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Networks - Fall 2005

Chapter 2

Play on networks 1: Strategic substitutes

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Summary

- Introduction
- Equilibria: characterization
- Equilibria: stability
- Welfare
- Link addition:

- $N = \{1, ..., n\}$, set of players.
- g an undirected network. That is: $g_{ij} \in \{0, 1\}, g_{ij} = g_{ji}, \forall i, j \in N$.
- $X_i = \Re^+, x_i \in X_i$ is player i's action.
- $u_i(x_1, ..., x_n; g) = b(x_i + \overline{x}_i) - cx_i$, with $c > 0$ and where $\overline{x}_i = \sum_{j \in N} g_{ij} x_j$.
- Assume $b' > 0, b'' < 0$ and there exists a unique x^* with $b'(x^*) = c$.
- Notice that $\frac{\partial^2 u_i}{\partial x_i \partial x_j} = g_{ij} b''(x_i + \overline{x}_i) \leq 0$. Strategic substitutes.

Equilibria: characterization *(1/2)*

Proposition 1 $x = (x_1, ..., x_n)$ *is a Nash equilibrium if (a)* $\overline{x}_i \geq x^*$ *and* $x_i = 0$ *or (b)* $\overline{x}_i < x^*$ *and* $x_i = x^* - \overline{x}_i$.

Remark 2 $BR_i(x_{-i}) = \max\{0, x^* - \overline{x}_i\}$.

Example 3 *Let a completely connected network with* $N = 4$, $x^* = 1$. *The following are NE: (a) (1/4,1/4,1/4,1/4) (b) (0,0,0,1) (c) (0,1/4,3/4,0).*

Example 4 *Let a circle with* $N = 4$, $x^* = 1$ *with an added link* $ij = 13$. *The following are NE: (a) (1,0,0,0) (b) (0,1,0,1) (c) (1/4,0,3/4,0).*

Proposition 5 $x = (x_1, ..., x_n)$ *is an expert Nash equilibrium if the corresponding set of experts is a maximal independent set of* g.

Let us explain this proposition:

Equilibria: characterization *(2/2)*

1. $x = (x_1, ..., x_n)$ is an *expert Nash equilibrium* if it is a Nash equilibrium and $x_i \in \{0, x^*\}$ for all $i \in N$.

2. Set of experts in x in an *expert Nash equilibrium* is $\{i \in N | x_i = x^*\}$.

3. $I \subseteq N$ is an *independent set* for g iff for all $i, j \in I, g_{ij} = 0$.

4. An independent set is called *maximal independent set,* if no additional member can be added without destroying independence (*maximal* with respect to set inclusion.)

Equilibria: stability

Definition 6 $x = (x_1, ..., x_n)$ *is a stable Nash equilibrium if there exists a* $\rho > 0$ *such that for any vector* ε *satisfying* $|\varepsilon_i| < \rho$ *for all* $i \in N$, *the sequence* $x^{(n)}$ *defined by* $x^{(0)} = x + \varepsilon$ *and* $x^{(n+1)} = BR(x^{(n)})$ *converges to* x.

Proposition 7 *For any network* g *an equilibrium is stable if and only if it is specialized and every non specialist is connected to (at least) two specialists.*

- Networks were all $x_i > 0$ are neutrally stable, it leads to limit cycles. If i increases, j matches the decrease and vice versa.
- Center-sponsored stars diverge. A decrease of ε is matched by simultaneous increase of many, which is amplified.
- Center-subsidized stars converge. A decrease of ε by the periphery is not matched and back to normal.

$W(x,g) = \sum_{i \in N} b(x_i + \overline{x}_i) - c \sum_{i \in N} x_i$, and notice $x_j > 0$ implies $x_j = x^* - \overline{x}_j$.

$$\left.\frac{\partial W}{\partial x_j}\right|_{x_j > 0} = \underbrace{b'(x_j + \overline{x}_j) - c}_{=0} + \sum_{k \neq j, jk \in g} b'(x_k + \overline{x}_k) > 0 \qquad (1)$$

- So any agent $j \in N$ with $x_j > 0$ would increase W by increasing x_j.
- What equilibrium has highest welfare?
- Let x be a Nash equilibrium for g. At equilibrium for all i, $x_i + \overline{x}_i \geq x^*$
- So $W(x,g) = n.b(x^*) + \sum_{i | x_i = 0} (b(\overline{x}_i) - b(x^*)) - c \sum_{i \in N} x_i$.

Welfare *(2/3)*

- $\sum_{i|x_i=0}(b(\overline{x}_i) - b(x^*))$ is premium from specialization.

- In a completely connected graph, with all making same effort $(1/N * x^*)$ no premium from specialization but minimum possible cost.

- Expert equilibria, premium from specialization but higher cost.

1. Distributed equilibria $W(x,g) = nb(x^*) - c\sum_{i \in N} x_i$
2. Expert equilibria. There are free riders $\sum_{i|x_i=0}(b(\overline{x}_i) - b(x^*))$ free rider premium.

In a 4 person circle:

Welfare *(3/3)*

1. $W(dist) = 4b(x^*) - \frac{4}{3}cx^*$,

2. $W(\exp) = 4b(x^*) + 2(b(2x^*) - b(x^*)) - 2cx^*$.

Heuristic 1: For low c expert equilibria are better than distributed ones.

Let expert equilibria with maximal independent set I, and s_j be the number of contacts in I for $j \notin I$

$W(x,g) = nb(x^*) + \sum_{j \notin I} \left(b(s_j x_j) - b(x^*) \right) - c|I|x^*$. But since $s_j \geq 2$

$W(x,g) \geq nb(x^*) + (n - |I|)\,(b(2x^*) - b(x^*)) - c|I|x^*$, decreasing with $|I|$.

Heuristic 2: Look for expert equilibria with maximum number of free-riders.

Compare the Second-best welfare when adding a link ij.

1. Suppose either $x_i = 0$ or $x_j = 0$ in g. Then x is still an equilibrium in $g + ij$, so welfare can only increase.

2. Suppose both $x_i \neq 0$ and $x_j \neq 0$. Then x is not an equilibrium in $g + ij$ and welfare could decrease.

- Take two three person stars. Second-best is two center-sponsored stars.
- Link two centers.
- New second best is one of the centers still specialist and the periphery of the other specialist.
- Welfare falls if increase in cost $2ce^*$ is bigger than new free-riding premium $b(4e^*) - b(e^*)$.

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