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Networks - Fall 2005

Chapter 2

Play on networks 2: Strategic complements

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October 31, 2005

Summary

- Introduction
- Nash equilibrium in pure strategies.
- Example
- Interpretation: Counting path length
- Policy: The Key Player
- Generalization of above set-up.

Introduction *(1/4)*

- Let network g with $g_{ij} \in \{0, 1\}$.

- For all $i \in N$, action $x_i \geq 0$.

- $\frac{\partial^2 u_i}{\partial x_i \partial x_j} = g_{ij} b''(x_i + \overline{x}_i) \leq 0$ in Bramoullé-Kranton.

- $\frac{\partial^2 u_i}{\partial x_i \partial x_j} = g_{ij} \lambda \geq 0$ here. Local strategic complements.

- Linear-quadratic utilities

$$u_i(x_1, ..., x_n; g) = \alpha x_i - \frac{1}{2} x_i^2 + \lambda \sum_{j \in N} g_{ij} x_i x_j; \lambda \geq 0, \alpha > 0.$$

- With $\lambda = 0$, no interdependence and $x_i^* = \alpha$.

- With $\lambda > 0$, interdependence.

- FOC:

$$\frac{\partial u_i}{\partial x_i} = \alpha - x_i + \lambda \sum_{j \in N} g_{ij} x_j = 0.$$

- FOC ($x_i - \lambda \sum_{j \in N} g_{ij} x_j = \alpha$) in general gives a system of equations

$$[I - \lambda G] \overrightarrow{x} = \alpha \overrightarrow{1}.$$

- Determinant of $[I - \lambda G]$ is a polynomial in λ, thus generically invertible matrix.

- We study this more in depth later.

- Now, suppose you have a regular network, where for all $i \in N$, $\sum_{j \in N} g_{ij} = k$.

- Then an equilibrium exists with $x_i = x$ for all $i \in N$. We must have $\alpha - x + \lambda k x = 0$, thus $x^* = \frac{\alpha}{1-\lambda k}$ (assuming $\lambda k < 1$).

- For $\lambda > 0$, $x^*(\lambda)$ is increasing in λ (when equilibrium exists).

- In general, outcome will depend on the network, when there is heterogeneity.

Remark 1 *We show here there is a generically unique Nash equilibrium in pure strategies.*

- Notice that $u_i(x_1, ..., x_n; g)$ is such that $\frac{\partial^2 u_i}{\partial x_i^2} = -1 < 0$. This implies:

- x^* is a Nash equilibrium iff for all $i \in N$ either

  1(a) $x_i^* = 0$ and $\frac{\partial u_i}{\partial x_i}(0, x_{-i}^*) \leq 0$

  (b) $x_i^* > 0$ and $\frac{\partial u_i}{\partial x_i}(x^*) = 0$.

- But notice that if $x_i^* = 0$, $\frac{\partial u_i}{\partial x_i}(0, x_{-i}^*) = \alpha + \lambda \sum_{j \in N} g_{ij} x_j^* > 0$.

- Thus only (b) is relevant and x^* is a Nash equilibrium iff:

$$[I - \lambda G] \overrightarrow{x}^* = \alpha \overrightarrow{1}, \text{ and } x_i^* > 0 \text{ for all } i \in N.$$

Nash equilibrium in pure strategies. *(2/5)*

- Solution of former equation exists and is unique iff det $[I - \lambda G] \neq 0$.

- There exists a finite number of values of λ such that $[I - \lambda G]$ is degenerate, and it has Lebesgue measure zero, thus generically unique Nash equilibrium.

- When a solution exists, is it necessarily in $\Re^{+}$?

- Debreu and Herstein (1953), the matrix $[I - \lambda G]^{-1} = M(g, \lambda)$ is well-defined and non-negative iff λ is smaller than the largest eigenvalue of G.

- If λ is small enough

$$[I - \lambda G]^{-1} = \sum_{k \geq 0} \lambda^k G^k$$

- To see this diagonalize $G = P^{-1} \begin{bmatrix} \mu_1 & \dots & 0 \\ \dots & \mu_i & \dots \\ 0 & \dots & \mu_n \end{bmatrix} P.$

- Thus $\lambda^k G^k = P^{-1} \begin{bmatrix} (\lambda\mu_1)^k & \dots & 0 \\ \dots & (\lambda\mu_i)^k & \dots \\ 0 & \dots & (\lambda\mu_n)^k \end{bmatrix} P.$

- So if $\lambda \max_i \{\mu_i\} < 1$, $\sum_{k \geq 0} \lambda^k G^k$ converges and

$$\overrightarrow{x}^* = \alpha \left[I - \lambda G\right]^{-1} \overrightarrow{1}$$

- Summarizing the above we have:

Proposition 2 *Let* $\mu_1(g)$ *be the largest positive eigenvalue of* G. *If* $\lambda\mu_1(g) < 1$, *the game has a unique interior pure strategy equilibrium given by*

$$\frac{x_i^*}{\alpha} = m_{i1}(g, \lambda) + \dots + m_{in}(g, \lambda)$$

Nash equilibrium in pure strategies. *(4/5)*

with $M(g, \lambda) = \left[m_{ij}(g, \lambda) \right] = [I - \lambda G]^{-1} = \sum_{k \geq 0} \lambda^k G^k$.

