Econometrics II - EXAM

Answer each question in separate sheets in three hours

1. *Consider the unobserved effects model for a randomly drawn cross section observation* i,

$$y_{it} = \mathbf{x}_{it}'\beta + c_i + u_{it}, \quad t = 1, \ldots, T.$$

Denote $x_i = (\mathbf{x}_{i1}', \ldots, \mathbf{x}_{iT}')'$ *and* $u_i = (u_{i1}, \ldots, u_{iT})'$. *Assume that the following conditions*

$$\begin{aligned} (i) \quad \mathbb{E}[u_{it}|\mathbf{x}_i, c_i] &= 0, \quad t = 1, \ldots, T \\ (ii) \quad \mathbb{E}[c_i|\mathbf{x}_i] &= \mathbb{E}[c_i] = 0 \\ (iii)\ \mathbb{E}[\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i, c_i] &= \sigma_u^2 \mathbf{I}_T \end{aligned}$$

hold, but $Var(c_i|\mathbf{x}_i) \neq Var(c_i) = \sigma_c^2$.

(a) *Describe the general nature of* $E(\mathbf{v}_i\mathbf{v}_i'|\mathbf{x}_i)$, $v_i = (v_{i1}, \ldots, v_{iT})'$,

$$v_{it} = u_{it} + c_i.$$

Writing

$$\mathbf{v}_i = \mathbf{u}_i + c_i\mathbf{j}_T,$$

we have that

$$\begin{aligned} E(\mathbf{v}_i\mathbf{v}_i'|\mathbf{x}_i) &= E(\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i) + E(\mathbf{u}_i c_i|\mathbf{x}_i)\mathbf{j}_T' + \mathbf{j}_T E(\mathbf{u}_i' c_i|\mathbf{x}_i) + \mathbf{j}_T\mathbf{j}_T' E\left(c_i^2|\mathbf{x}_i\right) \\ &= \sigma_u^2\mathbf{I}_T + 0 + 0 + \mathbf{j}_T\mathbf{j}_T' Var(c_i|\mathbf{x}_i) \\ &\neq \sigma_u^2\mathbf{I}_T + \mathbf{j}_T\mathbf{j}_T'\sigma_c^2. \end{aligned}$$

by $(iii), (i), (i)$ and $Var(c_i|\mathbf{x}_i) \neq Var(c_i) = \sigma_c^2$.

(b) *What are the asymptotic properties of the Random Effects estimator for this model? State any additional conditions you need.*

The random effects estimator is the FGLS estimate

$$\hat{\boldsymbol{\beta}}_{RE} = \mathbb{E}_n\left[\mathbf{X}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}'\right]^{-1}\mathbb{E}_n\left[\mathbf{X}\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y}\right]$$

where

$$\hat{\boldsymbol{\Omega}}_n = \hat{\sigma}_u^2\mathbf{I}_T + \hat{\sigma}_c^2\mathbf{j}_T\mathbf{j}_T'$$

for consistent estimates of σ_u^2 and σ_c^2. For the analysis we need that rank $\mathbb{E}\left[\mathbf{X}_i\boldsymbol{\Omega}^{-1}\mathbf{X}_i'\right] = K$, and

$$p\lim_{n\to\infty}\hat{\boldsymbol{\Omega}}_n = \boldsymbol{\Omega} = E(\mathbf{v}_i\mathbf{v}_i')$$

where

$$\hat{\boldsymbol{\Omega}}_n = \hat{\sigma}_u^2\mathbf{I}_T + \hat{\sigma}_c^2\mathbf{j}_T\mathbf{j}_T'$$

and $\hat{\sigma}_u^2$ and $\hat{\sigma}_c^2$ are consistent estimates of σ_u^2 and σ_c^2, resp. However $\boldsymbol{\Omega} \neq E(\mathbf{v}_i\mathbf{v}_i'|\mathbf{x}_i)$. We could also allow for $p\lim_{n\to\infty}\hat{\boldsymbol{\Omega}}_n = \boldsymbol{\Omega}^* \neq E(\mathbf{v}_i\mathbf{v}_i')$ Then the RE estimate is consistent and asymptotic normal. The asymptotic variance of $\hat{\boldsymbol{\beta}}_{RE}$ is

