Auctions with Heterogeneous Entry Costs*

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Abstract

It is well known that in standard auctions where buyers have independent private values and homogeneous entry costs a reserve price equal to the seller's value (and no entry fee) maximizes social surplus and seller revenue, and leaves bidders with no surplus. Further, in mixed strategy entry equilibria social surplus and seller revenue decrease with the number of potential bidders. In contrast, we show that when entry costs are heterogeneous the revenue maximizing reserve price is typically above the seller's value, an appropriate entry fee (and a reserve price equal to the seller's value) generates even more revenue, and bidders capture informational rents. Further, seller revenue and social surplus may either increase or decrease with the number of potential bidders. However, asymptotic seller revenue is the entire social surplus.

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1 Introduction

A classic result of the auction literature is that in a standard auction with an exogenously fixed number of bidders who have independent private values, the optimal (i.e., revenue maximizing) reserve price is above the seller's value, and is independent of the number of bidders – see Riley and Samuelson (1981) and Myerson (1981). In many instances, however, the number of bidders is the result of costly entry decisions. As noted by Milgrom (2004), "... auctions for valuable yet highly specialized assets often fail because of insufficient interest by bidders ... buyers are naturally reluctant to begin an expensive, time-consuming evaluation of an asset when they believe that they are unlikely to win at a favorable price." Entry decisions may thus be affected by the auction format (including the reserve price), making the number number of bidders endogenous. Indeed, McAfee and McMillan (1987) and Levin and Smith (1994) have shown that the endogenous entry of bidders has important implications for the seller's choice of a reserve price. Specifically, if all bidders have the same entry cost, then the optimal reserve price is the seller's value and an entry fee is not a useful instrument to increase seller revenue.

In this paper we study optimal reserve prices and entry fees in standard auctions with endogenous entry, but where bidders have heterogenous entry costs. In the sale of a firm, for example, prospective buyers may have different concerns regarding the regulatory restrictions they face; discovering the value of the firm for sale may involve substantially different costs for different bidders, as some bidders may have to seek approval by regulatory authorities while others may not face such constraints. Our model is identical to that of McMillan (1987) and Levin and Smith (1994), except that prior to deciding whether to enter the auction each bidder privately observes her entry cost, which is an independent draw from a common distribution. In the model, bidders, upon observing their entry costs, simultaneously choose whether to enter the auction. Each bidder who enters the auction observes his value for the object and then bids.

Heterogeneity in entry costs alters many of the conclusions obtained for the homogenous entry cost case. Specifically, we show that the optimal reserve price is typically above the seller's value, although it is below the reserve that is optimal when the number of bidders is fixed. In addition, the optimal reserve price depends on the number of bidders as well as the distribution of values and entry costs. Further, entry fees provide a valuable instrument to rise seller revenue: when an entry fee is feasible, then it is optimal to set a reserve equal to the seller's value in conjunction with a positive entry fee.

In order to understand the intuition for our results, it is useful to review the results and intuition when entry costs are homogeneous. Let us assume for simplicity that the seller's value for the object is zero. A key result in this setting is that in a standard auction with a reserve price equal to zero the contribution to social surplus of an additional bidder is exactly equal to the bidder's expected utility to entering.[1] Thus, when entry costs are homogeneous, the interests of a entrant and society are aligned: a bidder enters only if her expected utility to entering is above her entry cost; that is, if her contribution to social surplus is positive. Since bidders enter the auction so long as their contribution to social surplus is positive, the number of entering bidders maximizes social surplus provided there are sufficiently many bidders.[2] In equilibrium, the bidder surplus is competed away (a bidder is indifferent between entering or not), and therefore the seller captures the entire social surplus. Hence a reserve price equal to zero maximizes both seller revenue and social surplus, regardless of the distribution of values and the number of bidders (provided there are sufficiently many).

