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# Implementation in Adaptive Better-Response Dynamics

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# Summary

- Introduction
- The model
- Results: complete information
- Results: incomplete information

## MOTIVATION

- Implementation theory has produced many mechanisms.
  - Not easy to know which is more relevant.
- Dynamic approach to test their robustness and simplicity/learnability.
- Recent research (Cabrales 1999, Cabrales and Ponti 2000, Sandholm 2002) showed:
  - Canonical mechanism (when implementing in *strict* Nash) stable *and* learnable. *Integer* games nonessential
  - More "refined" mechanism (in iterative deletion of WD strategies) can stabilize "bad" equilibria.
- Are negative results purely mechanism-driven?
  - Negative (but qualified) answer in this paper.

## RESULTS

- *Quasimonotonicity* necessary for implementation when all kinds of mutations are allowed.
- *Quasimonotonicity* plus 3 players and $\varepsilon-$security also sufficient.
- More permissive sufficient conditions with other assumptions on mutations:
    - "Regret" makes more serious mistakes less likely.
    - Mutations are all same order of magnitude (and exploit myopy heavily).
- For incomplete information environments:
    - *Bayesian quasimonotonicity* plus *incentive compatibility* ncessary (and sufficient with 3 players and $\varepsilon-$security).

## PRELIMINARIES

- $N = \{1, ..., n\}$: set of agents.
- Environment: exchange economy.
- $X_i$: $i$'s consumption set, grid in $\Re^l_+$
- $\omega_i \in X_i$: $i$'s initial endowment.
- Set of allocations:

$$Z = \left\{ (x_i)_{i \in N} \in \prod X_i : \sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i \right\}.$$

# The model *(2/4)*

## PREFERENCES

- $\theta_i$: $i$'s preference ordering.
- Assumptions:
  1. No externalities.
  2. 0 is worst bundle.
  3. Increasing preference: For all $i$ and for all $x_i \in X_i$, if $y_i \gg x_i$, $y_i \succ_i^{\theta_i} x_i$.
- $\theta = (\theta_i)_{i \in N} \in \Theta$: preference profile.
- $f : \Theta \to Z$: social choice function (SCF).

## MECHANISMS AND IMPLEMENTATION

- $G = \left((M_i)_{i \in N}, g\right)$: mechanism, where $M_i$ is $i$'s message set and $g : \prod_{i \in N} M_i \to Z$ is the outcome function.

- Played simultaneously every period by boundedly rational agents.

- Better-response dynamics (unperturbed Markov process):
  - Let $m(t)$ message vector at time $t$.
  - $m_i(t+1)$ (if chosen to update) puts positive probability on any $m_i'$ such that

$$g\left(m_i', m_{-i}(t)\right) \succsim_i^{\theta} g(m(t))$$

- Better-response dynamics with mistakes (perturbed Markov process):
  - Irreducible and aperiodic perturbation of better-response dynamics.

- An SCF is *implementable in stochastically stable strategies* if there is a mechanism $G$ such that a perturbation of the better response dynamics applied to its induced game when the preference profile is $\theta$ has $f(\theta)$ as the unique outcome supported by stochastically stable message profiles.

### PROPERTIES OF SCF

- An SCF is $\varepsilon-$*secure* if for each $\theta$, and for each $i \in N$, $f(\theta) \geq (\varepsilon, ..., \varepsilon)$.
- An SCF is *quasimonotonic* if, whenever it is true that for every $i \in N$, $f(\theta) \succ_i^{\theta} z$ implies that $f(\theta) \succ_i^{\phi} z$, we have that $f(\theta) = f(\phi)$ for all $\theta, \phi \in \Theta$.

## NECESSITY AND SUFFICIENCY

**Theorem 1:** If $f$ is implementable in SSS of any perturbed better-response dynamics, $f$ is *quasimonotonic*.

**Proof:**

- Let true preference profile be $\theta$.
- $f$ implementable in SSS implies only $f(\theta)$ is in set of recurrent classes.
- Let $\phi$ such that for all $i$, $f(\theta) \succ_i^\theta z$ implies that $f(\theta) \succ_i^\phi z$.
- Since $f(\theta)$ is only outcome in recurrent class when preference is $\theta$, when message profile gives $\theta$:
    - Unilateral deviations for $i$ must give either $f(\theta)$ again,
    - or $z$ with $f(\theta) \succ_i^\theta z$.
- But this implies $f(\theta)$ must also be in recurrent class when preferences are $\phi$.
- And therefore $f(\theta) = f(\phi)$, thus $f$ is *quasimonotonic*.