$$\mathbb{E}\left[\mathbf{X}\boldsymbol{\Omega}^{-1}\mathbf{X}'\right]^{-1}\mathbb{E}\left[\mathbf{X}\boldsymbol{\Omega}^{-1}\mathbf{v}\mathbf{v}'\boldsymbol{\Omega}^{-1}\mathbf{X}\right]\mathbb{E}\left[\mathbf{X}\boldsymbol{\Omega}^{-1}\mathbf{X}'\right]^{-1}$$

where $\mathbb{E}\left[\mathbf{X}\boldsymbol{\Omega}^{-1}\mathbf{v}\mathbf{v}'\boldsymbol{\Omega}^{-1}\mathbf{X}\right] \neq \mathbb{E}\left[\mathbf{X}\boldsymbol{\Omega}^{-1}\mathbf{X}\right]$.

(c) *How should the random effects Wald test statistics be modified to have standard properties?*

The estimate of the Avar should take into account the previous result, where now

$$\mathbb{E}\left[\mathbf{X}\widehat{\boldsymbol{\Omega}^{-1}\mathbf{v}\mathbf{v}'}\boldsymbol{\Omega}^{-1}\mathbf{X}\right] = \mathbb{E}_n\left[\mathbf{X}\hat{\boldsymbol{\Omega}}_n^{-1}\hat{\mathbf{v}}\hat{\mathbf{v}}'\hat{\boldsymbol{\Omega}}_n^{-1}\mathbf{X}\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i\hat{\boldsymbol{\Omega}}_n^{-1}\hat{\mathbf{v}}_i\hat{\mathbf{v}}_i'\hat{\boldsymbol{\Omega}}_n^{-1}\mathbf{X}_i$$

where $\hat{\boldsymbol{\Omega}}_n^{-1}$ is a consistent estimate of Ω, maybe the same one as in the estimate calculation, and $\hat{\mathbf{v}}$ are the vector of POLS residuals for each individual.

(d) *Propose an efficient estimate of the parameter β.*

We should use a FGLS estimate that takes into account the conditional heteroskedasticity,

$$\hat{\boldsymbol{\beta}}_{RE} = \mathbb{E}_n\left[\mathbf{X}\hat{\boldsymbol{\Omega}}\left(\mathbf{x}\right)^{-1}\mathbf{X}'\right]^{-1}\mathbb{E}_n\left[\mathbf{X}\hat{\boldsymbol{\Omega}}\left(\mathbf{x}\right)^{-1}\mathbf{y}\right],$$

so that its asymptotic variance is

$$\mathbb{E}\left[\mathbf{X}\boldsymbol{\Omega}\left(\mathbf{x}\right)^{-1}\mathbf{X}'\right]^{-1}.$$

(e) *Is the Fixed Effects estimate efficient under $(i)-(iii)$ and $Var(c_i|\mathbf{x}_i) \neq Var(c_i) = \sigma_c^2$?*

No, because despite the fact that c_i is removed by mean centering and the conditional heteroskedasticy condition does not affect asymptotic inference, under RE conditions the previous (conditional heteroskedasticity) GLS estimate should be better.

When not using information on $\mathrm{Var}(c_i|x_i)$, FE under the given conditions is efficient. (Only if $\mathbb{E}\left[\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i, c_i\right] = \Omega \neq \sigma_u^2\mathbf{I}_T$ then FE FGLS using information on $E\left[\ddot{\mathbf{u}}\ddot{\mathbf{u}}'\right]$ should be more efficient than FE.)

Does your analysis depend on condition (ii)? That is, does your conclusion hold if (ii) fails: $E\left[c_i|\mathbf{x}_i\right] \neq E\left[c_i\right] = 0$?

Yes, it does depend, because in this case any GLS estimate is no longer consistent, but FE is still consistent and now efficient.

2. *In the linear regression model*

$$y_t = \boldsymbol{\beta}'\mathbf{z}_t + v_t,$$

where z_t is strictly exogenous and stationary and contains an intercept and v_t follows a stationary AR(1) model

$$v_t = \alpha v_{t-1} + e_t,$$

with $|\alpha| < 1$ and the e_t is white noise with variance σ^2.