We show that when entry costs are heterogeneous a version of the key result described above also holds: in a standard auction with a reserve price equal to zero the contribution to social surplus of a marginal increase of the equilibrium entry threshold is proportional to the bidder's expected utility; that is, the interests of bidders and society are also aligned when entry costs are heterogenous. Consequently, a standard auction with a zero reserve maximizes social surplus whether entry costs are homogeneous or heterogeneous. With heterogeneous entry costs, however, not all bidder surplus is competed away by entry: whereas the net expected utility of a bidder with an entry cost equal to the equilibrium threshold is exactly zero, the

[1] A version of this result is established in Engelbrech-Wiggans (1993)'s Proposition 1, and is also observed in both MM and LS.

[2] The maximum social surplus differs if we consider asymmetric equilibria where some bidders enter and others stay out of the auction, or we restrict attention to symmetric equilibria where all bidders enter with the same probability – see section 3.

expected entry cost of an entering bidder is below the equilibrium threshold, and therefore bidders capture a positive share of the surplus. Hence, even though setting a positive reserve price reduces total surplus, it may increase the seller's share of social surplus and therefore may increase revenue.

Thus, when entry costs are heterogeneous the optimal reserve price may be positive (i.e., above the seller's value). Indeed, we show this is the case when the bidders' values are uniformly distributed, regardless of the distribution of entry costs. Interestingly, the optimal reserve is always below the reserve price that is optimal when the number of bidders is exogenously fixed. In addition, when the optimal reserve price is above the seller's value, then an even greater revenue can be obtained by employing an appropriate entry fee and setting the reserve price to zero. (In contrast, it is well-known that an entry fee is equivalent to a reserve price when the number of bidders is fixed, and that the optimal entry fee is zero when bidders have homogenous entry costs.) Further, the optimal reserve price depends on the number of bidders, as well as on the distribution of values and entry costs.

There is another important difference between homogeneous and heterogeneous entry costs. For homogeneous entry costs, LS show that seller revenue decreases with the number of bidders in an entry equilibrium in mixed strategies. We describe simple examples that show that a direct extension of this result does not hold when entry costs are heterogeneous: even if the number of bidders is such that a bidder enters with probability less than one, an increase in the number of bidders may either increase or decrease seller revenue depending upon the distribution of values and entry costs.

Equilibrium with homogenous and heterogeneous entry costs are, however, closely related as the number of bidders grows large. In particular, asymptotic seller revenue is the same when (i) bidders have homogenous entry costs, $c > 0$, and (ii) when bidders have heterogenous entry costs and the lower bound of entry costs is c. In other words, heterogeneity of entry costs does not matter asymptotically. We also show that if bidders' values are uniformly distributed and the lower bound of entry costs is zero, then seller revenue approaches the maximum surplus as the number of bidders becomes large.

Other models of auctions with endogenous entry have been studied in the literature. Samuelson (1985) studies a procurement sealed-bid auction with entry where bidders have a homogeneous entry cost, but make entry decisions after observing their procurement costs. He shows that when the reserve is equal to the buyer's value, equilibrium is socially optimal. He also shows by means of examples that an increase in the number of bidders may either an increase or decrease procurement costs. Analogous results are obtained by Menezes and Monteiro (2000) who study the equilibria of first- and second-price sealed-bid auctions in this framework – see also Tan and Yilankaya (2007). Kaplan and Sela (2003) study auctions where entry costs are private information, but the bidders' values are commonly known. Green and Laffont (1984) study the existence of equilibrium in a model where, as in our setting, both entry costs and values are private information, but they assume, as in Samuelson (1985), that a bidder makes entry decisions having observed both her entry cost and her value.

