Networks - Fall 2005

Chapter 2

Play on networks 3: Coordination and social action

Morris (2000) and Chwe (2000)

September 16, 2005

Summary

- Introduction: Morris 2000
- Questions
- Cohesion
- Introduction (Chwe 2000)
- Sufficient networks and cliques

- Set of players N on a network g.
- Agents on nodes play a coordination game with neighbors. Use same action on all.
- Game Γ is:

$s_1 \backslash s_2$	0	1
0	$u(0,0); u(0,0)$	$u(0,1); u(1,0)$
1	$u(1,0); u(0,1)$	$u(1,1); u(1,1)$

- Assume $u(0,0) > u(1,0)$ and $u(1,1) > u(0,1)$.

- If agent 2 chooses strategy 1 with probability p, agent 1 prefers 1 to 0 if:

$$(1-p)\cdot u(0,0)+p\cdot u(0,1)>(1-p)\cdot u(1,0)+p\cdot u(1,1).$$

- That is agent 2 prefers 1 to 0 if $q<p$, where

$$q\equiv\frac{u(0,0)-u(1,0)}{(u(0,0)-u(1,0))+(u(1,1)-u(0,1))}$$

- Then, let the game Γ' :

$s_1 \backslash s_2$	0	1
0	q, q	$0, 0$
1	$0, 0$	$1 - q, 1 - q$

- The game Γ' is strategically equivalent to Γ.

- In effect notice that agent 2 prefers 1 to 0 if:

$$(1 - p) \cdot 0 + p \cdot (1 - q) > (1 - p) \cdot q + p \cdot 0 \Leftrightarrow p > q.$$

- So we will use the simpler Γ'.

- We let g given, $n \to \infty$.

Questions *(1/3)*

- Suppose initially everybody plays $s_i(0) = 0$: $s(0) = (0, 0, ..., 0)$.
- Suppose that a finite group of players switches to $s_i = 1$.
- *Can the whole network switch to* $s_j = 1$?
- *It depends on the value of* q *and the network* g.
- Suppose some play 1 and some play zero at time $t - 1$.
  - Payoff for player i playing 0 is:

$$u_i(0, s_{-i}(t-1) = q \cdot \sharp\{j \in N | ij \in g, s_j(t-1) = 0\}.$$

Questions *(2/3)*

- Payoff for player i playing 1 is:

$$u_i(1, s_{-i}(t-1)) = (1-q) \cdot \sharp\{j \in N | ij \in g, s_j(t-1) = 1\}.$$

- A switch occurs if $u_i(1, s_{-i}(t-1)) > u_i(0, s_{-i}(t-1))$:

$$q < \frac{\sharp\{j \in N | ij \in g, s_j(t-1) = 1\}}{\sharp\{j \in N | ij \in g\}} = \frac{\sharp\{j \in N | ij \in g, s_j(t-1) = 1\}}{\sum_{j \in N} g_{ij}}$$

- Take a line. A few people switch to play 1. Then for somebody in the boundary of the "switchers" the condition is $q < \frac{1}{2}$.

- For a regular m-dimensional grid interacting with 1 step away in at most 1 dimension (interaction between x and x' if $\sum_{i=1}^{m} \left| x_i - x'_i \right| = 1$).
  - Then contagion occurs if $q < \frac{1}{2n}$.

Questions *(3/3)*

- Now take m-dimensional grid, but interaction with agents situated n-steps away at most in all dimensions (interaction between x and x' if $\max_{i=1,\dots,n} \left| x_i - x_i' \right| = n$).
  - Contagion if $q < \frac{n(2n+1)^{m-1}}{(2n+1)^m - 1}$.
    - Denominator: The $2n+1$ combinations in m dimensions (-1 as you do not count yourself).
    - Numerator: Any advancing "frontier" has to be one-dimension less, but has a "depth" n.

- Important property for contagion.

- Intuition: how likely it is that friends of my friends are also my friends (in physics lit. "clustering.")

