\left(x_t x_{t-1}-\hat{\rho}_{i-1}\hat{\sigma}_{i-1}^2\right) \\ \hat{\sigma}_{i-1}^2\left(x_t x_{t-1}-\hat{\rho}_{i-1}\hat{\sigma}_{i-1}^2\right)\end{array}\right]\end{aligned}$$

or alternatively

$$\begin{aligned}
\hat{\beta}_i &= \hat{\beta}_{i-1} - \left(\begin{bmatrix} -1 & -\hat{\rho}_{i.1} \\ 0 & -\hat{\sigma}^2_{i-1} \end{bmatrix} \right)^{-1} \frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} x_t^2 - \hat{\sigma}^2_{i-1} \\ x_t x_{t-1} - \hat{\rho}_{i-1}\hat{\sigma}^2_{i-1} \end{bmatrix} \\
&= \hat{\beta}_{i-1} - \frac{1}{\hat{\sigma}^2_{i-1}} \begin{bmatrix} -\hat{\sigma}^2_{i-1} & \hat{\rho}_{i.1} \\ 0 & -1 \end{bmatrix} \frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} x_t^2 - \hat{\sigma}^2_{i-1} \\ x_t x_{t-1} - \hat{\rho}_{i-1}\hat{\sigma}^2_{i-1} \end{bmatrix} \\
&= \hat{\beta}_{i-1} + \frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} x_t^2 - \hat{\sigma}^2_{i-1} - \hat{\rho}_{i.1}\left(x_t x_{t-1} - \hat{\rho}_{i-1}\hat{\sigma}^2_{i-1}\right)/\hat{\sigma}^2_{i-1} \\ \left(x_t x_{t-1} - \hat{\rho}_{i-1}\hat{\sigma}^2_{i-1}\right)/\hat{\sigma}^2_{i-1} \end{bmatrix}
\end{aligned}$$

Since under the null $T^{1/2}\hat{\rho}_T \to_d N(0,1)$ we have that

$$Wald = T\hat{\rho}_T^2 \to_d \chi_1^2$$

(d) *Consider now the enlarged set of moment conditions*

$$M_t(\theta) = \begin{bmatrix} x_t^2 - \sigma^2 \\ x_t x_{t-1} - \rho_1 \sigma^2 \\ x_t x_{t-2} - \rho_2 \sigma^2 \end{bmatrix}$$

where $\theta = \left(\sigma^2, \rho_1, \rho_2\right)'$ *with* ρ_1 *and* ρ_2 *the first and second order autocorrelation coefficients, respectively. Consider the restricted estimation of* θ *using* $M_t(\theta)$ *under* H_0^*,

$$H_0^* : \rho_2 = 0$$

and propose a Lagrange Multiplier test for H_0^*.

The restricted estimation fixes $\tilde{\rho}_2 = 0$, and the estimation of ρ_1 and σ^2 is the same as before, since we have to consider the GMM objective function

$$\begin{aligned}
\arg\min_{s^2,\rho} Q_T\left(\left(s^2,\rho,0\right)\right) &= \arg\min_{s^2,\rho} \frac{1}{T}\sum_{t=1}^{T} M_t\left(\left(s^2,\rho,0\right)\right)' \frac{1}{T}\sum_{t=1}^{T} M_t\left(\left(s^2,\rho,0\right)\right) \\
&= \arg\min_{s^2,\rho} \frac{1}{T}\sum_{t=1}^{T} m_t\left(\left(s^2,\rho\right)\right)' \frac{1}{T}\sum_{t=1}^{T} m_t\left(\left(s^2,\rho\right)\right)
\end{aligned}$$

where

$$M_t\left(\left(s^2,\rho,0\right)\right) = \begin{bmatrix} x_t^2 - s^2 \\ x_t x_{t-1} - \rho\sigma^2 \\ x_t x_{t-2} \end{bmatrix}, \quad m_t\left(\left(s^2,\rho\right)\right) = \begin{bmatrix} x_t^2 - s^2 \\ x_t x_{t-1} - \rho\sigma^2 \end{bmatrix},$$

because the last moment in M_t does not depend on σ^2 or ρ.

