Econometrics II - EXAM

Answer each question in separate sheets in three hours

1. Let u_1 and u_2 be jointly Gaussian and independent of z in all the equations.

   (a) Investigate the identification of the following system of simultaneous equations

   $$\begin{aligned} y_1 + \gamma_{12} y_2 + \delta_{11} + \delta_{12} z &= u_1 \\ y_2 + \delta_{21} + \delta_{22} z &= u_2. \end{aligned}$$

   How does your answer change if $\delta_{12} = 0$ but this is ignored? If some of the equations are not identified, propose a minimum set of additional restrictions that would exactly identify them.

   (b) Obtain the reduced form for the following system with nonlinear endogenous variables,

   $$y_1 + \gamma_{12} y_2 + \delta_{11} + \delta_{12} z = u_1 \quad (1)$$
   $$\log y_2 + \delta_{21} + \delta_{22} z = u_2, \quad (2)$$

   and check that it produces the following conditional expectations

   $$\begin{aligned} E[y_1|z] &= -\gamma_{12} \exp\left\{\frac{1}{2}\sigma_{22} - \delta_{21} - \delta_{22} z\right\} - \delta_{11} - \delta_{12} z \\ E[y_2|z] &= \exp\left\{\frac{1}{2}\sigma_{22} - \delta_{21} - \delta_{22} z\right\} \end{aligned}$$

   where $\sigma_{22} = Var[u_2]$ by using that for a Gaussian r.v. $X \sim N(0, \sigma^2)$ we have that $E[e^X] = \exp\left\{\frac{1}{2}\sigma^2\right\}$.

   (c) Analyze the identification of the two equations (1)-(2), proposing appropriate instrumental variables (IV) and studying the corresponding moment conditions needed.

   (d) If δ_{22} where known, which one would be your optimal choice of IV's for the first equation? Provide the asymptotic variance of the 2SLS estimates for this equation.

   (e) Analyze the identification when $\delta_{22} = 0$.

2. Consider the time series model

$$y_t = \beta y_{t-1} + v_t$$

wherein

$$v_t = \theta e_{t-1} + e_t$$

is a sequence of disturbances generated by a first-order moving average process which is driven by a white-noise sequence e_t of independently and identically distributed random variables with zero mean and variance σ_e^2.

   (a) Assuming that $|\beta| < 1$ obtain the first three elements of the autocovariance sequence of y_t, $\gamma_y(0), \gamma_y(1), \gamma_y(2)$. For which values of θ is y_t stationary?

   (b) Show that the estimate of β obtained by applying the ordinary least-squares procedure on the first equation, would tend to the value of $\beta + \theta\left(1 - \beta^2\right) / \left(1 + \theta\left(2\beta + \theta\right)\right)$ as the size of the sample increases.

   (c) Propose a test of $H_0 : \theta = 0$.

   (d) Imagine that the model

   $$y_t = \beta_1 y_{t-1} + \beta_2 y_{t-2} + error_t$$

   is fitted to the data by OLS. What values would you expect to obtain for β_1 and β_2 in the limit, as the size of the sample increases indefinitely, when $\beta = 0$?

   (e) If $\theta \neq 0$ were known, propose a consistent estimate of β and find its asymptotic distribution.

3. Consider the following nonlinear regression model,

$$y = h(\mathbf{z}; \theta_0) + v$$

with $E[y|\mathbf{z}] = h(z; \theta_0)$ and usual identification conditions hold.

(a) Consider the estimation by OLS of a misspecified linear regression model

$$y = \mathbf{z}'\beta + u$$

and find the solution for β that minimizes the variance of u. Would the OLS estimate of β be consistent for it? And for θ_0?