**Theorem 2:** Let $n \geq 3$. If an SCF $f$ is $\varepsilon-$*secure* and *quasimonotonic*, it is implementable in SSS of any perturbed better-response dynamics.

**Proof: Canonical mechanism**

- *Message set*: $M_i = \Theta \times Z$.
- *Outcome function*:

  i If $\forall i$, $m_i = (\theta, f(\theta))$, $g(m) = f(\theta)$.

  ii If $\forall j \neq i$, $m_j = (\theta, f(\theta))$ and $m_i = (\phi, z) \neq (\theta, f(\theta))$ :

  1.(a) If $z \succeq_i^\theta f(\theta)$, $g(m) = (f_i(\theta) - \varepsilon, f_{-i}(\theta))$.

  (b) If $f(\theta) \succ_i^\theta z$, $g(m) = z$.

  iii In all other cases, $g(m) = 0$.

# Results: complete information *(3/11)*

*Let $\theta$ be the true preference profile.*

**Step 1** *No message profile in rule (iii) is part of a recurrent class.*

- W.l.o.g., suppose $m_1 = (\phi, z) \neq (\theta, f(\theta))$.
- Change one by one strategies of $i \neq 1$, to $(\theta, f(\theta))$.
- Outcome is still 0, so better response, until $(n-1)$ messages are $(\theta, f(\theta))$.
- Then outcome switches to either $z$ or $(f_1(\theta) - \beta, f_{-1}(\theta))$, both better-response.
- In last step agent 1 switches from $(\phi, z)$ to $(\theta, f(\theta))$. This yields $f(\theta)$, a better response and contradiction.

**Step 2** *No message profile under rule (ii.a) is part of a recurrent class.*

- $m_j = (\phi, f(\phi))$, for all $j \neq i$, and $m_i = (\phi', z')$ such that $z' \succeq_i^{\phi} f(\phi)$, leading to $f_i(\phi) - \beta$ for $i$.
- Agent $i$ switches to $(\phi, z)$, where $z_i = f_i(\phi) - \beta'$ (for $\beta' < \beta$) and $z_j = 0$ for every $j \neq i$, which yields outcome $z$.
- From here each $j \neq i$ can switch to $(\phi^j, z^j)$ (for some $(\phi^j, z^j) \neq (\phi, f(\phi))$), leading to rule (iii), contradiction.

**Step 3** *No recurrent class contains profiles under rule (ii.b).*

- For all $j \neq i$ $m_j = (\phi, f(\phi))$, whereas $m_i = (\phi', z')$, satisfying that $f_i(\phi) \succ_i^{\phi} z'_i$. This implies outcome is $z'$.
- Agent $i$ switches, if necessary, to $(\phi', z)$, where $z_i = z'_i$ and for all $j \neq i$, $z_j = 0$, after which the outcome is $z$.
- As before, any of the other agents can switch to rule (iii), and contradiction.

## Results: complete information *(5/11)*

**Step 4** *Only the truthful profile* $(\theta, f(\theta))$ *is a member of a recurrent class.*

- Thus, all recurrent classes contain only profiles under rule (i). One cannot abandon rule (i) to get to another without passing through rule (ii). Thus, recurrent classes are singletons.
- Each recurrent class, a singleton under rule (i), must consist of a Nash equilibrium of the game when true preferences are $\theta$, by better-response dynamics.
- One such Nash equilibrium is the truthful profile $(\theta, f(\theta))$ reported by every agent. Unilateral deviations lead to rule (ii.a) or rule (ii.b). Not possible under better-response dynamics.
- One may have other (non-truthful) Nash equilibria under rule (i). Let $(\phi, f(\phi))$ be such NE.
- For this to be a NE, for all $i \in N$, $f(\phi) \succ_i^{\phi} z$ implies that $f(\phi) \succeq_i^{\theta} z$.
- Moreover, since profile is a absorbing state of the dynamics, we must also have for all $i \in N$, $f(\phi) \succ_i^{\phi} z$ implies that $f(\phi) \succ_i^{\theta} z$.
- Thus, because $f$ is quasimonotonic, we must have that $f(\theta) = f(\phi)$.