In a concurrent paper, Lu (2007) studies optimal entry fees in a model similar to ours. He shows that the seller's optimal auction is a second-price sealed-bid auction with an entry fee, and provides an interesting characterization of optimal entry fees. While a seller can generally set a reserve price, in many settings it is not feasible for the seller to set an entry fee. Characterizing the optimal reserve price is difficult as, unlike entry fees, reserve prices not only influences the bidders' entry decisions but also reduce the efficiency of the auction. In Internet auctions a bidder's cost of discovering his value is the opportunity cost of his time, and it varies significantly across bidders. Since reserve prices are commonly used in such auctions (and entry fees are not possible), our results are useful to understanding the effect of reserve prices in empirical studies of reserve prices in Internet auctions using either naturally occurring data or data obtained from field experiments – e.g., Reiley (2006).

The paper is organized as follows. In Section 2 we layout the basic setting. Section 3 reviews the results for homogenous entry costs. Section 4 presents our results for heterogenous entry costs. Section 5 concludes. Proofs are in the Appendix.

2 Preliminaries

Consider a market for a single object for which there are N risk-neutral bidders and a risk-neutral seller. In this market the object is allocated using an unspecified *standard* auction with a reserve price. Each bidder must decide whether to enter the auction, and thereby incur an entry cost. A bidder who enters the auction learns her value (and perhaps the number of bidders who entered the auction), and then bids. The bidders' values $X_1, \ldots, X_N$ are independently and identically distributed on $[0, \omega]$ according to an increasing and differentiable *c.d.f.* F with an increasing hazard rate. The seller's value for the object is zero.

In order to focus on the analysis of the "entry game," we assume throughout that bidding strategies conform to the assumptions required to apply the Revenue Equivalence Principle; that is, we assume that following entry decisions, *for each reserve price the bidding strategies form an increasing symmetric equilibrium of the auction such that the expected payment of a bidder with value zero is zero* – see Myerson (1981), Riley and Samuelson (1981). (It is well-known that the Revenue Equivalence Principle applies even when there is uncertainty about the number of bidders in the auction, provided that bidders have symmetric expectations – see Krishna (2002), Section 3.2.2, whose notation we follow closely.) Under this assumption, the seller's revenue and the bidders' expected utilities following entry decisions can be calculated as if the auction were a second-price sealed-bid auction. Thus, if the reserve price is $r \in [0, \omega]$ and exactly $n \in \{1, \ldots, N\}$ bidders enter the auction, then seller revenue is

$$\pi(r,n) = n\left[r(1-F(r))F^{n-1}(r) + (n-1)\int_r^\omega y(1-F(y))F^{n-2}(y)f(y)dy\right],$$

and the expected utility of a bidder is

$$u(r,n) = \int_r^\omega \left(\int_r^y F(x)^{n-1}dx\right) f(y)dy.$$

Also, the gross social surplus, i.e., the social surplus ignoring entry costs, can be calculated as

$$s(r,n) = \int_r^\omega y dF^n(y).$$

Note that

$$s(0,n) = E(Y_1^{(n)}),$$

where for $n \in \{1, \ldots, N\}$, $Y_1^{(n)}$ is the highest order statistic. It is easy to see that $\pi(r, n)$ is increasing in n, $u(r, n)$ is decreasing in both r and n, and $s(r, n)$ is decreasing in r and increasing in n. The convention $s(0, 0) = 0$ will be useful in what follows.

Proposition 1 below establishes that in a standard auction with a zero reserve price and n bidders the expected utility of each bidder is equal to the gross social contribution of the n-th bidder. (We provide a simple proof in the Appendix. A version of this formula is established in Proposition 1 of Engelbrecht-Wiggans (1993).) As will be seen later, this fact is key to understanding the intuition for our results.

Proposition 1. *For* $n \in \{1, \ldots, N\}$: $u(0, n) = s(0, n) - s(0, n-1)$.