- Take a finite set V, and $i \in V$. Let the proportion of i's contacts in V.

$$B_i(V) = \frac{\sharp\{\{j \in N|ij \in g\} \cap V\}}{\sharp\{j \in N|ij \in g\}}$$

Definition 1 *The cohesion of V, denoted by $B(V) = \min_{i \in V} B_i(V)$*

- That is, the cohesion of V is the minimum proportion of contacts in V among all members of V, or the minimum proportion of inner links (resp. outer links) is at least $B(V)$ (resp. $1 - B(V)$.)

Definition 2 *A finite set of nodes V is $(1-q)$-cohesive if $B(V) \geq 1-q$*

- V is $(1-q)$-cohesive if the proportion of outer links is at most q.

- A set is cofinite if its complementary is finite.

Lemma 3 *Diffusion is not possible if every cofinite set contains a finite $(1-q)$-cohesive subset.*

Remark 4 *Decreasing q increases possibility of contagion.*

- Contagion by definition starts in a finite set X.

- So take its complement X^c. This is a cofinite set.

Cohesion *(3/5)*

- By the assumption of the lemma, X^c contains a finite $(1-q)$-cohesive subset. Call it V.

- $q \geq 1 - B(V)$, so even if all people around V switch to playing 1, the people in V will not switch. Thus contagion is not possible.

Remark 5 *If there exists a cofinite set such that none of its subsets is* $(1-q)$*-cohesive, then contagion is possible.*

- This will happen if the "epidemic" starts in the complement of the cofinite set which has no $(1-q)$-cohesive subsets.

Definition 6 *Contagion threshold* ξ *is the largest* q *such that action* 1 *spreads to the whole population starting by best-response from some finite group.*

Proposition 7 *The contagion threshold is the smallest* p *(call it* p^**) such that every co-finite group contains an infinite* $(1-p)$*-cohesive subgroup.*

- Suppose not. Then $\xi(g) > p^*$. Let $\xi(g) > q > p^*$. For such q contagion is possible.

- But for q there by the contradiction assumption there is a cofinite group which contains an infinite $(1-q)$-cohesive subgroup. But by previous lemma, contagion is not possible. A contradiction.

Proposition 8 *Let* D *such that for all* $i \in N$, $\sharp\{j \in N | ij \in g\} \leq D$*. Then* $\xi(g) \geq \frac{1}{D}$.

- Suppose not. Then $\xi(g) < \frac{1}{D}$. Then let $\xi(g) < q < \frac{1}{D}$.

- But every person who comes in contact with one 1-player will switch over to 1.

- This is true since for that person $\sharp\{j \in N | ij \in g, s_j(t-1) = 1\} \geq 1$, and for everybody $\sharp\{j \in N | ij \in g\} \leq D$.

- Thus $q < \frac{1}{D} \leq \frac{\sharp\{j \in N | ij \in g, s_j(t-1)=1\}}{\sharp\{j \in N | ij \in g\}}$.

Corollary 9 *If players are connected within g, in the long-run co-existence of conventions is possible if $\xi(g) < q < 1 - \xi(g)$.*

Remark 10 *In the line, co-existence is not possible since $\xi(g) = 1/2$.*

Remark 11 *If you want to get rid of coexistence, you should change q or the structure of the network,*

- **Question**: Why are all of a sudden people interested in collective action?

- N set of players.

- $N = \{1, ..., n\}$, set of players.

- $X_i = \{0, 1\}, x_i \in X_i$ is player i's action.

- Types are $\theta_i \in \Theta_i = \{w, y\}$ (*willing, unwilling*), private information.

- $\theta = (\theta_1, ..., \theta_n) \in \Theta = \{w, y\}^n$.

Introduction (Chwe 2000) *(2/4)*

- $u_i(x_i, y) = \begin{cases} 0 \text{ if } x_i = 0 \\ 1 \text{ if } x_i = 0 \end{cases}$. So *unwilling* do not revolt no matter what.