(a) *Find the autocorrelation function of v_t and of $v_t z_t$. Study the asymptotic properties of the OLS estimate of β based on $\{y_t, \mathbf{z}_t\}_{t=1}^{T}$.*

$$\gamma_v\left(j\right) = \sigma_e^2\alpha^j, \quad \gamma_v\left(j\right) = \frac{\alpha^j}{1-\alpha}.$$

$$\begin{aligned}\Gamma_{v\mathbf{z}}\left(j\right) &= E\left[v_t\mathbf{z}_t v_{t+j}\mathbf{z}_{t+j}'\right]\\ &= E\left[\mathbf{z}_t\mathbf{z}_{t+j}'\right]E\left[v_t v_{t+j}\right]\\ &= \gamma_v\left(j\right)\Gamma_{\mathbf{z}}\left(j\right),\end{aligned}$$

if $E\left[v_t v_{t+j}|\ldots,\mathbf{z}_{t-1},\mathbf{z}_t,\mathbf{z}_{t+1},\ldots\right] = E\left[v_t v_{t+j}\right]$.

We have that under standard conditions,

$$E\left[\mathbf{z}_t\mathbf{z}_t'\right] = \mathbf{M} > 0$$

and a law of large numbers and a central limit theorem for time averages of $v_t\mathbf{z}_t$,

$$\beta_T^{OLS} = \left(\sum_{t=1}^{T}\mathbf{z}_t\mathbf{z}_t'\right)^{-1}\sum_{t=1}^{T}\mathbf{z}_t y_t$$

is consistent and asymptotic normal with

$$T^{1/2}\left(\beta_T^{OLS} - \beta\right) \to_d N\left(0, \mathbf{M}^{-1}\sum_{j=-\infty}^{\infty}\Gamma_{v\mathbf{z}}(j)\,\mathbf{M}^{-1}\right).$$

(b) *Write the log-likelihood*

$$L_{\sigma^2}(e_1, \ldots, e_T)$$

of $(e_1, \ldots, e_T)$ *assuming Gaussianity.*

$$L_{\sigma^2}(e_1, \ldots, e_T) = -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log\left(\sigma^2\right) - \frac{1}{2}\sum_{t=1}^{T}\frac{e_t^2}{\sigma^2}$$

(c) *Using that* $e_t = e_t(\beta, \alpha) = v_t - \alpha v_{t-1} = y_t - \alpha y_{t-1} - \beta'(\mathbf{z}_t - \alpha\mathbf{z}_{t-1})$, $t \geq 2$, *use the previous question to write the log-likelihood*

$$L_{\boldsymbol{\beta},\alpha,\sigma^2}(y_2, \ldots, y_T|y_1, \mathbf{z}_1, \ldots, \mathbf{z}_T)$$

conditional on $(y_1, \mathbf{z}_1, \ldots, \mathbf{z}_T)$.

$$\begin{aligned}
L_{\boldsymbol{\beta},\alpha,\sigma^2}(y_2, \ldots, y_T|y_1, \mathbf{z}_1, \ldots, \mathbf{z}_T) &= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log\left(\sigma^2\right) - \frac{1}{2}\sum_{t=2}^{T}\left(\frac{y_t - \alpha y_{t-1} - \beta'(\mathbf{z}_t - \alpha\mathbf{z}_{t-1})}{\sigma}\right)^2 \\
&= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log\left(\sigma^2\right) - \frac{1}{2}\sum_{t=2}^{T}\left(\frac{y_t^* - \beta'\mathbf{z}_t^*}{\sigma}\right)^2
\end{aligned}$$

where, $t = 2, \ldots, T$,

$$\begin{aligned}
y_t^* &= y_t - \alpha y_{t-1} \\
\mathbf{z}_t^* &= \mathbf{z}_t - \alpha\mathbf{z}_{t-1}.
\end{aligned}$$