It will be useful to calculate the expected revenue of the seller and the expected utility of a bidder when the number of bidders in the auction follows a binomial distribution $B(N, p)$, where p is the probability that a single bidder enters, and $p_n^N(p)$ is the probability that exactly $n \in \{0, 1, \ldots, N\}$ bidders enter. The expected revenue of the seller is

$$\Pi(r, p) = \sum_{n=1}^{N} p_n^N(p)\pi(r, n), \tag{1}$$

and the expected utility to a bidder entering the auction is

$$U(r, p) = \sum_{n=0}^{N-1} p_n^{N-1}(p)u(r, n+1).$$

It is easy to see that $U(r, p)$ is decreasing in p: If $p'' > p'$, then $B(N, p'')$ first order stochastically dominates $B(N, p')$, and therefore since $u(r, n)$ is decreasing with respect to n, we have $U(r, p'') < U(r, p')$.

3 Homogenous entry costs

In this section we derive existing results and simple extensions that identify the *optimal* reserve price (i.e., the reserve price that maximizes seller revenue) for the case of homogenous entry costs. Assume that all bidders have the same fixed entry cost $c > 0$. We assume that $u(0, N) < c$; i.e., when the reserve price is zero, if all N bidders enter, then the expected utility of each bidder is less than c. This assumption rules out the uninteresting case where every bidder enters the auction with probability one. We further assume that $c < u(0,1)$, which rules out an equilibrium in which no bidder enters.

In this setting McAfee and McMillan (1987) establish that in a pure strategy entry equilibrium of a first-price sealed-bid auction with a zero reserve price (i) the maximum social surplus is realized (i.e., the optimal number of bidders enters the auction and the object is allocated to the bidder with the maximum value), and (ii) the seller captures the entire surplus; hence (iii) the optimal reserve price is zero. Levin and Smith (1994) show that results analogous to (i)-(iii) hold in a symmetric mixed strategy equilibria of any standard auction. These results are easily derived in our setting, and extended to any standard auction in the case of McAfee and McMillan (1987)'s results. This exercise will help provide intuition for our results for the perhaps more realistic case where entry costs are heterogenous.

The maximum social surplus that can be achieved by *any* mechanism with a fixed number n of bidders is

$$w(n) = E(Y_1^{(n)}) - nc = s(0,n) - nc.$$

A standard auction with a zero reserve price attains this maximum. Write $w^* = \max_{n \in \{0,1,\ldots,N\}} w(n)$.

Since $u(0,n) = s(0,n) - s(0,n-1)$ by Proposition 1, then the social contribution of the n-th bidder is

$$\begin{aligned} w(n) - w(n-1) &= s(0,n) - s(0,n-1) - c \\ &= u(0,n) - c. \end{aligned}$$

Since $u(0,n)$ is decreasing in n this contribution is decreasing in n.

Consider the incentives of bidders when they sequentially decide whether to enter a standard auction with a zero reserve price. The n-th bidder enters if her payoff to entering is at least her cost, i.e., if

$$u(0,n) - c \geq 0. \tag{2}$$

As shown above, the left hand side of this expression is just the social contribution of the n-th bidder. Hence, when the reserve price is zero a bidder enters if and only if her entry raises social surplus. Therefore in a pure strategy entry equilibrium the number of entering bidders n^* maximizes social surplus; i.e., $w(n^*) = w^*$. If we ignore that n^* must be an integer, then n^* satisfies (2) with equality, and bidders capture none of the surplus.

This argument establishes that a standard auction with a zero reserve price maximizes social surplus and, moreover, the seller captures the entire social surplus. A positive reserve price reduces social surplus and, because seller revenue is at most the social surplus, also reduces seller revenue. Hence the optimal reserve price is zero.

