1. MOTIVATIONS

- Attainability of "socially desirable" equilibria (convergence *vs.* stability)

- Old wine in new bottles: the "equilibrium selection" problem

- (Iterated) deletion of weakly dominated strategies and evolutionary dynamics

- RELATED LITERATURE:

  • THE DYNAMICS OF IMPLEMENTATION: Muench and Walker (1984), de Trenquayle (1988), Cabrales (1996)
  • ITERATIVE WEAK DOMINANCE: Dekel and Fudenberg (1990), Ben-Porath (1993), Börgers (1994)
  • (ITERATED) DELETION OF DOMINATED STRATEGIES AND EVOLUTIONARY DYNAMICS: Samuelson and Zhang (1992), Cressman and Schlag (1996)
  • EVOLUTIONARY DYNAMICS WITH DRIFT: Binmore, Gale and Samuelson (1995), Binmore and Samuelson (1996)

THE IMPLEMENTATION PROBLEM

- Consider an environment with a finite set of agents $\Im = \{1, 2, \ldots\ldots, I\}$ with typical element *i*, and a set of feasible outcomes *A*, with typical element *a;*

- Agents are endowed with a preference relation over *A* (e.g. a VNM utility function) $v_i : A \to \Re$, with $v = \{v_i\}_{i \in \Im} \in V$. We shall assume that the *state* of the environment, $v \in V$ is common knowledge among the agents

- A *social choice function* is a mapping

$$f : V \to A$$

- A (game form) mechanism is defined as $G = \{S_i, \alpha\}$, with $\alpha : S \to A$. for a given $v \in V$, the couple (G, v) defines a *game* $\Gamma(v)$.

- Take a *solution concept* $\Sigma(\Gamma)$. We say that the (game form) mechanism $G = \{S_i, \alpha\}$ implements the social choice function *f* if, for any $v \in V$:

$$\Sigma(\Gamma(v)) = f(v)$$

AN EXAMPLE

- A unit of a good has to be divided among three players: 1, 2, and 3.
- Player 3's preferences can either be of type "0", or type "1".
- The mechanism is constructed as follows:
  - Each player has to make a (simultaneous) statement about player 3's preferences (either m_i^0 or m_i^1):
  - If the true preferences are of type 1, the game Γ can be represented as follows:

$m_3 = m_3^0$

	m_2^0	m_2^1
m_1^0	$\frac{1}{4},\frac{1}{4},\frac{1}{2}$	$\frac{1}{3},0,\frac{1}{3}$
m_1^1	$0,\frac{1}{3},\frac{1}{3}$	$0,0,\frac{1}{3}$

$m_3 = m_3^1$

	m_2^0	m_2^1
m_1^0	$0,0,\frac{1}{2}$	$0,\frac{1}{3},\frac{1}{2}$
m_1^1	$\frac{1}{3},0,\frac{1}{2}$	$\frac{1}{3},\frac{1}{3},\frac{1}{3}$

- NOTICE:
  - The game Γ is weakly dominance solvable
  - Only one round of deletion of weakly dominated strategies is required (the first)
  - Player 3 is indifferent only against “totally consistent” strategy profiles

2. CONTINUOUS-TIME DYNAMICS: SOME TERMINOLOGY

Let:

- $r_i^k(t)$ = the probability with which player i selects her pure strategy k at time t;

- $r_i(t) \equiv \left(r_i^{1+\delta_i^2}(t),\ldots,r_i^{\Theta}(t)\right)$ denoting the (column) vector collecting such probabilities;

- $r_i(t) \in \Delta^{|S_i|-1}$ (hereto referred as Δ_i)

- $r_{-i}(t) \in \Delta_{-i}$

- $r(t) \equiv \left(r_I(t), r_{II}(t)\right)$ denoting the mixed strategy profile played at each point in time has to be interpreted as

the *state of the system* at time t

defined over the state space $\Delta \equiv \Delta^{|S_I|-1} \times \Delta^{|S_{II}|-1}$, with $\overset{\leftrightarrow}{\Delta}$ denoting the relative interior of Δ, that is, the set of completely mixed strategy profiles.

- ASSUMPTION The evolution of $r(t)$ over time, is given by the following system of (autonomous) differential equations:

$$\dot{r}_i^k(t) = f_i^k(r(t)) + \lambda_i\left(\beta_i^k - r_i^k(t)\right)$$

3. REGULAR MONOTONIC SELECTION DYNAMICS

• DEFINITION. *f* is said to yield a *regular* dynamic if the following hold:

- Lipschitz continuity
- $\sum_{k \in S_i} f_i^k(r(t)) = 0; i = I, II;$
- the following limit $\frac{f_i^k(r(t))}{0} \equiv \lim_{r_i^k \to 0} \frac{f_i^k(r(t))}{r_i^k(t)}$ exists and is finite.