- $u_i(x_i, w) = \begin{cases} -1 \text{ if } x_i = 1, \text{ and } \sharp\{j \in N | x_j = 1\} < e_i \\ 1 \text{ if } x_i = 1, \text{ and } \sharp\{j \in N | x_j = 1\} \geq e_i \\ 0 \text{ if } x_i = 0 \end{cases}$. So the *willing* revolt if enough other people do so.

- The game is denoted by $\Gamma_{e_1, e_2, \ldots, e_n}$

- The communication *network* is *directed*: $g_{ji} = 1$ means that i knows j's type.

- So each individual i knows the people in her ball: $B(i) = \{j | g_{ji} = 1\}$.

Introduction (Chwe 2000) *(3/4)*

- The state of the world is θ, but each i only knows that:

$$\theta \in P_i(\theta) = \{(\theta_{B(i)}, \phi_{N \setminus B(i)}) : \phi_{N \setminus B(i)} \in \{w, y\}^{n - \sharp B(i)}\}$$

- The union of sets $\cup_{\theta \in \Theta} \{P_i(\theta)\}$ is a partition of Θ, which we denote $\mathcal{P}_i$.

- A *strategy* is a function $f_i : \Theta \to \{0, 1\}$, which is measurable with respect to $\mathcal{P}_i$.

- That is, if both $\theta, \theta' \in P$ and $P \in \mathcal{P}_i$, then $f_i(\theta) = f_i(\theta')$.

- F_i is the set of all strategies for i.

- Let prior beliefs $\pi \in \Delta(\Theta)$.

- Then ex-ante expected utility of strategy profile f is

$$EU_i(f) = \sum_{\theta \in \Theta} \pi(\theta) u_i(f(\theta), \theta).$$

- A strategy profile f is an *equilibrium* if

$$EU_i(f) \geq EU_i(g_i, f_{N \setminus \{i\}}) \text{ for all } g_i \in F_i.$$

- A pure strategy equilibrium exists (use supermodularity.) One can even talk of a "maximal" equilibrium.

- It is important that the information on types only travels one link.

Sufficient networks and cliques *(1/6)*

- What are *sufficient networks* so that "all go" for *all priors*?

Definition 12 *We say that g is a sufficient network if for all $\pi \in \Delta(\Theta)$, there exists an equilibrium f of $\Gamma(g, \pi)$ such that $f_i(w, ..., w) = 1$ for all $i \in N$.*

- Sufficient networks exist since the complete network is sufficient.

- In a complete network, types are common knowledge, so if $\theta_i = w$ for all $i \in N$, then if all willing types except i revolt, then i prefers to revolt.

- Priors do not matter at this point since types are common knowledge.

- What are the *minimal* sufficient networks?

Definition 13 *We say that g is a minimal sufficient network if for all g, if $g' \subset g$ and g' is a sufficient network, then $g' = g$.*

Definition 14 *A clique of g is a set $M_k \subset N$ such that $g_{ij} = 1$ for all $i, j \subset M_k$.*

- A *clique* is, then, a component of a network of fully intraconnected individuals.

Proposition 15 *Say g is a minimal sufficient network. Then there exist cliques $M_1, ..., M_z$ such that $N = M_1 \cup ... \cup M_z$ and a binary relation $\rightarrow$ over the M_i such that:*

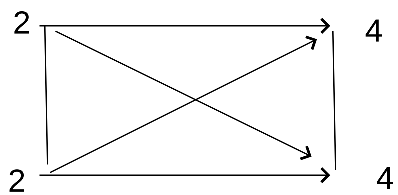
1. *$g_{ji} = 1$ iff there exist M_k and M_l such that $i \in M_k$ and $j \in M_l$ and $M_k \rightarrow M_l$*

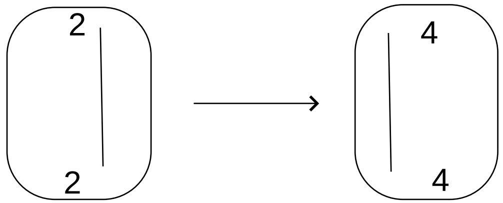
2. *If $M_{i_y-1} \to M_{i_y}$ then there exists a totally ordered set $M_{i_1}, ..., M_{i_y-1}, M_{i_y}$, where M_{i_1} is maximal.*

- Fact 1: in a minimal sufficient network if I talk to you everybody in my clique also talks to you/knows your type.