## PERMISSIVE RESULTS

### 1. REGRET DYNAMICS

- Suppose agent $i$ moves at time $t$.
- $z_i^0$: bundle at period $t$.
- $y_i$: bundle that $i$ proposes.
- $z_i$: bundle that he receives in new outcome.
- Resistance of such transition:

$$\left[u_i(z_i^0) - u_i\left(z_i\right)\right] - \lambda\left[u_i(y_i) - u_i\left(z_i\right)\right],$$

where $0 < \lambda < 1$ is small enough. Call these *better-response regret dynamics*.

**Theorem 3:** Let $n \geq 3$. Then, any $\varepsilon-$*secure* SCF $f$ is implementable in SSS of any perturbed *better-response regret dynamics*.

- Proof based on (modified) canonical mechanism of Theorem 2.
- Quasimonotonicity of $f$ implies again recurrent classes are singletons under rule (i).
- Let $\theta$ denote the true preferences.
- We classify recurrent classes of unperturbed process into:

  $E_0$ truth-telling profile, for each $i \in N$, $m_i = (\theta, f(\theta))$.

  $E_j$ for $j = 1, \ldots, J$ is coordinated lie on profile $\theta^j$: for each $i \in N$, $m_i = (\theta^j, f(\theta^j))$, a Nash equilibrium of the mechanism under $\theta$. These require that for all $i \in N$, $f(\theta^j) \succ_i^{\theta^j} z$ implies that $f(\theta^j) \succ_i^{\theta} z$.

- Modify outcome function of proof of Theorem 2:

**(ii.a'.)** Replace $\beta$ with $(\Delta, 0, \ldots, 0)$, punishment is smallest unit of nummeraire.

- Profile in $E_0$ is only stochastically stable profile:

  [a] To get out of $E_0$, through rule (ii.a') paying $(1 + \lambda)\Delta$ or through (ii.b) paying no less than $(1 + \lambda)\Delta$.

  - After that, a mistake to rule (iii), costs $K$, takes us to 0.
  - From there for free to any equilibria in $E_j$.

  [b] To get out of any $E_j$, two paths but cheapest under rule (ii.a') again.

  - In this case, resistance is strictly smaller than $(1 + \lambda)\Delta$, because of the relief term.
  - After that, to rule (iii) paying also $K$, and from there for free to $E_0$.

## 2. UNIFORM MUTATIONS

- An SCF $f$ is *(strongly) Pareto efficient* if for all $\theta$ and for all $z \neq f(\theta)$, there exists an $i(\theta, z)$ such that $f(\theta) \succ^{\theta}_{i(\theta,z)} z$.

- For every $\theta$ and $\phi$, there is an $j(\theta, \phi)$ and $x(\theta, \phi)$ and $y(\theta, \phi)$ such that

$$x(\theta, \phi) \succ^{\theta}_{j(\theta,\phi)} y(\theta, \phi) \qquad \text{and} \qquad y(\theta, \phi) \succeq^{\phi}_{j(\theta,\phi)} x(\theta, \phi). \tag{$*$}$$

  Denote by $J(\theta, \phi)$ the set of agents $j(\theta, \phi)$ for whom there exists a preference reversal between a pair of alternatives across states $\theta$ and $\phi$, as specified in (*).

**(5)** For each $\theta$ and $\phi$, there is $j(\theta, \phi) \in J(\theta, \phi)$ such that $j(\theta, \phi) \neq i(\theta, x(\theta, \phi))$, where $x(\theta, \phi)$ is an alternative for which agent $j(\theta, \phi)$ has a preference reversal as in (*).

## Results: complete information *(10/11)*

**Theorem 4.** Suppose environment satisfies (1), (2) and (5). Let $n \geq 5$. Any $\epsilon$-secure and strongly Pareto efficient SCF $f$ is implementable in SSS, when mutations are uniform.