• DEFINITION A regular selection dynamics is said to be *monotonic* if pure strategies which yield to higher payoffs grow faster:

$$\forall k, k' \in S_i \; ; \; u_i(s_i^k, r_{-1}) \ge u_1(s_i^{k'}, r_{-1}) \Leftrightarrow \frac{f_i^k(r)}{r_i^k} \ge \frac{f_i^{k'}(r)}{r_i^{k'}}$$

• DEFINITION. A regular selection dynamics is said to be *aggregate monotonic* if *mixed* strategies which yield to higher payoffs "grow faster":

$$\forall r_i, r_i' \in \Delta_i \; ; \; u_i(y_i, r_{-1}) \ge u_1(y_i', r_{-1}) \Rightarrow \sum_{k \in S_i} \frac{f_i^k(r)}{r_i^k} \cdot y_i^k \ge \sum_{k \in S_i} \frac{f_i^k(r)}{r_i^k} \cdot y_i'^k$$

• DEFINITION. The *continuous-time Replicator Dynamics* is a (payoff-positive) dynamics defined in which the difference in growth rates equals the difference in payoffs:

$$\dot{r}_i^k(t) = r_i^k(t)\left(u_i\left(s_i^k, r_{-i}(t)\right) - u_i(r(t))\right)$$

STABILITY: STANDARD DEFINITIONS

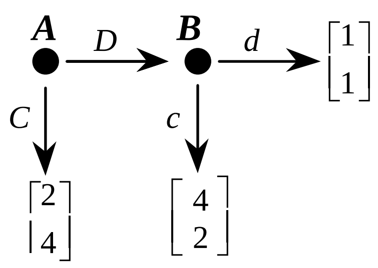
- DEFINITION. Let $x(t,x(0))$ be the solution of a differential equation on state space Δ given initial condition $x(0)$. Let also C denote a closed set of restpoints in Δ of the same differential equation. Then:

  (i) C is (interior) *stable* if, for every neighbourhood O of C, there is another neighbourhood U of C, with $U \subset O$, such that, for any $x(0) \in U \cap \Delta$ ($U \cap \Delta^0$), we have $x(t,x(0)) \in O$;

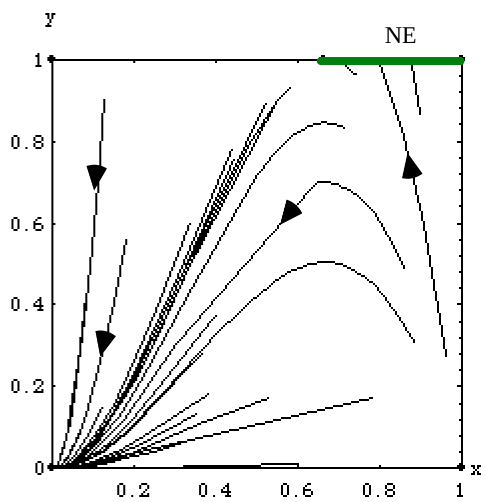
  (ii) C is (interior) *attracting* if is contained in an open set O such for any $x(0) \in O \cap \Delta$ ($O \cap \Delta^0$) we have $\lim_{t\to\infty} x(t,x(0)) \in C$.;

  (iii) C is *globally* (interior) attracting if for any $x(0) \in \Delta$ (Δ^0) we have $\lim_{t\to\infty} x(t,x(0)) \in C$;

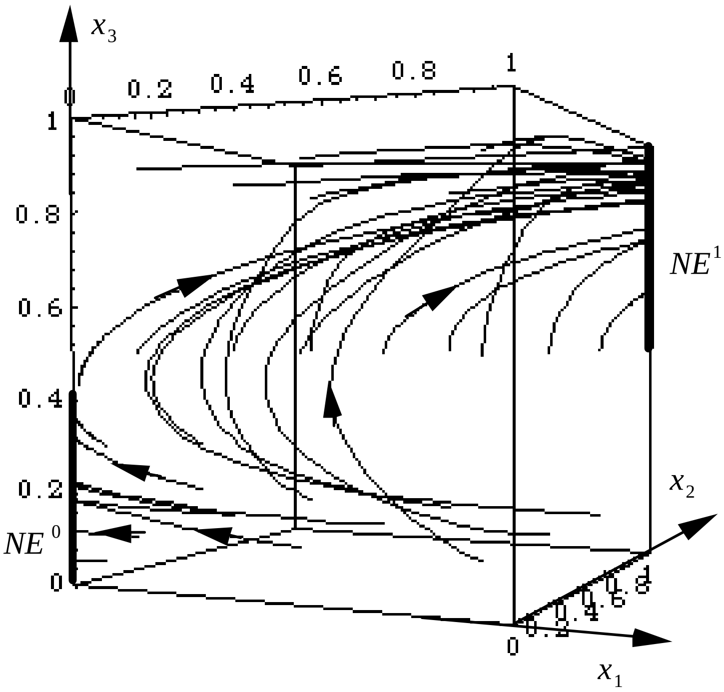
  (iv) C (interior) *asymptotically stable* if it is (interior) attracting and (interior) stable.