Notice differences with previous model:

1. Equilibrium unique with complement - multiplicity with substitutes.

2. Equilibrium interior with complement - interior equilibria unstable with substitutes.

Example *(1/3)*

Suppose a 3 person network, with 1 connected to 2 and 3.

- $G = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow G^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, G^3 = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

- By induction $G^{2p} = \begin{bmatrix} 2^p & 0 & 0 \\ 0 & 2^{p-1} & 2^{p-1} \\ 0 & 2^{p-1} & 2^{p-1} \end{bmatrix}, G^{2p+1} = \begin{bmatrix} 0 & 2^p & 2^p \\ 2^p & 0 & 0 \\ 2^p & 0 & 0 \end{bmatrix}$

- $x_1^* = \sum_{p=0}^{\infty} \left[\lambda^{2p} 2^p + \lambda^{2p+1} 2^p + \lambda^{2p+1} 2^p\right] = \frac{1}{1-2\lambda^2} + \frac{2\lambda}{1-2\lambda^2} = \frac{1+2\lambda}{1-2\lambda^2}$

- $x_2^* = x_3^* = \sum_{p=0}^{\infty} \left[\lambda^{2p+1} 2^p + \lambda^{2p} 2^{p-1} + \lambda^{2p} 2^{p-1}\right] = \frac{1+\lambda}{1-2\lambda^2}.$

Example *(2/3)*

- Condition for existence $1 - 2\lambda^2 > 0$, $\lambda < 1/\sqrt{2}$.

- In general for a star with n nodes, largest eigenvalue of $G = \sqrt{n-1}$.

Interpretation: Counting path length *(1/3)*

- How many paths are there (in example) starting at node i between individuals i and j with length 2 (not repeating traveled through nodes)?

- Between 1&1 - 2, between 1&2 or 1&3 - 0.

- Between 2&1 - 0, between 2&2 or 2&3 -1.

- Between 3&1 - 0, between 3&2 or 3&3 -1.

- Notice that $G^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

- This is general. For $G^k = \left[g_{ij}^{[k]}\right]$ counts total number of paths in g of length k starting at node i between individuals i and j.

- Now $\sum_{k\geq 0} \lambda^k g_{ij}^{[k]}$ is the total number of paths in g *of all lengths* between individuals i and j but discounting paths of length k by λ^k.

- Remember $m_{ij}(g,\lambda) = \sum_{k\geq 0} \lambda^k g_{ij}^{[k]}$.

Definition 3 *Bonacich (1987). Take network g and parameter λ small enough. The network centrality of individual i in g of parameter λ is*

$$b_i(g,\lambda) \equiv \sum_{j=1}^{n} m_{ij}(g,\lambda) = \underbrace{m_{ii}(g,\lambda)}_{\textit{self-loops}} + \underbrace{\sum_{j\neq i} m_{ij}(g,\lambda)}_{\textit{outer-paths}}$$

- Since $\frac{x_i^*}{\alpha} = \sum_{j=1}^{n} m_{ij}(g,\lambda) = b_i(g,\lambda)$, the equilibrium action is proportional so Bonacich centrality.

Policy: The Key Player *(1/7)*

In first place one must propose a planner's objective.

1. $F(g;\lambda,\alpha) = \sum_{j=1}^{n} x_j^* = \alpha \sum_{j=1}^{n} b_i(g,\lambda)$. This may be the measure if the network is simply a "factor of production" of a "good" or a "bad" (the model was originally created to study crime.)

2. $G(g;\lambda,\alpha) = \sum_{j=1}^{n} u_j(x^*;g)$. This is more useful if we think of a "public good" setup.

For the second measure notice that by FOC $\alpha - x_i^* + \lambda \sum_{j\in N} g_{ij} x_j^* = 0$. Thus

$$u_j(x^*;g) = x_i^* \left(\alpha - \frac{1}{2}x_i^* + \lambda \sum_{j\in N} g_{ij} x_j^*\right) = x_i^* \left(0 + \frac{1}{2}x_i^*\right) = \frac{1}{2}{x_i^*}^2$$

And thus

$$G(g;\lambda,\alpha) = \frac{1}{2} b_i(g,\lambda)^2.$$

PLANNER'S TOOLS-THE KEY PLAYER

- Classical public economics tools (tax subsidy) modify: λ, α.
- To the extent she can control it $\rightarrow$ Modify g
  - Reshuffle network.
  - Eliminate link(s).