(d) *Obtain the scores for* β, α, σ^2 *based on* $L_{\boldsymbol{\beta},\alpha,\sigma^2}$. *Interpret the corresponding moment conditions implied by the score vector.*

$$\begin{aligned}
\frac{\partial}{\partial\beta}L_{\boldsymbol{\beta},\alpha,\sigma^2} &= \sum_{t=2}^{T}\left(\frac{y_t^* - \beta'\mathbf{z}_t^*}{\sigma}\right)\frac{\mathbf{z}_t^*}{\sigma} = \frac{1}{\sigma^2}\sum_{t=2}^{T}e_t(\beta, \alpha)\mathbf{z}_t^* \\
\frac{\partial}{\partial\alpha}L_{\boldsymbol{\beta},\alpha,\sigma^2} &= \sum_{t=2}^{T}\left(\frac{y_t - \alpha y_{t-1} - \beta'(\mathbf{z}_t - \alpha\mathbf{z}_{t-1})}{\sigma}\right)\left(\frac{y_{t-1} - \beta'\mathbf{z}_{t-1}}{\sigma}\right) \\
&= \frac{1}{\sigma^2}\sum_{t=2}^{T}e_t(\beta, \alpha)v_t(\beta) \\
\frac{\partial}{\partial\sigma^2}L_{\boldsymbol{\beta},\alpha,\sigma^2} &= -\frac{T}{2\sigma^2} + \frac{1}{2\sigma^2}\sum_{t=2}^{T}\left(\frac{y_t^* - \beta'\mathbf{z}_t^*}{\sigma}\right)^2
\end{aligned}$$

If we know rewrite the sample moment conditions $\frac{\partial}{\partial\theta}L\left(\hat{\theta}\right)=0$ as

$$\begin{aligned}
\frac{1}{T}\sum_{t=2}^{T}e_t(\hat{\beta},\hat{\alpha})\mathbf{z}_t^*\left(\hat{\alpha}\right) &= 0\\
\frac{1}{T}\sum_{t=2}^{T}e_t(\hat{\beta},\hat{\alpha})v_t\left(\hat{\beta}\right) &= 0\\
\frac{1}{T}\sum_{t=2}^{T}e_t(\hat{\beta},\hat{\alpha})^2 &= \hat{\sigma}^2
\end{aligned}$$

where the first condition is the sample equivalent of $E\left[e_t\mathbf{z}_t^*\right]=0$, the identifying moment condition for β in the linear regression model

$$y_t^*\left(\alpha\right)=\boldsymbol{\beta}'\mathbf{z}_t^*\left(\alpha\right)+e_t,$$

that leads to the GLS type estimate

$$\hat{\boldsymbol{\beta}}=\left(\sum_{t=2}^{T}\mathbf{z}_t^*\left(\alpha\right)\mathbf{z}_t^*\left(\alpha\right)'\right)^{-1}\sum_{t=2}^{T}\mathbf{z}_t^*\left(\alpha\right)y_t^*\left(\alpha\right)$$

the second condition is the sample equivalent of $E\left[e_tv_t\right]=0$, the identifying moment condition for α in the linear regression model

$$v_t\left(\beta\right)=\alpha v_{t-1}\left(\beta\right)+e_t,$$

which leads to the estimate of α,

$$\tilde{\alpha}=\left(\sum_{t=2}^{T}v_{t-1}\left(\beta\right)^2\right)^{-1}\sum_{t=2}^{T}v_{t-1}\left(\beta\right)v_t\left(\beta\right)$$

and the last one is the definition of the sample residual variance, $E\left[e_t^2-\sigma^2\right]=0$.

3. *Assume that the sequence x_t generated by the following nonlinear autoregressive model*

$$x_t=m\left(x_{t-1};\delta_0\right)+\varepsilon_t$$

is strict stationary and the possibly nonlinear function $m\left(\cdot\right)$ is smooth, $E\left[\varepsilon_t|x_{t-1},x_{t-2},\ldots\right]=0$ and we know that $\delta_0\in\Delta\subset R^k$.

   (a) *Given observations $x_1,\ldots,x_T$ consider the estimation of δ_0 by nonlinear LS. Provide the asymptotic properties of the NLS estimate $\hat{\delta}_T^{NLS}$ under the assumption that*

   $$E\left[\varepsilon_t^2|x_{t-1},x_{t-2},\ldots\right]=\sigma^2\left(x_{t-1}\right).$$

   Under suitable identification and consistency properties $\hat{\delta}_T^{NLS}$ will be asymptotically normal with asymptotic variance given by

   $$E\left[\dot{m}_{t-1,0}\dot{m}_{t-1,0}'\right]^{-1}E\left[\sigma^2\left(x_{t-1}\right)\dot{m}_{t-1,0}\dot{m}_{t-1,0}'\right]E\left[\dot{m}_{t-1,0}\dot{m}_{t-1,0}'\right]^{-1}$$

   where $\dot{m}_{t-1,0}=\dot{m}\left(x_{t-1};\delta_0\right)$

   Propose a suitable iterative scheme to approximate such estimate starting from an initial value $\tilde{\delta}_T$. Establish any additional condition you use.