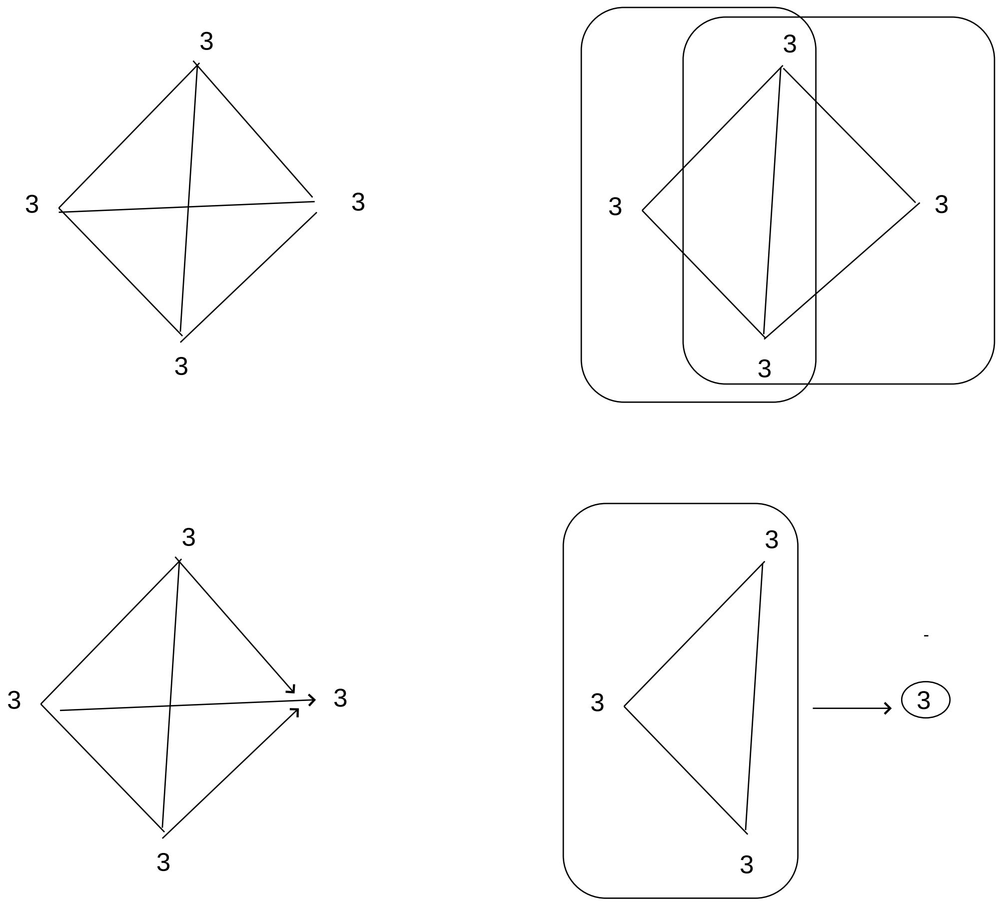
- Fact 2: the cliques are arranged in a hierarchical order, that is, all cliques are ordered in "chains."