(b) Consider the following linearization around an arbitrary value θ^*,

$$\begin{aligned} y &= h(\mathbf{z}; \theta_0) + v \\ &\approx h(\mathbf{z}; \theta^*) + \frac{\partial h(\mathbf{z}; \theta^*)}{\partial \theta'} (\theta_0 - \theta^*) + v \end{aligned} \quad (3)$$

and set the following linear regression model,

$$y^* = \mathbf{z}^{*\prime}\theta_0 + error, \quad (4)$$

where

$$y^* = y - h(\mathbf{z}; \theta^*) + \frac{\partial h(\mathbf{z}; \theta^*)}{\partial \theta'}\theta^*, \quad \mathbf{z}^* = \frac{\partial h(\mathbf{z}; \theta^*)}{\partial \theta}.$$

Assuming that the approximation in (3) is perfect if we include a further term

$$a = (\theta_0 - \theta^*)' \frac{\partial^2 h\left(\mathbf{z}; \theta^\dagger\right)}{\partial \theta \partial \theta'} (\theta_0 - \theta^*),$$

study the probability limit of the OLS estimate $\tilde{\theta}_n$ of θ_0 in (4).

(c) Show that if $\theta^* \to_p \theta_0$, then $\tilde{\theta}_n \to_p \theta_0$ under standard identification assumptions, and that $\tilde{\theta}_n$ corresponds to a Gauss-Newton iteration of Nonlinear Least Squares (NLS) starting with $\theta = \theta^*$.

(d) Assume that $h(\mathbf{z}; \theta) = h(\mathbf{z}'\theta)$ and that $\mathbf{z}'\theta = \mathbf{z}_1'\theta_1 + \mathbf{z}_2'\theta_2$. Consider testing the hypothesis

$$H_0 : \theta_{10} = 0,$$

under the condition that $\mathrm{E}\left[v^2|\mathbf{z}\right] = \sigma^2$, by means of a Wald test based on NLS estimates. Provide a test statistic and the asymptotic distribution under the null.

(e) Find the distribution of the restricted estimate of θ_0 under H_0 and compare it with that of the unrestricted one. Propose now a Lagrange Multiplier test for H_0 and show that this amounts to a linear regression of restricted NLS residuals on an appropriate set of regressors.

Econometrics II - EXAM

SOLUTIONS

1. *Let u_1 and u_2 be jointly Gaussian and independent of z in all the equations.*

   (a) *Investigate the identification of the following system of simultaneous equations*

$$\begin{aligned} y_1 + \gamma_{12} y_2 + \delta_{11} + \delta_{12} z &= u_1 \\ y_2 + \delta_{21} + \delta_{22} z &= u_2. \end{aligned}$$

   2nd equation: is a reduced form, always identified.

   1st equation: the order condition is not satisfied, no omitted exogenous variables, is not identified in any case.

   How does your answer change if $\delta_{12} = 0$ but this is ignored?

   If this is ignored and not used, it would not help. If it is known, then eq. 1 will be identified if $\delta_{22} \neq 0$.

   If some of the equations are not identified, propose a minimum set of additional restrictions that would exactly identify them.

   Alternative conditions would be that it is known that $\delta_{12} = 0$, or that $\delta_{11} = 0$, or that $Cov(u_1, u_2) = 0$.

   (b) *Obtain the reduced form for the following system with nonlinear endogenous variables,*

$$\begin{aligned} y_1 + \gamma_{12} y_2 + \delta_{11} + \delta_{12} z &= u_1 \\ \log y_2 + \delta_{21} + \delta_{22} z &= u_2, \end{aligned}$$

   and check that it produces the following conditional expectations

$$\begin{aligned} E[y_1|z] &= -\gamma_{12} \exp\left\{\frac{1}{2}\sigma_{22} - \delta_{21} - \delta_{22} z\right\} - \delta_{11} - \delta_{12} z \\ E[y_2|z] &= \exp\left\{\frac{1}{2}\sigma_{22} - \delta_{21} - \delta_{22} z\right\} \end{aligned}$$

   where $\sigma_{22} = Var[u_2]$ by using that for a Gaussian r.v. $X \sim N(0, \sigma^2)$ we have that $E[e^X] = \exp\left\{\frac{1}{2}\sigma^2\right\}$.

   Using that $y_2 = \exp\{u_2 - \delta_{21} - \delta_{22} z\}$ the reduced form is given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} u_1 - \gamma_{12} \exp\{u_2 - \delta_{21} - \delta_{22} z\} - \delta_{11} - \delta_{12} z \\ \exp\{u_2 - \delta_{21} - \delta_{22} z\} \end{bmatrix},$$

   so that

$$E[y_2|z] = \exp\{-\delta_{21} - \delta_{22} z\} E[\exp\{u_2\}|z] = \exp\left\{\frac{1}{2}\sigma_{22} - \delta_{21} - \delta_{22} z\right\}.$$

   (c) *Analyze the identification of the two equations (1)-(2), proposing appropriate instrumental variables (IV) and studying the corresponding moment conditions needed.*

   The second equation is still identified, just defining $y_2^* = \log y_2$.