Definition 4 *Node i is a* Key Player *iff*

$$i \in \arg\max_{j \in N} \left\{ \sum_{k=1}^{n} b_k(g,\lambda) - \sum_{k \neq j} b_k(g^{-j},\lambda) \right\}$$

- Notice that

$$\sum_{k=1}^{n} b_k(g,\lambda) - \sum_{k\neq j} b_k(g^{-j},\lambda) = \underbrace{b_i(g)}_{i\text{'s direct contribution}} + \underbrace{\sum_{k\neq j}\Big(b_k(g,\lambda) - b_k(g^{-j},\lambda)\Big)}_{i\text{'s indirect contribution}}.$$

- Thus Key Player need not be the player with highest centrality, since indirect contribution also matters.

- Example:

Proposition 5 *Node i is a* Key Player *iff*

$$i \in \arg\max_{j\in N} \left\{ \frac{b_j(g,\lambda)^2}{m_{jj}(g)} \right\}$$

To show this we first prove:

Lemma 6 $m_{ij}(g) \cdot m_{ik}(g) = \underbrace{m_{ii}(g)\left[m_{jk}(g) - m_{jk}(g^{-1})\right]}_{B}$

Proof. $m_{ii}(g) = \sum_{p\geq 0} \lambda^p g_{ii}^{[p]}$

$$m_{jk}(g) - m_{jk}(g^{-1}) = \sum_{\substack{p\geq 0 \\ p\geq 2 \text{ (at least need 2 steps)}}} \lambda^p \underbrace{\left[g_{jk}^{[p]} - g_{j(-i)k}^{[p]}\right]}_{\substack{g_{j(i)k}^{[p]} \\ \text{paths } jk \text{ through } i}}$$

Thus

$$B = \sum_{p=2}^{\infty} \lambda^p \left[\sum_{\substack{r+s=p \\ r\geq 0, s\geq 2}} g_{ii}^{[r]} \cdot g_{j(i)k}^{[s]}\right]$$

Notice that $\left(\sum_{p\geq 1}\lambda^p x^p\right)\left(\sum_{p\geq 1}\lambda^p y^p\right) = \sum_{p\geq 2}\lambda^p\left(\sum_{r+s=p} x^r y^s\right)$

Thus

$$\sum_{p\geq 2}\lambda^p \sum_{r^i+s^i=p} g_{ji}^{[r^i]}\cdot g_{ik}^{[s^i]} = \left(\sum_{p\geq 1}\lambda^p g_{ji}^{[p]}\right)\left(\sum_{p\geq 1}\lambda^p g_{ji}^{[p]}\right)$$

■

Now to prove the proposition. By lemma:

$$\begin{aligned}\sum_{k\neq j}\left(b_k(g,\lambda)-b_k(g^{-j},\lambda)\right) &= \sum_{j\neq i}\sum_k\left[m_{jk}(g)-m_{jk}(g^{-1})\right]\\ &= \sum_{j\neq i}\sum_k \frac{m_{ij}(g)\cdot m_{ik}(g)}{m_{ii}(g)}\\ &= \sum_{j\neq i}\frac{m_{ij}(g)}{m_{ii}(g)}\underbrace{\sum_k m_{ik}(g)}_{b_i(g,\lambda)}\end{aligned}$$

Thus:

$$
\begin{aligned}
b_i(g) + \sum_{k \neq j} \left(b_k(g, \lambda) - b_k(g^{-j}, \lambda) \right) &= b_i(g) \left[1 + \sum_{j \neq i} \frac{m_{ij}(g)}{m_{ii}(g)} \right] \\
&= b_i(g) \left[\frac{m_{ii}(g) + \sum_{j \neq i} m_{ij}(g)}{m_{ii}(g)} \right] \\
&= \frac{b_i(g)^2}{m_{ii}(g)}
\end{aligned}
$$

- Note that $\frac{b_i(g)^2}{m_{ii}(g)} = b_i(g) \left[1 + \sum_{j \neq i} \frac{m_{ij}(g)}{m_{ii}(g)} \right]$,
- Thus what matters is not only centrality, but also the composition of the contribution.
- If the relative weight of outer paths to self loops is larger, more likely to be Key Player.

Generalization of above set-up.

Let

$$u_i(x_1, ..., x_n; g) = \alpha x_i + \sum_{j \in N} \sigma_{ij} x_i x_j; \lambda \geq 0, \alpha > 0.$$

$\underline{\sigma} = \min_{ij \in g} \sigma_{ij}; \overline{\sigma} = \max_{ij \in g} \sigma_{ij}; \frac{\partial^2 u_i}{\partial x_i^2} = \sigma_{ii} < 0$

Conditions: $\sigma_{ii} = \sigma < \min\{0, \underline{\sigma}\}$, concavity on myself is highest.

In Bramoullé-Kranton: $\frac{\partial^2 u_i}{\partial x_i^2} = b''(x_i + \overline{x}_i) = \frac{\partial^2 u_i}{\partial x_i \partial x_j}$; if $g_{ij} \neq 0$.

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