Microeconomics II - Winter 2005

Chapter 3

Games with Incomplete Information - Bayes-Nash equilibrium

Antonio Cabrales

January 18, 2005

Summary

- Examples
- Bayesian games
- Bayesian equilibria for examples

Examples *(1/4)*

A Entry and capacity building game.

I,E	e	n
B	0,-1	2,0
N	2,1	3,0

I,E	e	n
B	1.5,-1	3.5,0
N	2,1	3,0

In the left-hand side game, the cost of building capacity is 3, in the left-hand side, it is 1.5. Nature chooses left-hand side with probability p. Player 1 is informed (and only him), then both players choose actions simultaneously.

B Contribution game.

1,2	C	N
C	$1 - c_1, 1 - c_2$	$1 - c_1, 1$
N	$1, 1 - c_2$	$0, 0$

Cost, if contribution is chosen, c_i is private information and distributed $U[0, 2]$. Benefit is 1 if at least one contributes.

Examples *(3/4)*

C Second price auction. Two players, strategies $b_i \in \Re^+, i = 1, 2$.

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 \text{ if } b_1 > b_2 \\ \frac{v_1 - b_2}{2} \text{ if } b_1 = b_2 \\ 0 \text{ if } b_1 < b_2 \end{cases}, u_2(b_1, b_2) = \begin{cases} v_2 - b_1 \text{ if } b_2 > b_1 \\ \frac{v_2 - b_1}{2} \text{ if } b_2 = b_1 \\ 0 \text{ if } b_2 < b_1 \end{cases}$$

v_i is private information and distributed $D[0, 1]$.

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v_i is private information and distributed $U[0, 1]$.

Bayesian games *(1/5)*

1. Players

   $N = \{1, ..., n\}$

2. Set of types (for each player)

   $\forall i \in N$, there is a set T_i of types representing what player i knows (preferences, technology, information) $T \equiv \prod_{i=1}^{n} T_i$. That is, i knows which $t_i \in T_i$ is true, but only knows that t_{-i} is some member of T_{-i}.

3. Probability distribution of types

   $P : T \rightarrow [0, 1]$

4. Possible actions

   $\forall i \in N$, there is a set A_i, a generic member is a_i. The generic profile of actions $a = (a_1, ..., a_n) \in A \equiv \prod_{i=1}^{n} A_i$.

5. Payoffs

   $\forall i \in N$, there is a function $\Pi_i : T \times A \longrightarrow \Re$.

To transform this into a game, we need a strategy set

Strategies

$\forall i \in N$, a strategy $\gamma_i \in \Gamma_i$ is a function $\gamma_i : T_i \rightarrow A_i$.

Bayes-Nash Equilibrium

A strategy profile $\gamma^* = (\gamma_1^*, ..., \gamma_n^*) = (\gamma_i^*, \gamma_{-i}^*) \in \Gamma$ is a Bayes-Nash equilibrium if, for all $i \in N$, $\gamma_i \in \Gamma_i$

$$\sum_{t_1 \in T_1} \cdots \sum_{t_n \in T_n} P(t_1, ..., t_n)\pi_i(t_1, ..., t_n, \gamma_1^*(t_1), ..., \gamma_i^*(t_i), ..., \gamma_n^*(t_n)) \geq$$
$$\sum_{t_1 \in T_1} \cdots \sum_{t_n \in T_n} P(t_1, ..., t_n)\pi_i(t_1, ..., t_n, \gamma_1^*(t_1), ..., \gamma_i(t_i), ..., \gamma_n^*(t_n))$$

We can rewrite this as

$$\sum_{t_i \in T_i} \sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) \pi_i(t_i, t_{-i}, \gamma_i^*(t_i), \gamma_{-i}^*(t_{-i})) \geq$$
$$\sum_{t_i \in T_i} \sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) \pi_i(t_i, t_{-i}, \gamma_i(t_i), \gamma_{-i}^*(t_{-i}))$$

More importantly, this definition is equivalent to:

A strategy profile $\gamma^* = (\gamma_1^*, ..., \gamma_n^*) = (\gamma_i^*, \gamma_{-i}^*) \in \Gamma$ is a Bayes-Nash equilibrium if, for all $i \in N$, $a_i \in A_i$ and for all $t_i \in T_i$

$$\sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) \pi_i(t_i, t_{-i}, \gamma_i^*(t_i), \gamma_{-i}^*(t_{-i})) \geq \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) \pi_i(t_i, t_{-i}, a_i, \gamma_{-i}^*(t_{-i}))$$

Bayesian games *(5/5)*

Why?

First note that the definition below certainly implies the one above. Remember that $P(t_i, t_{-i}) = P(t_i)P(t_{-i}|t_i)$, then add over t_i.

Now, suppose the one above did not imply the one above. Then there must exist a type t_i' and an action a_i' with

$$\sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i')\pi_i(t_i', t_{-i}, a_i', \gamma_{-i}^*(t_{-i})) > \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i')\pi_i(t_i', t_{-i}, \gamma_i^*(t_i'), \gamma_{-i}^*(t_{-i}))$$

But then let us construct γ_i', such that $\gamma_i'(t_i) = \gamma_i^*(t_i)$ for $t_i \neq t_i'$ and $\gamma_i'(t_i') = a_i'$. Then by noticing again that $P(t_i, t_{-i}) = P(t_i)P(t_{-i}|t_i)$ and adding over t_i. we see that

$$\sum_{t_i \in T_i} \sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i})\pi_i(t_i, t_{-i}, \gamma_i'(t_i), \gamma_{-i}^*(t_{-i})) > \sum_{t_i \in T_i} \sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i})\pi_i(t_i, t_{-i}, \gamma_i^*(t_i$$

and so we reach a contradiction.