		$1-x$	x
		C	D
y	C	2,4	2, 4
1- y	D	4,2	1,1



3. Convergence and Stability of the Solution



- PROPOSITION. The set NE of Nash equilibria of is the union of precisely two disjoint components:

$$NE^0 \equiv \left\{ x \in \Delta \middle| x_1 = x_2 = 0, x_3 \leq \frac{3}{7} \right\}$$
$$NE^1 \equiv \left\{ x \in \Delta \middle| x_1 = x_2 = 1, x_3 \geq \frac{1}{2} \right\}$$

- PROPOSITION. Any solution $x(t, x(0))$ of a monotonic selection dynamic $\dot{x} = D(x)$ with completely mixed initial conditions converges asymptotically to NE.

- PROPOSITION. Under the replicator dynamics, NE^1 is interior asymptotically stable whereas NE^0 is not.

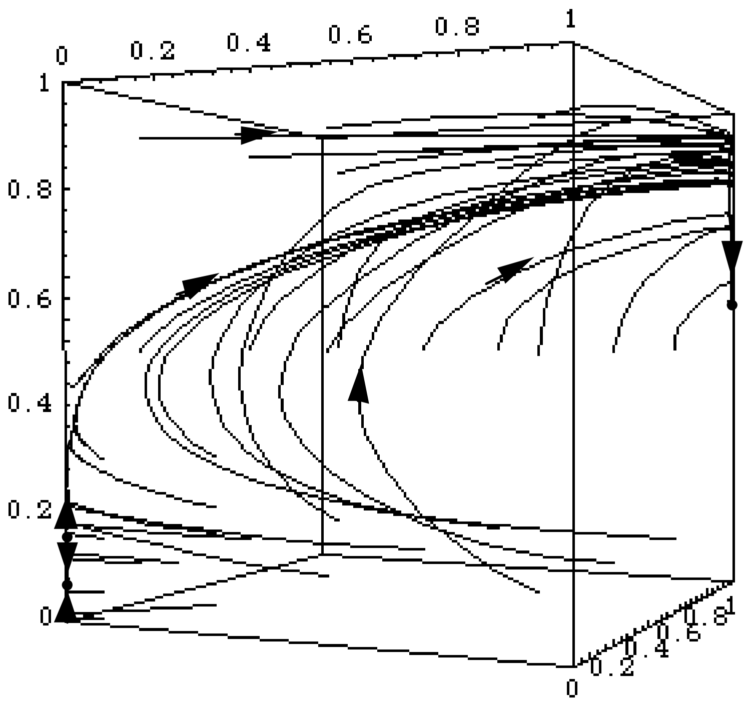
4. STRUCTURAL STABILITY: REPLICATOR DYNAMICS WITH DRIFT

- ASSUMPTION. The evolution of $x_i(t)$ is given by the following system of differential equations:

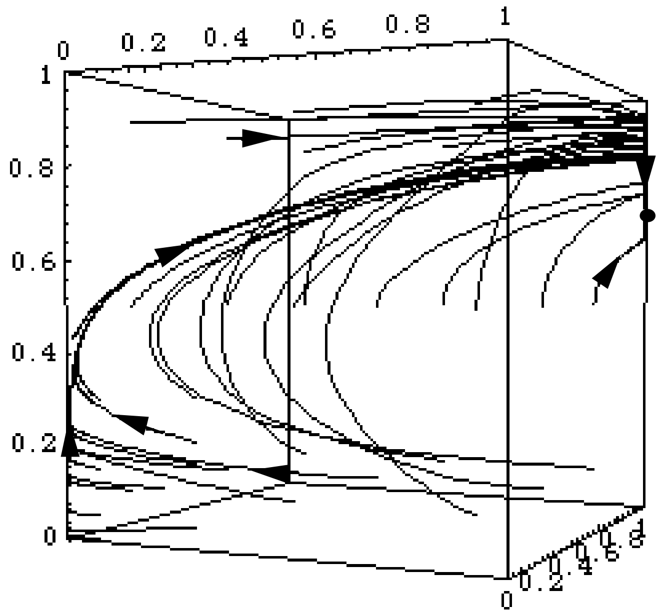
$$\dot{x}_i(t) = x_i(t)(1 - x_i(t))\Delta\Pi_i(\cdot) + \lambda(\beta_i - x_i(t))$$

with $\lambda > 0$, $\beta_1 = \beta_2 = \frac{1}{2}$ and $\beta_3 \equiv \beta \in (0,1)$