**Proof:** Let $M_i = \Theta \times Z$, $m_i = (m_i^1, m_i^2)$, $m = (m^1, m^2)$.

**(i.)** If for every $i \in N$, $m_i^1 = \theta$, $g(m) = f(\theta)$.

**(ii.a.)** If exactly $(n-1)$ messages $m_i$ are such that $m_i^1 = \theta$ and $m_{i(\theta,x(\theta,\phi))} = (\phi, x(\theta,\phi))$, $g(m) = (x_{i(\theta,x(\theta,\phi))}(\theta,\phi), x_{j(\theta,\phi)}(\theta,\phi), 0, 0, \dots, 0)$.

**(ii.b.)** If exactly $(n-1)$ messages $m_i$ are such that $m_i^1 = \theta$, but the odd man out, say agent $k$, does not satisfy the requirements of rule (ii.a), $g(m) = (f_k(\theta) - \beta, f_{-k}(\theta))$, where $f_k(\theta) \geq f_k(\theta) - \beta \geq (\epsilon, \dots, \epsilon)$.

**(iii.a.)** If exactly $(n-2)$ messages $m_i$ are such that $m_i^1 = \theta$, $m_{i(\theta,x(\theta,\phi))} = (\phi, x(\theta,\phi))$ and $m_{j(\theta,\phi)} = (\phi, y(\theta,\phi))$, $g(m) = (y_{i(\theta,x(\theta,\phi))}(\theta,\phi), y_{j(\theta,\phi)}(\theta,\phi), 0, 0, \dots, 0)$.

**(iii.b.)** If exactly $(n-2)$ messages $m_i$ are such that $m_i^1 = \theta$, but we are not under rule (iii.a), for all $k \in N$, $g_k(m) = (\epsilon, \dots, \epsilon)$.

**(iv.)** In all other cases, $g(m) = 0$.

# Results: complete information *(11/11)*

$E_0^j$ All $n$ agents report the true state $\theta$ as the first part of their announcement.

$E_1^j$ Agents' reported state is not $\theta$, the true state.

**[a]** To get out of $E_0^j$, $i(\theta, x(\theta, \phi))$

- imposes one reversal $x(\theta, \phi)$ – one mistake.
- Next, $j(\theta, \phi)$ imposes $y(\theta, \phi)$ – second mutation.
- Finally, anyone changes to (iv) where 0 is the outcome – third mutation.
- From 0, for free to any other absorbing state.

**[b]** To get out of an untruthful profile, say $m^1 = \phi$:

- $i(\phi, x(\phi, \theta))$ can impose $x(\phi, \theta)$. If $f(\phi) \succ^{\theta}_{i(\phi, x(\phi,\theta))} x(\phi, \theta)$, this requires a first mutation. If $x(\phi, \theta) \succeq^{\theta}_{i(\phi, x(\phi,\theta))} f(\phi)$, zero resistance.
- Next, $j(\phi, \theta)$ changes to $y(\phi, \theta)$ for free.
- Finally, someone changes to 0 under rule (iv), at most a second mutation.
- From there,for free to any other absorbing state.

## ENVIRONMENT

- Each agent knows $\theta_i \in \Theta_i$.
- Let $\Theta = \prod_{i \in N} \Theta_i$ and $\Theta_{-i} = \prod_{j \neq i} \Theta_j$.
- We assume the set of states with ex-ante positive probability is $\Theta$.
- Let $q_i(\theta_{-i}|\theta_i)$ be type $\theta_i$'s interim probabilityover $\theta_{-i}$ .
- An SCF is a mapping $f : \Theta \mapsto Z$ .
- Let $A$ denote the set of SCFs.
- We shall $\theta_i$'s interim expected utility over an SCF $f$:

$$U_i(f|\theta_i) \equiv \sum_{\theta_{-i} \in \Theta_{-i}} q_i(\theta_{-i}|\theta_i) u_i(f(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})).$$

- $G = ((M_i)_{i \in N}, g)$, $m_i : \Theta_i \to M_i)$, and $g : \Theta \mapsto Z$.