To identify the second equation we can now use $E[y_2|z]$ and expand $\exp\{-\delta_{22}z\}$ in power series, so any higher moment of z could be an IV for y_2 in eq. (1). Set $Z = \left(1, z, z^2\right)'$ and $X = (1, z, y_2)'$. Then we need

$$rank\{E[ZX']\} = 3,$$

where, $a = \exp\left\{\frac{1}{2}\sigma_{22} - \delta_{21}\right\}$,

$$\begin{aligned} E[ZX'] &= E\begin{bmatrix} 1 & z & y_2 \\ z & z^2 & zy_2 \\ z^2 & z^3 & z^2 y_2 \end{bmatrix} = E\begin{bmatrix} 1 & z & \exp\left\{\frac{1}{2}\sigma_{22} - \delta_{21} - \delta_{22}z\right\} \\ z & z^2 & z\exp\left\{\frac{1}{2}\sigma_{22} - \delta_{21} - \delta_{22}z\right\} \\ z^2 & z^3 & z^2\exp\left\{\frac{1}{2}\sigma_{22} - \delta_{21} - \delta_{22}z\right\} \end{bmatrix} \\ &= E\begin{bmatrix} 1 & z & a\exp\{-\delta_{22}z\} \\ z & z^2 & za\exp\{-\delta_{22}z\} \\ z^2 & z^3 & z^2 a\exp\{-\delta_{22}z\} \end{bmatrix}. \end{aligned}$$

(d) *If δ_{22} where known, which one would be your optimal choice of IV's for the first equation? Provide the asymptotic variance of the 2SLS estimates for this equation.*

In general the optimal IV's are $Z^* = \Omega(z) E[X|z] \doteq E[X|z] = (1, z, E[y_2|z])'$. Therefore in this case the optimal choice would be to take as extra IV $\exp\{-\delta_{22}z\}$ itself, so $Z^* = (1, z, \exp\{-\delta_{22}z\})'$ and

$$\begin{aligned} E[Z^*X'] &= E\begin{bmatrix} 1 & z & y_2 \\ z & z^2 & zy_2 \\ \exp\{-\delta_{22}z\} & z\exp\{-\delta_{22}z\} & \exp\{-\delta_{22}z\} y_2 \end{bmatrix} \\ &= E\begin{bmatrix} 1 & z & \exp\left\{\frac{1}{2}\sigma_{22} - \delta_{21} - \delta_{22}z\right\} \\ z & z^2 & z\exp\left\{\frac{1}{2}\sigma_{22} - \delta_{21} - \delta_{22}z\right\} \\ \exp\{-\delta_{22}z\} & z\exp\{-\delta_{22}z\} & \exp\{-\delta_{22}z\}\exp\left\{\frac{1}{2}\sigma_{22} - \delta_{21} - \delta_{22}z\right\} \end{bmatrix} \\ &= E\begin{bmatrix} 1 & z & a\exp\{-\delta_{22}z\} \\ z & z^2 & az\exp\{-\delta_{22}z\} \\ \exp\{-\delta_{22}z\} & z\exp\{-\delta_{22}z\} & a\exp\{-\delta_{22}z\}\exp\{-\delta_{22}z\} \end{bmatrix}, \end{aligned}$$

which produces almost the same result as OLS.

(e) *Analyze the identification when $\delta_{22} = 0$.*

In this case

$$E[ZX'] = E\begin{bmatrix} 1 & z & 1 \\ z & z^2 & za \\ z^2 & z^3 & z^2 a \end{bmatrix},$$

and clearly the first and third columns are proportional, so rank is at most 2. Identification relies on z appearing in the second equation.

2. *Consider the time series model*

$$y_t = \beta y_{t-1} + v_t$$

wherein

$$v_t = \theta e_{t-1} + e_t$$

is a sequence of disturbances generated by a first-order moving average process which is driven by a white-noise sequence e_t of independently and identically distributed random variables with zero mean and variance σ_e^2.