CASE A: $\beta = \frac{1}{100}$



CASE B: $\beta = \frac{1}{2}$



- We are interested in the convergence and stability properties of the replicator dynamic with drift when $\lambda \to 0$ under two different configurations of the drift parameter β:

$$\text{Case A: } \beta \in \left(0, \frac{23 - 4\sqrt{30}}{49}\right)$$

$$\text{Case B: } \beta \in \left(\frac{23 - 4\sqrt{30}}{49}, 1\right)$$

with $\frac{23 - 4\sqrt{30}}{49} \cong .0222673$.

- PROPOSITION. Let $\hat{R}(\beta)$ be the set of restpoints of the dynamic with drift when $\lambda \to 0$

    - $\forall \beta \in (0,1)$, $\hat{R}(\beta)$ contains an element of NE^1, which is also asymptotically stable;

- Under CASE A, $\hat{R}(\beta)$ contains two additional restpoints, both belonging to NE^0, one of which is asymptotically stable.

FICTITIOUS PLAY AND SJOSTROM'S EXAMPLE

- Suppose that the players are now endowed with some *belief* about their opponents' strategies, which are constantly updated along the sequence of plays $\zeta(t) = (m(1), m(2), \ldots, m(t))$ which defines the (discrete-time) *history* of the game.

- Each player *i*, after having put initial (arbitrary) weights $\chi_i(0): M_{-i} \varnothing (0, \infty)$ to any pure strategy profile of the opponents (which constitutes her initial non-normalized beliefs), will update these beliefs as follows:

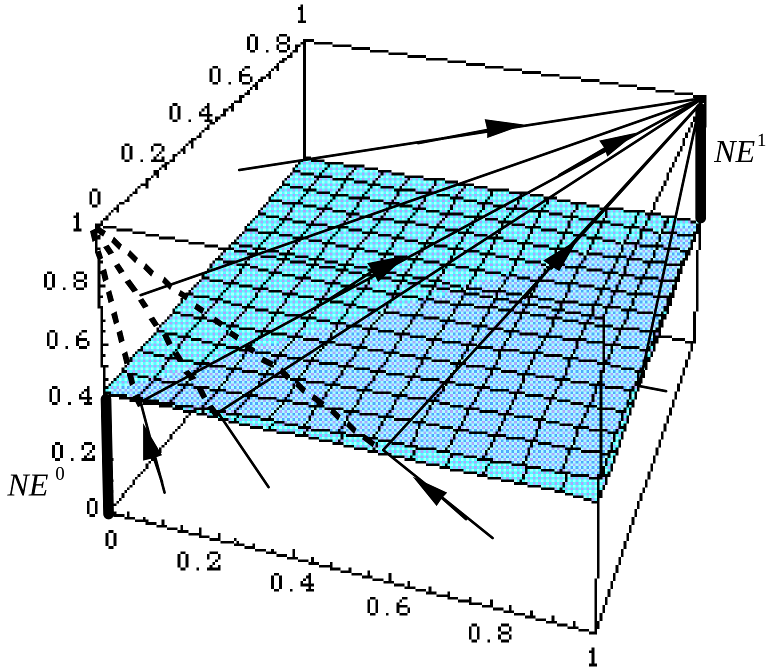
$$\xi_i^{m_{-i}}(t) = \frac{\chi_i^{m_{-i}}(0) + \kappa^{m_{-i}}(\zeta(t))}{\sum_{M_{-i}} \chi_i^{m_{-i}}(0) + \kappa^{m_{-i}}(\zeta(t))}$$

with $\kappa^{m_{-i}}(\zeta(t))$ denoting the number of times the pure strategy profile m_{-i} has been observed for a given history $\zeta(t)$.

- Each player selects, at each point in time, the pure strategy which maximizes her expected payoff, given her current beliefs (with ties broken at random):

$$m_i(t) \quad \arg\max \sum_{M_{-i}} v_i(m_i, m_{-i})$$

- PROPOSITION. For any possible history $\zeta(t)$ of Γ, if players behave according with fictitious play and initial beliefs are completely mixed, there will be a time T after which $m(t)=(m_1^1, m_2^1, m_3^1)$ for all $t>T$.



- PROPOSITION. For any possible history $\zeta(t)$ of the game $(\alpha, \hat{R})$, if players behave according with fictitious play and initial beliefs are completely mixed, there will be a time T after which $m(t)=\{\hat{s}_i\}, i \;\; \Upsilon$, for all $t>T$.