Game A

I,E	e	n
BB	$1.5(1-p), -1$	$2p + 3.5(1-p), 0$
BN	$2(1-p), -p + (1-p)$	$2p + 3(1-p), 0$
NB	$2p + 1.5(1-p), p - (1-p)$	$3p + 3.5(1-p), 0$
NN	2,1	3,0

- Note first that BB and BN are strictly dominated for player I.

- If $p > 0.5$, n is strictly dominated for E and then there is only one equilibrium (NN,e).

- If $p \leq 0.5$, there are two equilibria in pure strategies (NN,e) and (NB,n). There is also an equilibrium in mixed strategies (if $p < 0.5$), namely $((1/2(1-p), 1 - 1/2(1-p)), (1/2, 1/2))$.

Game B

$\gamma_i : [0,2] \to \{C, N\}$. Let $z_j = \Pr(\gamma_j(c_j) = C)$. Then

$$\pi_1(C, \gamma_2(c_2)|c_1) = 1 - c_1; \pi_1(N, \gamma_2(c_2)|c_1) = z_2$$

Thus the optimal strategy (best-response) is:

$$\gamma_1^*(c_1) = \begin{cases} C \text{ if } c_1 \leq 1 - z_2 \\ N \text{ if } c_1 > 1 - z_2 \end{cases}$$

and similarly

$$\gamma_2^*(c_2) = \begin{cases} C \text{ if } c_2 \leq 1 - z_1 \\ N \text{ if } c_2 > 1 - z_1 \end{cases}$$

The indifferent type is c_i^*. Thus, $1 - z_2 = c_1^*$, and $1 - z_1 = c_2^*$. From the definition of z_i we have $1 - \frac{c_2^*}{2} = c_1^*, 1 - \frac{c_1^*}{2} = c_2^*$. Thus $c_1^* = c_2^* = \frac{2}{3}$.

No contribution, even though, $\frac{2}{3} < c_i < 1$.

Game C

$\gamma_1(v_1) = v_1$ is a weakly dominant strategy.

1. Let $b'_1 > v_1$.

   (a) If $b_2 > b'_1$

   $u_1(v_1, b_2) = 0 = u_1(b'_1, b_2)$

   (b) If $b_2 < b'_1, b_2 \geq v_1$

   $u_1(v_1, b_2) = 0 > u_1(b'_1, b_2) = v_1 - b_2$

   (c) If $b_2 = b'_1, b_2 \geq v_1$

   $u_1(v_1, b_2) = 0 > u_1(b'_1, b_2) = \frac{v_1 - b_2}{2}$

(d) If $b_2 < b'_1, b_2 < v_1$

$$u_1(v_1, b_2) = v_1 - b_2 = u_1(b'_1, b_2)$$

2. Let $b'_1 < v_1$.

(a) If $b_2 \geq v_1, b_2 > b'_1$

$$u_1(v_1, b_2) = 0 = u_1(b'_1, b_2)$$

(b) If $b_2 < v_1, b_2 > b'_1$

$$u_1(v_1, b_2) = v_1 - b_2 > 0 = u_1(b'_1, b_2)$$

(c) If $b_2 < v_1, b_2 < b'_1$

$$u_1(v_1, b_2) = v_1 - b_2 = u_1(b'_1, b_2)$$

Game D

Equilibrium:

1. In pure strategies.
2. Strategies are affine functions: $\gamma_i(v_i) = \max\{\alpha_i + \beta_i v_i, 0\}$
3. Symmetric: $\alpha_i = \alpha, \beta_i = \beta, \forall i = 1, 2.$

The equilibrium is like this, but strategies are best against anything else.

- $\alpha \geq 0$. Otherwise, let $v_1 < \frac{-\alpha_1}{\beta_1}$. This type must bid 0. But then

$$u_1(0, \gamma_2|v_1) = \frac{v_1}{2}\Pr(v_2 \leq \frac{-\alpha_2}{\beta_2}) < (v_1 - \varepsilon)\Pr(v_2 \leq \frac{-\alpha_2}{\beta_2}) \leq u_1(\varepsilon, \gamma_2|v_1)$$

  Thus $b_1 = 0$ is not optimal for $v_1 < \frac{-\alpha_1}{\beta_1}$.

- $\alpha \leq 0$. Otherwise some types v_i have $\gamma_i(v_i) = \alpha_i + \beta_i v_i > v_i$ (v_i small enough is such that $\alpha_i > (1 - \beta_i)v_i$).

- So we have $\gamma_i(v_i) = \beta v_i$ ($\beta > 0$).
  $u_1(b_1, \gamma_2|v_1) = (v_1 - b_1)\Pr(v_2 < b_1/\beta|v_1) + \frac{(v_1 - b_1)}{2}\Pr(v_2 = b_1/\beta|v_1)$.

  Given $v_i\tilde{}U[0,1]$, $\Pr(v_2 = b_1/\beta|v_1) = 0$, thus $u_1(b_1, \gamma_2|v_1) = (v_1 - b_1)\frac{b_1}{\beta}$. Thus the optimal strategy for agent 1 is: $\gamma_1(v_1) = \frac{v_1}{2}$, and thus, identifying coefficients $\beta = \frac{1}{2}$.

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