(a) *Assuming that* $|\beta| < 1$ *obtain the first three elements of the autocovariance sequence of* y_t, $\gamma_y(0), \gamma_y(1), \gamma_y(2)$. *For which values of* θ *is* y_t *stationary?*

We can write, since $|\beta| < 1$,

$$y_t = \sum_{j=0}^{\infty} \beta^j v_{t-j},$$

so that if v_t is covariance/strict stationary, so is y_t. But v_t is a MA on iid w.n., so it is always s.s. and cov. stationary.

Then

$$\begin{aligned}\gamma_y(0) &= Var[y_t] = \beta^2 Var[y_{t-1}] + Var[v_t] + 2\beta Cov[y_{t-1}, v_t] \\ &= \frac{1}{1-\beta^2}\left\{\sigma_e^2\left(1+\theta^2\right) + 2\beta\theta\sigma_e^2\right\} \\ &= \frac{\sigma_e^2}{1-\beta^2}\left\{1+\theta^2+2\beta\theta\right\}\end{aligned}$$

since

$$Cov[y_{t-1}, v_t] = Cov\left[\sum_{j=0}^{\infty}\beta^j v_{t-1-j}, v_t\right] = Cov\left[\beta^0 v_{t-1-0}, v_t\right] = Cov[v_{t-1}, v_t] = \theta\sigma_e^2.$$

Next,

$$\begin{aligned}\gamma_y(1) &= Cov[y_t, y_{t-1}] = \beta Var[y_{t-1}] + Cov[y_{t-1}, v_t] = \beta\gamma_y(0) + Cov[y_{t-1}, v_t] \\ &= \beta\frac{\sigma_e^2}{1-\beta^2}\left\{1+\theta^2+2\beta\theta\right\} + \theta\sigma_e^2 \\ &= \sigma_e^2\left\{\frac{\beta}{1-\beta^2}\left[1+\theta^2+2\beta\theta\right]+\theta\right\}\end{aligned}$$

and

$$\begin{aligned}\gamma_y(2) &= Cov[y_t, y_{t-2}] = \beta Cov[y_{t-1}, y_{t-2}] + Cov[y_{t-2}, v_t] = \beta\gamma_y(1) + 0 \\ &= \beta\gamma_y(1).\end{aligned}$$

(b) *Show that the estimate of* β *obtained by applying the ordinary least-squares procedure on the first equation, would tend to the value of* $\beta + \theta\left(1-\beta^2\right)/\left(1+\theta\left(2\beta+\theta\right)\right)$ *as the size of the sample increases.*

We find that

$$\begin{aligned}p\lim\hat{\beta} &= \beta + \frac{Cov(y_{t-1}, v_t)}{Var(y_{t-1})} \\ &= \beta + \frac{\theta\sigma_e^2}{\gamma_y(0)} = \beta + \frac{\theta\left(1-\beta^2\right)}{1+\theta(\theta+2\beta)}\end{aligned}$$

or

$$\begin{aligned}p\lim\hat{\beta} &= \frac{Cov(y_{t-1}, y_t)}{Var(y_{t-1})} = \frac{\gamma(1)}{\gamma(0)} \\ &= \frac{\beta\gamma(0)+\theta\sigma_e^2}{\gamma(0)} = \beta + \frac{\theta\left(1-\beta^2\right)}{1+\theta(\theta+2\beta)}.\end{aligned}$$

(c) *Propose a test of* $H_0 : \theta = 0$.

If $\theta = 0$, then the OLSE of β is consistent, so that the residuals $\hat{v}_t = y_t - \hat{\beta} y_{t-1}$, should be approximately uncorrelated. This can be tested using uncorrelation tests based on $\hat{\rho}_{\hat{v}}(j)$, such as Box-Pierce. Alternatively, LM tests could be derived for an MA model, which implies testing for $\rho(1) = 0$.