Then the LM test is

$$LM_T = TQ_{T,\rho_2}\left(\left(\hat{\sigma}^2,\hat{\rho}_1,0\right)\right)' \widehat{AVar}\left(Q_{T,\rho_2}\left(\left(\hat{\sigma}^2,\hat{\rho}_1,0\right)\right)\right) Q_{T,\rho_2}\left(\left(\hat{\sigma}^2,\hat{\rho}_1,0\right)\right),$$

where

$$\begin{aligned}
Q_{T,\rho_2}\left(\left(\hat{\sigma}^2,\hat{\rho}_1,0\right)\right) &= \frac{\partial}{\partial\rho_2} Q_T(\theta)_{\theta=(\hat{\sigma}^2,\hat{\rho}_1,0)'} \\
&= -2\hat{\sigma}^2 \frac{1}{T}\sum_{t=1}^{T} x_t x_{t-2}
\end{aligned}$$

and $AVar\left(Q_{T,\rho_2}\left(\left(\hat{\sigma}^2,\hat{\rho}_1,0\right)\right)\right)$ is given, under $H_0^*:\rho_2=0$ (so that $E\left[x_t x_{t-2}\right]=0$) by

$$\begin{aligned} 4\sigma^4 Var\left[\frac{1}{T^{1/2}}\sum_{t=1}^{T}x_t x_{t-2}\right] &= 4\sigma^4\sum_{j=-\infty}^{\infty}Cov\left(x_t x_{t-2},x_{t-j}x_{t-2-j}\right) \\ &= 4\sigma^4 W. \end{aligned}$$

Then, using $\widehat{AVar}=4\hat{\sigma}^4\hat{W}$,

$$LM_T=T\hat{W}^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}x_t x_{t-2}\right)^2=\left(\frac{1}{T^{1/2}}\sum_{t=1}^{T}\frac{x_t x_{t-2}}{\hat{W}^{1/2}}\right)^2\to_d\chi_1^2,$$

under H_0^*, which is the second autocorrelation standardized coefficient squared.

(e) *Study the asymptotic distribution of the GMM estimates of σ^2 and ρ_1 defined by $m_t(\beta)$ in (b) when H_0 does not hold, $\rho_1\neq 0$, and any additional conditions you may require.*

Now we have to consider the distribution of

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T}m_t\left(\beta_0\right)=\frac{1}{T^{1/2}}\sum_{t=1}^{T}\left[\begin{array}{c} x_t^2-\sigma_0^2 \\ x_t x_{t-1}-\rho_{10}\sigma^2\end{array}\right].$$

Now, even if x_t is Gaussian since $\rho_1\neq 0$, then x_t, nor x_t^2 or $x_t x_{t-1}$ are iid or even uncorrelated, so we need a new CLT for

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T}m_t\left(\beta_0\right)\to N\left(0,V\right)$$

where V depends on the (cross) autocorrelations of x_t^2 and $x_t x_{t-1}$. Then, noting that still

$$\Xi\left(\beta_0\right)=-\left[\begin{array}{cc}1 & 0\\ 0 & \sigma_0^2\end{array}\right]$$

we get

$$T^{1/2}\left(\hat{\beta}-\beta_0\right)\to_d N\left(0,\mathrm{Avar}\left(\hat{\beta}\right)\right)$$

where $\mathrm{Avar}\left(\hat{\beta}\right)$ is

$$\begin{aligned} &\left(\left[\begin{array}{cc}1 & 0\\ 0 & \sigma_0^2\end{array}\right]'\left[\begin{array}{cc}1 & 0\\ 0 & \sigma_0^2\end{array}\right]\right)^{-1}\left[\begin{array}{cc}1 & 0\\ 0 & \sigma_0^2\end{array}\right]'V\left[\begin{array}{cc}1 & 0\\ 0 & \sigma_0^2\end{array}\right]\left(\left[\begin{array}{cc}1 & 0\\ 0 & \sigma_0^2\end{array}\right]'\left[\begin{array}{cc}1 & 0\\ 0 & \sigma_0^2\end{array}\right]\right)^{-1} \\ =& \left[\begin{array}{cc}1 & 0\\ 0 & \sigma_0^{-2}\end{array}\right]V\left[\begin{array}{cc}1 & 0\\ 0 & \sigma_0^{-2}\end{array}\right]=\left[\begin{array}{cc}V_{11} & 0\\ 0 & V_{12}/\sigma_0^2\end{array}\right]. \end{aligned}$$