   We can propose Gauss-Newton,

$$\hat{\delta}_{T(k)}^{NLS}=\hat{\delta}_{T(k-1)}^{NLS}-E_n\left[\dot{m}\left(x_{t-1};\hat{\delta}_{T(k-1)}^{NLS}\right)\dot{m}\left(x_{t-1};\hat{\delta}_{T(k-1)}^{NLS}\right)\right]^{-1}E_n\left[\left(x_t-m\left(x_{t-1};\hat{\delta}_{T(k-1)}^{NLS}\right)\right)\dot{m}\left(x_{t-1};\hat{\delta}_{T(k-1)}^{NLS}\right)\right]$$

4. *It is known now that*
$$\sigma^2(x_{t-1}) = \sigma^2(x_{t-1}, \gamma_0)$$
for some $\gamma_0 \in \Gamma \subset R^p$. *Analyze the asymptotic properties of the nonlinear GLS estimate of* $\theta = (\delta', \gamma')'$ *minimizing the objective function*
$$Q_T(\theta) = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{x_t - m(x_{t-1};\delta)}{\sigma(x_{t-1},\gamma)}\right)^2 + \frac{1}{T}\sum_{t=1}^{T}\log\sigma^2(x_{t-1},\gamma).$$
Now we have that
$$S_T(\theta) = -\frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} 2\left(\frac{x_t - m(x_{t-1};\delta)}{\sigma^2(x_{t-1},\gamma)}\right)\dot{m}(x_{t-1};\delta) \\ \left\{\left(\frac{x_t - m(x_{t-1};\delta)}{\sigma(x_{t-1},\gamma)}\right)^2 - 1\right\}\frac{\dot{\sigma}^2(x_{t-1},\gamma)}{\sigma^2(x_{t-1},\gamma)}\end{bmatrix}$$
so that under conditions guaranteing a CLT for time series (the vector is a MD sequence, so we would need also ergodicity)
$$T^{1/2}S_T(\theta_0) \to_d N\left(0, \begin{bmatrix} 4E\left[\frac{\dot{m}(x_{t-1};\delta_0)\dot{m}(x_{t-1};\delta_0)'}{\sigma^2(x_{t-1},\gamma_0)}\right] & 0 \\ 0 & \mu_4 E\left[\frac{\dot{\sigma}^2(x_{t-1},\gamma_0)}{\sigma^2(x_{t-1},\gamma_0)}\frac{\dot{\sigma}^2(x_{t-1},\gamma_0)'}{\sigma^2(x_{t-1},\gamma_0)}\right]\end{bmatrix}\right)$$
if
$$\begin{aligned} E\left[\left(\frac{x_t - m(x_{t-1};\delta_0)}{\sigma^2(x_{t-1},\gamma_0)}\right)^3 |x_{t-1}\right] &= 0 \\ E\left[\left\{\left(\frac{x_t - m(x_{t-1};\delta_0)}{\sigma(x_{t-1},\gamma_0)}\right)^2 - 1\right\}^2 |x_{t-1}\right] &= \mu_4 \end{aligned}$$
which holds under (conditional) symmetry and constant curtosis, and
$$H_T(\theta) = -\frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} \begin{matrix} 2\left(\frac{x_t - m(x_{t-1};\delta)}{\sigma^2(x_{t-1},\gamma)}\right)\ddot{m}(x_{t-1};\delta) \\ -2\frac{\dot{m}(x_{t-1};\delta_0)\dot{m}(x_{t-1};\delta)'}{\sigma^2(x_{t-1},\gamma)} \end{matrix} & \\ 2\left(\frac{x_t - m(x_{t-1};\delta)}{\sigma^2(x_{t-1},\gamma)}\right)\frac{\dot{\sigma}^2(x_{t-1},\gamma)}{\sigma^2(x_{t-1},\gamma)}\dot{m}(x_{t-1};\delta)' & \begin{matrix} \left\{\left(\frac{x_t - m(x_{t-1};\delta)}{\sigma(x_{t-1},\gamma)}\right)^2 - 1\right\}\frac{\ddot{\sigma}^2(x_{t-1},\gamma)}{\sigma^2(x_{t-1},\gamma)} \\ -\left(\frac{x_t - m(x_{t-1};\delta)}{\sigma(x_{t-1},\gamma)}\right)^2\frac{\dot{\sigma}^2(x_{t-1},\gamma)}{\sigma^2(x_{t-1},\gamma)}\frac{\dot{\sigma}^2(x_{t-1},\gamma)'}{\sigma^2(x_{t-1},\gamma)} \\ \left\{\left(\frac{x_t - m(x_{t-1};\delta)}{\sigma(x_{t-1},\gamma)}\right)^2 - 1\right\}\frac{\dot{\sigma}^2(x_{t-1},\gamma)}{\sigma^2(x_{t-1},\gamma)}\frac{\dot{\sigma}^2(x_{t-1},\gamma)'}{\sigma^2(x_{t-1},\gamma)} \end{matrix}\end{bmatrix}$$
so that
$$H_T(\theta_0) \to_p \begin{bmatrix} 2E\left[\frac{\dot{m}(x_{t-1};\delta_0)\dot{m}(x_{t-1};\delta)'}{\sigma^2(x_{t-1},\gamma)}\right] & 0 \\ 0 & \mu_4 E\left[\frac{\dot{\sigma}^2(x_{t-1},\gamma)}{\sigma^2(x_{t-1},\gamma)}\frac{\dot{\sigma}^2(x_{t-1},\gamma)'}{\sigma^2(x_{t-1},\gamma)}\right]\end{bmatrix}$$