(d) *Imagine that the model*

$$y_t = \beta_1 y_{t-1} + \beta_2 y_{t-2} + error_t$$

is fitted to the data by OLS. What values would you expect to obtain for β_1 *and* β_2 *in the limit, as the size of the sample increases indefinitely, when* $\beta = 0$*?*

We have that in this case y_t is just a MA(1),

$$\begin{aligned} \gamma_y(0) &= \sigma_e^2 \left\{1 + \theta^2\right\} \\ \gamma_y(1) &= \sigma_e^2 \theta \\ \gamma_y(2) &= 0, \end{aligned}$$

so that the OLSE will converge to the corresponding population parameters,

$$\begin{aligned} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} &= \begin{pmatrix} \gamma_y(0) & \gamma_y(1) \\ \gamma_y(1) & \gamma_y(0) \end{pmatrix}^{-1} \begin{pmatrix} \gamma_y(1) \\ \gamma_y(2) \end{pmatrix} \\ &= \begin{pmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{pmatrix}^{-1} \begin{pmatrix} \theta \\ 0 \end{pmatrix} \\ &= \frac{1}{\left(1+\theta^2\right)^2 - \theta^2} \begin{pmatrix} 1+\theta^2 & -\theta \\ -\theta & 1+\theta^2 \end{pmatrix} \begin{pmatrix} \theta \\ 0 \end{pmatrix} \\ &= \frac{1}{\left(1+\theta^2\right)^2 - \theta^2} \begin{pmatrix} \left(1+\theta^2\right)\theta \\ -\theta^2 \end{pmatrix}, \end{aligned}$$

and in particular $\beta_1 \neq \beta = 0$.

(e) *If* $\theta \neq 0$ *were known, propose a consistent estimate of* β *and find its asymptotic distribution.*

The idea would be to implement a GLS estimate, so that

$$y_t^* = \beta y_{t-1}^* + v_t^*$$

and the v_t^*, $t = 1, \ldots, T$ are uncorrelated. Note that in this case $v_t \sim MA(1) \sim AR(\infty)$ with coefficients $\alpha_j = (-\theta)^j$ as far as $|\theta| < 1$, so v_t^* depends on v_j, $j = 1, \ldots, t$, and when $t \to \infty$, $v_t^* \to e_t$. Then this regression is asymptotically equivalent to regress

$$y_t^\dagger = \beta y_{t-1}^\dagger + e_t$$

where $y_t^\dagger = \sum_{j=0}^{\infty} (-\theta)^j y_{t-j}$.

3. *Consider the following nonlinear regression model,*

$$y = h(\mathbf{z}; \theta_0) + v$$

with $E[y|\mathbf{z}] = h(z; \theta_0)$ *and usual identification conditions hold.*

(a) *Consider the estimation by OLS of a misspecified linear regression model*

$$y = \mathbf{z}'\beta + u$$

and find the solution for β that minimizes the variance of u. Would the OLS estimate of β be consistent for it? And for θ_0?

As usual

$$\begin{aligned}\beta_0 &= \arg\min_b E\left[(y-\mathbf{z}'\beta)^2\right] = E[\mathbf{z}\mathbf{z}']^{-1}E[\mathbf{z}y] \\ &= E[\mathbf{z}\mathbf{z}']^{-1}E[\mathbf{z}h(\mathbf{z};\theta_0)],\end{aligned}$$

which can be estimated consistently by OLS, while

$$\theta_0 = \arg\min_\theta E\left[(y-h(\mathbf{z};\theta))^2\right].$$

Of course $\beta_0 \neq \theta_0$ in general unless $h(\mathbf{z};\theta) = \mathbf{z}'\theta$.

(b) *Consider the following linearization around an arbitrary value θ^*,*

$$\begin{aligned} y &= h(\mathbf{z};\theta_0) + v \\ &\approx h(\mathbf{z};\theta^*) + \frac{\partial h(\mathbf{z};\theta^*)}{\partial\theta'}(\theta_0-\theta^*) + v\end{aligned} \tag{5}$$

and set the following linear regression model,

$$y^* = \mathbf{z}^{*\prime}\theta_0 + error, \tag{6}$$

where

$$y^* = y - h(\mathbf{z};\theta^*) + \frac{\partial h(\mathbf{z};\theta^*)}{\partial\theta'}\theta^*, \quad \mathbf{z}^* = \frac{\partial h(\mathbf{z};\theta^*)}{\partial\theta}.$$

Assuming that the approximation in (5) is perfect if we include a further term $a = (\theta_0-\theta^*)'\frac{\partial^2 h(\mathbf{z};\theta^\dagger)}{\partial\theta\partial\theta'}(\theta_0-\theta^*)$, study the probability limit of the OLS estimate $\tilde\theta_n$ of θ_0 in (6).