- Strategy revision using the interim better-response logic. That is, letting $m^t$ profile at period $t$, type $\theta_i$ switches from $m_i^t(\theta_i)$ to any $m_i'$ such that:

$$\sum_{\theta_{-i}\in\Theta_{-i}} q_i(\theta_{-i}|\theta_i)u_i(g(m_i', m_{-i}^t(\theta_{-i})), (\theta_i, \theta_{-i})) \geq \sum_{\theta_{-i}\in\Theta_{-i}} q_i(\theta_{-i}|\theta_i)u_i(m^t(\theta), \theta).$$

- An SCF $f$ is *implementable in asymptotically stable strategies* if there exists $G$ such that interim better-response process has $f$ as unique outcome of the recurrent classes of the process.

- An SCF $f$ is *implementable in stochastically stable strategies* if there exists $G$ such that a perturbation of the interim better-response process has $f$ as unique outcome supported by stochastically stable strategy profiles.

### NECESSITY

An SCF $f$ is *strictly incentive compatible* if for all $i$ and for all $\theta_i$,

$$\sum_{\theta_{-i}\in\Theta_{-i}} q_i(\theta_{-i}|\theta_i)u_i(f(\theta),\theta) > \sum_{\theta_{-i}\in\Theta_{-i}} q_i(\theta_{-i}|\theta_i)u_i(f(\theta_i',\theta_{-i}),(\theta_i,\theta_{-i}))$$

for every $\theta_i' \neq \theta_i$.

**Theorem 5.** If $f$ is implementable in SSS of any perturbation of interim better-response dynamics, $f$ is incentive compatible. If at least one recurrent class is a singleton, $f$ is strictly incentive compatible.

- Consider a mapping $\alpha_i = (\alpha_i(\theta_i))_{\theta_i \in \Theta_i} : \Theta_i \mapsto \Theta_i$. A *deception* $\alpha = (\alpha_i)_{i \in N}$ is a collection of such mappings where at least one differs from the identity mapping.
- Given an SCF $f$ and a deception $\alpha$, let $[f \circ \alpha]$ denote the following SCF: $[f \circ \alpha](\theta) = f(\alpha(\theta))$ for every $\theta \in \Theta$.
- Finally, for a type $\theta'_i \in \Theta_i$, and an arbitrary SCF $y$, let $y_{\theta'_i}(\theta) = y(\theta'_i, \theta_{-i}))$ for all $\theta \in \Theta$.
- An SCF $f$ is *Bayesian quasimonotonic* if for all deceptions $\alpha$, for all $i \in N$, and for all $\theta_i \in \Theta_i$, whenever

$$U_i(f \mid \theta_i) > U_i(y_{\theta'_i} \mid \theta_i) \forall \theta'_i \in \Theta_i \quad \text{implies} \quad U_i(f \circ \alpha \mid \theta_i) > U_i(y \circ \alpha \mid \theta_i), \qquad (**)$$

  one must have that $f \circ \alpha = f$.

**Theorem 6.** If $f$ is implementable in asymptotically stable strategies of an unperturbed interim better-response dynamic process, $f$ is Bayesian quasimonotonic.

## SUFFICIENCY

**Theorem 7.** Suppose the environments satisfy Assumptions (1) and (2) in each state. Let $n \geq 3$. If an SCF $f$ is $\epsilon$-secure, strictly incentive compatible and Bayesian quasimonotonic, $f$ is implementable in asymptotically stable strategies of interim better-response dynamics.

**Proof:** $G = ((M_i)_{i \in N}, g)$, $M_i = \Theta_i \times A$. $m_i = (m_i^1, m_i^2)$. Outcome function $g$ is:

**(i.)** If for every agent $i \in N$, $m_i^2 = f$, $g(m) = f(m^1)$.

**(ii.)** If for all $j \neq i$ $m_j^2 = f$ and $m_i^2 = y \neq f$, one can have two cases:

- **(ii.a.)** If there exist types $\theta_i, \theta_i' \in \Theta_i$ such that $U_i(y_{\theta_i'} \mid \theta_i) \geq U_i(f \mid \theta_i)$, $g(m) = (f_i(m^1) - \beta, f_{-i}(m^1))$, where $f_i(m^1) \geq f_i(m^1) - \beta \in X_i$.
- **(ii.b.)** If for all $\theta_i, \theta_i' \in \Theta_i$, $U_i(y_{\theta_i'} \mid \theta_i) < U_i(f \mid \theta_i)$, $g(m) = y(m^1)$.

**(iii.)** In all other cases, $g(m) = 0$.

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