- Take the threshold game $\Gamma_{2,2,4,4}$. We represent below the minimal sufficient network and the hierarchy of cliques:



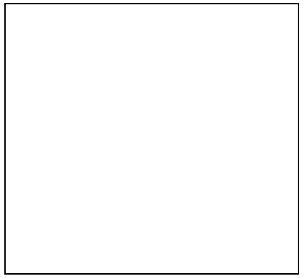


- For the game $\Gamma_{3,3,3,3}$ there are two minimal sufficient networks, represented below:



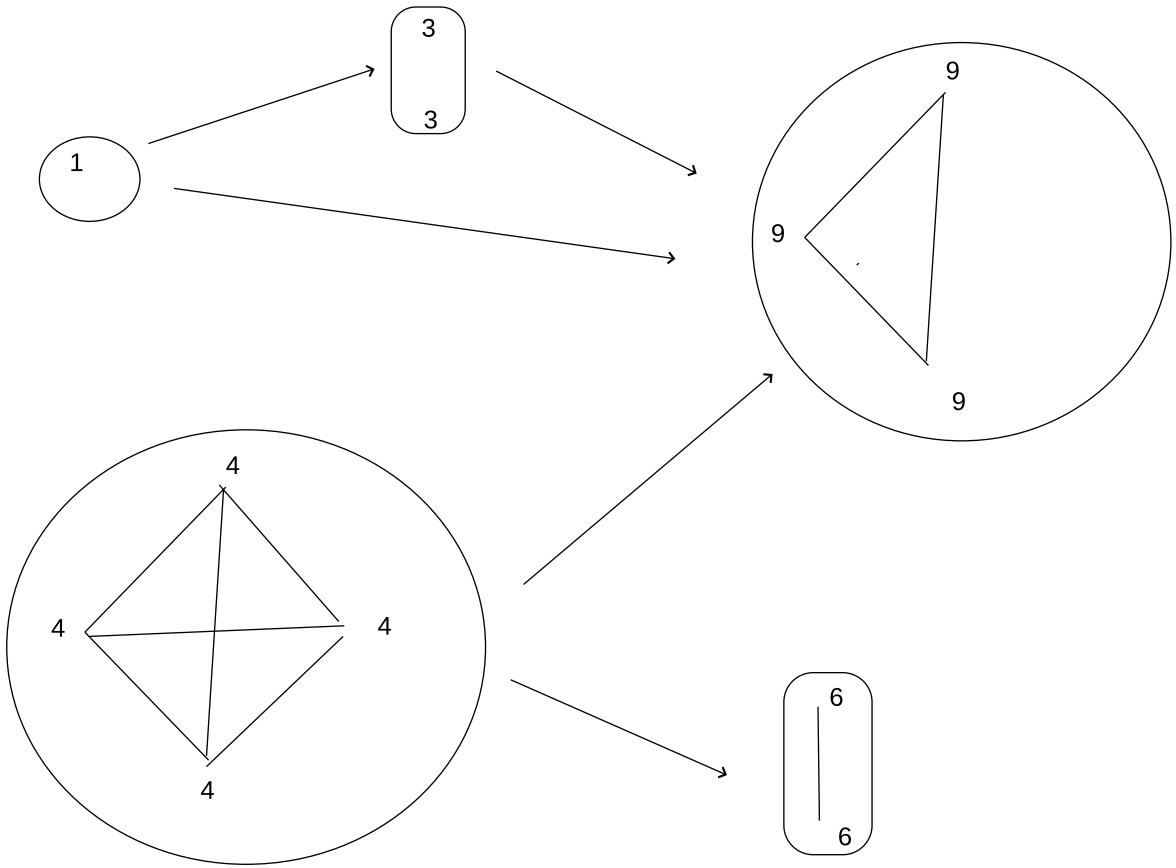
Sufficient networks and cliques *(5/6)*

- In that same game it is interesting to see why the following graph is not a sufficient network (even though all people know there is sufficient "impetus" for revolt):



Sufficient networks and cliques *(6/6)*

- For the game $\Gamma_{1,3,3,4,4,4,4,6,6,9,9,9}$ the minimal sufficient network has two leading cliques.



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