We have that

$$\begin{aligned}\tilde\theta_n &= E_n[\mathbf{z}^*\mathbf{z}^{*\prime}]^{-1}E_n[\mathbf{z}^*y^*] \\ &= E_n[\mathbf{z}^*\mathbf{z}^{*\prime}]^{-1}E_n\left[\mathbf{z}^*\left(y-h(\mathbf{z};\theta^*)+\frac{\partial h(\mathbf{z};\theta^*)}{\partial\theta'}\theta^*\right)\right] \\ &= E_n[\mathbf{z}^*\mathbf{z}^{*\prime}]^{-1}E_n\left[\mathbf{z}^*\left(\frac{\partial h(\mathbf{z};\theta^*)}{\partial\theta'}\theta_0+a+v\right)\right] \\ &= E_n[\mathbf{z}^*\mathbf{z}^{*\prime}]^{-1}E_n[\mathbf{z}^*\mathbf{z}^{*\prime}]\theta_0 + E_n[\mathbf{z}^*\mathbf{z}^{*\prime}]^{-1}E_n[\mathbf{z}^*a] + E_n[\mathbf{z}^*\mathbf{z}^{*\prime}]^{-1}E_n[\mathbf{z}^*v] \\ &= \theta_0 + E[\mathbf{z}^*\mathbf{z}^{*\prime}]^{-1}E[\mathbf{z}^*a] + E[\mathbf{z}^*\mathbf{z}^{*\prime}]^{-1}E[\mathbf{z}^*v] + o_p(1) \\ &= \theta_0 + b + o_p(1),\end{aligned}$$

where $b = E[\mathbf{z}^*\mathbf{z}^{*\prime}]^{-1}E[\mathbf{z}^*a]$ is not necessarily negligible since θ^* could be far away from θ_0.

(c) *Show that if $\theta^* \to_p \theta_0$, then $\tilde\theta_n \to_p \theta_0$ under standard identification assumptions, and that $\tilde\theta_n$ corresponds to a Gauss-Newton iteration of Nonlinear Least Squares (NLS) starting with $\theta = \theta^*$.*

In this case we can show that $E_n[\mathbf{z}^*a] = O_p\left(\|\theta^*-\theta_0\|^2\right)E_n\left[\mathbf{z}^*\left\|\frac{\partial^2 h(\mathbf{z};\theta^\dagger)}{\partial\theta\partial\theta'}\right\|\right] = o_p(1)\,O_p(1) = o_p(1)$ under standard moment conditions, so $\tilde\theta_n = \theta_0 + o_p(1)$. Then for $Q_n(\theta) = \frac{1}{2}E_n\left[(y-h(\mathbf{z};\theta_0))^2\right]$,

$$\begin{aligned}\frac{\partial}{\partial\theta}Q_n(\theta^*) &= -E_n\left[(y-h(\mathbf{z};\theta^*))\frac{\partial h(\mathbf{z};\theta^*)}{\partial\theta}\right] \\ &= -E_n\left[\left(y^*-\frac{\partial h(\mathbf{z};\theta^*)}{\partial\theta'}\theta^*\right)\mathbf{z}^*\right] \\ &= -E_n[y^*\mathbf{z}^*] + E_n[\mathbf{z}^{*\prime}\theta^*\mathbf{z}^*] \\ &= -E_n[\mathbf{z}^*y^*] + E_n[\mathbf{z}^*\mathbf{z}^{*\prime}]\theta^*\end{aligned}$$

while

$$\begin{aligned}\frac{\partial^2}{\partial\theta\partial\theta'}Q_n(\theta^*) &= E_n\left[\frac{\partial h(\mathbf{z};\theta^*)}{\partial\theta}\frac{\partial h(\mathbf{z};\theta^*)}{\partial\theta'}\right] - E_n\left[(y - h(\mathbf{z};\theta^*))\frac{\partial^2 h(\mathbf{z};\theta^*)}{\partial\theta\partial\theta^2}\right] \\ &\approx E_n[\mathbf{z}^*\mathbf{z}^{*\prime}] := G_n\end{aligned}$$