(a) *Take* $\sigma^2(x_{t-1},\gamma) = \exp(\gamma_1 x_{t-1} + \gamma_2)$ *and derive the LM test statistic for testing the null hypothesis of* $\gamma_1 = 0$ *against* $\gamma_1 \neq 0$ *based on* $Q_T(\theta,\gamma)$.

In this case
$$\frac{\partial}{\partial\gamma}\dot{\sigma}^2(x_{t-1},\gamma) = \begin{bmatrix} \exp(\gamma_1 x_{t-1} + \gamma_2)x_{t-1} \\ \exp(\gamma_1 x_{t-1} + \gamma_2)\end{bmatrix}$$

and therefore

$$\begin{aligned} S_{T,\gamma_1}\left(\tilde{\theta}_n\right) &= -\frac{1}{T}\sum_{t=1}^{T}\left[\left\{\left(\frac{x_t - m\left(x_{t-1};\tilde{\delta}_T\right)}{\sigma\left(x_{t-1},\tilde{\gamma}_T\right)}\right)^2 - 1\right\}\frac{\dot{\sigma}^2\left(x_{t-1},\tilde{\gamma}_T\right)}{\sigma^2\left(x_{t-1},\tilde{\gamma}_T\right)}\right] \\ &= -\frac{1}{T}\sum_{t=1}^{T}\left[\left\{\left(\frac{x_t - m\left(x_{t-1};\tilde{\delta}_T\right)}{\exp\left(\tilde{\gamma}_2\right)}\right)^2 - 1\right\}\frac{\exp\left(\tilde{\gamma}_2\right)x_{t-1}}{\exp\left(\tilde{\gamma}_2\right)}\right] \\ &= -\frac{1}{T}\sum_{t=1}^{T}\left[\left\{\varepsilon\left(\tilde{\delta}_T,\tilde{\gamma}_2\right) - 1\right\}^2 x_{t-1}\right]. \end{aligned}$$

Then

$$LM_T = T\frac{S_{T,\gamma_1}\left(\tilde{\theta}_n\right)^2}{\tilde{V}}$$

where $\tilde{V}$ is a consistent estimate of the asymptotic variance of $S_{T,\gamma_1}\left(\tilde{\theta}_n\right)$.