so that the Gauss Newton iterations are

$$\begin{aligned}\tilde{\theta}_n &= \theta^* - (G_n)^{-1}\frac{\partial}{\partial\theta}Q_n(\theta^*) \\ &= \theta^* - (E_n[\mathbf{z}^*\mathbf{z}^{*\prime}])^{-1}\{-E_n[\mathbf{z}^*y^*] + E_n[\mathbf{z}^*\mathbf{z}^{*\prime}]\theta^*\} \\ &= (E_n[\mathbf{z}^*\mathbf{z}^{*\prime}])^{-1}E_n[\mathbf{z}^*y^*].\end{aligned}$$

(d) *Assume that* $h(\mathbf{z};\theta) = h(\mathbf{z}'\theta)$ *and that* $z'\theta = z_1'\theta_1 + z_2'\theta_2$. *Consider testing the hypothesis*

$$H_0 : \theta_{10} = 0,$$

under the condition that $E[v^2|\mathbf{z}] = \sigma^2$, *by means of a Wald test based on NLS estimates. Provide a test statistic and the asymptotic distribution under the null.*

Wald test. We assume that the unrestricted estimate satisfies

$$n^{1/2}\left(\hat{\theta}_n - \theta_0\right) \to_d N\left(0, \mathbf{E}^{-1}\mathbf{D}\mathbf{E}^{-1}\right) \equiv \left(0, \sigma^2\mathbf{E}^{-1}\right)$$

where

$$\begin{aligned}\mathbf{E} &= p\lim\frac{\partial^2}{\partial\theta\partial\theta'}Q_n(\theta_0) = E[\mathbf{z}^*\mathbf{z}^{*\prime}] \\ \mathbf{D} &= Var\left(\frac{\partial}{\partial\theta}Q_n(\theta_0)\right) = E\left[v^2\mathbf{z}^*\mathbf{z}^{*\prime}\right] = \sigma^2 E[\mathbf{z}^*\mathbf{z}^{*\prime}] = \sigma^2\mathbf{E}.\end{aligned}$$

Note that $\partial h(\mathbf{z};\theta^*)/\partial\theta = \partial h(\mathbf{z}'\theta^*)/\partial\theta = \dot{h}(\mathbf{z}'\theta^*)\mathbf{z}$, where $\dot{h}(x) = \partial h(x)/\partial x$, so that $\mathbf{z}^* = \dot{h}(\mathbf{z}'\theta^*)\mathbf{z}$. Then the test statistics is

$$W_n = \frac{n}{\hat{\sigma}^2}\hat{\theta}_{1n}'\left\{\hat{\mathbf{E}}^{-1}\right\}_{1,1}^{-1}\hat{\theta}_{1n} \to_d \chi^2_{\#(\theta_1)}$$

where $\hat{\mathbf{E}} := E_n[\mathbf{z}^*\mathbf{z}^{*\prime}]$ and $\hat{\sigma}^2$ is the residual variance.

(e) *Find the distribution of the restricted estimate of* θ_0 *under* H_0 *and compare it with that of the unrestricted one. Propose now a Lagrange Multiplier test for* H_0 *and show that this amounts to a linear regression of restricted NLS residuals on an appropriate set of regressors.*

LM Test. In this case we need to study the restricted estimates of θ_0, so that

$$\bar{\theta}_n = \arg\min_{H_0} Q_n(\theta)$$

where $\bar{\theta}_n = \left(0', \bar{\theta}_{2n}'\right)'$ and we set $\mathcal{L}_n(\theta) = Q_n(\theta) - \lambda'\theta_1$. Then

$$\begin{aligned}\frac{\partial}{\partial\theta_1}Q_n(\bar{\theta}_n) &= \bar{\lambda} = \frac{\partial}{\partial\theta_1}Q_n(\theta_0) + \frac{\partial^2}{\partial\theta_1\partial\theta_2'}Q_n(\theta_0)\left(\bar{\theta}_{2n} - \theta_{20}\right) + o_p(1) \\ \frac{\partial}{\partial\theta_2}Q_n(\bar{\theta}_n) &= 0 = \frac{\partial}{\partial\theta_2}Q_n(\theta_0) + \frac{\partial^2}{\partial\theta_2\partial\theta_2'}Q_n(\theta_0)\left(\bar{\theta}_{2n} - \theta_{20}\right) + o_p(1)\end{aligned}$$

so that

$$\begin{aligned}n^{1/2}\left(\bar{\theta}_{2n} - \theta_{20}\right) &= -\left(\frac{\partial^2}{\partial\theta_2\partial\theta_2'}Q_n(\theta_0)\right)^{-1} n^{1/2}\frac{\partial}{\partial\theta_2}Q_n(\theta_0) + o_p(1) \\ &\to_d N\left(0, \mathbf{E}_{22}^{-1}\mathbf{D}_{22}\mathbf{E}_{22}^{-1}\right) = N\left(0, \sigma^2\mathbf{E}_{22}^{-1}\right),\end{aligned}$$

while $n^{1/2}\left(\hat{\theta}_{2n}-\theta_{20}\right)\rightarrow N\left(0,\sigma^2\left\{\mathbf{E}^{-1}\right\}_{2,2}\right)$ and $\left\{\mathbf{E}^{-1}\right\}_{2,2}\neq\mathbf{E}_{22}^{-1}$ unless $\mathbf{E}_{12}=\mathbf{E}_{21}'=0$. Also

$$
\begin{aligned}
n^{1/2}\bar{\lambda} &= n^{1/2}\frac{\partial}{\partial\theta_1}Q_n\left(\theta_0\right)+\mathbf{E}_{12}\left(\bar{\theta}_{2n}-\theta_{20}\right)+o_p\left(1\right)\\
&= n^{1/2}\frac{\partial}{\partial\theta_1}Q_n\left(\theta_0\right)-\mathbf{E}_{12}\mathbf{E}_{22}^{-1}n^{1/2}\frac{\partial}{\partial\theta_2}Q_n\left(\theta_0\right)+o_p\left(1\right)\\
&= \left(I \quad -\mathbf{E}_{12}\mathbf{E}_{22}^{-1}\right)n^{1/2}\frac{\partial}{\partial\theta}Q_n\left(\theta_0\right)+o_p\left(1\right)\\
\rightarrow_d\quad & N\left(0,\sigma^2\left(I \quad -\mathbf{E}_{12}\mathbf{E}_{22}^{-1}\right)\mathbf{E}\left(I \quad -\mathbf{E}_{12}\mathbf{E}_{22}^{-1}\right)'\right)\\
&= N\left(0,\sigma^2\left\{\mathbf{E}_{11}-\mathbf{E}_{12}\mathbf{E}_{22}^{-1}\mathbf{E}_{21}\right\}\right)\\
&= N\left(0,\sigma^2\left\{\mathbf{E}^{-1}\right\}_{11}^{-1}\right)
\end{aligned}
$$

and the test statistic is

$$
LM_n=n\bar{\lambda}'\left\{\hat{\mathbf{E}}^{-1}\right\}_{11}\bar{\lambda}\rightarrow_d\chi^2_{\#(\theta_1)}.
$$

Now note that

$$
\begin{aligned}
\bar{\lambda} &= \frac{\partial}{\partial\theta_1}Q_n\left(\bar{\theta}_n\right)=-E_n\left[\left(y-h\left(\mathbf{z}'\bar{\theta}_n\right)\right)\frac{\partial h\left(\mathbf{z}'\bar{\theta}_n\right)}{\partial\theta_1}\right]\\
&= -E_n\left[\left(y-h\left(\mathbf{z}_2'\bar{\theta}_{2n}\right)\right)\dot{h}\left(\mathbf{z}_2'\bar{\theta}_{2n}\right)\mathbf{z}_1\right]\\
&= -E_n\left[\bar{v}\left(\bar{\theta}_n\right)\dot{h}\left(\mathbf{z}_2'\bar{\theta}_{2n}\right)\mathbf{z}_1\right]
\end{aligned}
$$

which can be obtained -up to scale- from running a linear regression of the residuals $\bar{v}\left(\bar{\theta}_n\right)$ on the additional linearized regressors $\dot{h}\left(\mathbf{z}_2'\bar{\theta}_{2n}\right)\mathbf{z}_1$.