The key insight above was that the private and social benefit of the entry of a bidder coincide in a standard auction with a zero reserve price. The same logic applies to symmetric entry equilibria in mixed strategies. If each of N bidders enters with probability p, then the number of bidders follows the binomial distribution $B(N, p)$ and the maximum (constrained) social surplus that can be achieved by any mechanism is

$$W(p) = \sum_{n=1}^{N} p_n^N(p) s(0, n) - Npc. \tag{3}$$

A standard auction with a zero reserve price attains this maximum. Write $W^* = \max_{p \in [0,1]} W(p)$. Note that W^* is a "constrained" maximum surplus; i.e., it is the maximum surplus when all bidders enter with the same probability.[3]

Since $u(0, n) = s(0, n) - s(0, n-1)$, then we have[4]

$$\begin{aligned} W'(p) &= N \left(\sum_{n=1}^{N} p_{n-1}^{N-1}(p) s(0, n) - \sum_{n=1}^{N-1} p_n^{N-1}(p) s(0, n) - c \right) \\ &= N \left(\sum_{n=0}^{N-1} p_n^{N-1}(p) u(0, n+1) - c \right) \\ &= N(U(0, p) - c), \end{aligned}$$

i.e., the marginal social contribution of an increase in the probability of entry is proportional to the payoff of an entering bidder. In a symmetric mixed strategy entry equilibrium bidders are indifferent between entering and not;[5] i.e., bidders enter with

[3]It is easy to show that our assumption $u(0, N) < c < u(0, 1)$ implies the number of bidders n^* that maximizes social surplus $w(n)$ satisfies $1 < n^* < N$. This in turn implies that if bidders use a symmetric entry rule, then social surplus is below w^*. Hence $w^* > W^*$.

[4]A version of this formula can be found in Milgrom (2004)'s proof of Theorem 6.5.

[5]It is easy to see that a symmetric mixed-strategy equilibrium p^* exists, is unique, and satisfies $p^* > 0$.

a probability p^* satisfying

$$U(0, p^*) - c = 0.$$

Hence $W'(p^*) = 0$. Since U is decreasing in p, then $W''(p) < 0$, i.e., W is a concave function. Therefore $W^* = W^*(p^*)$; i.e., a symmetric mixed strategy entry equilibrium maximizes the social surplus. Since the seller captures all the social surplus, the optimal reserve price is zero.

Since the seller captures the entire social surplus with a zero reserve price, whether bidders follow a symmetric mixed strategy equilibrium or a pure strategy equilibrium, there is no advantage to the seller to setting an entry fee.[6]

These results are summarized in the Proposition below.

Proposition *(Homogeneous entry costs – McAfee and McMillan (1987), Levin and Smith (1994).) In a standard auction with a reserve price equal to zero, if bidders follow a (symmetric mixed) pure strategy entry equilibrium, the (constrained) maximum social surplus is realized and is captured by the seller. Hence the optimal reserve price and the optimal entry fee are zero.*

4 Heterogenous entry costs

In this section we study the general case where bidders have heterogenous entry costs. Specifically, each bidder i has a privately known entry cost Z_i. Bidders' entry costs $Z_1, \ldots, Z_N$ are independently and identically distributed on $\mathbb{R}_+$ according to a *c.d.f.* H with support $[\underline{c}, \bar{c}]$, where $0 \leq \underline{c} < \bar{c} \leq \infty$. As in the homogenous entry cost case (i.e., the case where H is degenerate), we assume that $u(0, N) < \bar{c}$ and $\underline{c} < u(0, 1)$ in order to rule out uninteresting equilibria. For simplicity, we assume also that H is increasing, differentiable, and satisfies $H(\underline{c}) = 0$.

Under these assumptions, an entry strategy for a bidder can be described by a number $t \in [\underline{c}, \bar{c}]$ indicating the threshold (the maximum entry cost) for which the

[6] However, our argument ignores that in the pure strategy equilibria studied by McAfee and McMillan (1987) the number of entrants is an integer, and therefore bidder surplus will be typically positive, and may be nonnegligible. When this is the case, an entry fee equal to bidder surplus allows the seller to capture the entire social surplus. If an entry fee is not feasible, then the optimal reserve price is positive.

bidder enters the auction; that is, the bidder enters if her entry cost is less than t, and does not enter if it is greater than t – whether the bidder enters when her entry cost is exactly t is inconsequential.[7] If all bidders employ the same threshold t, then the number of bidders in the auction is distributed according to the binomial distribution $B(N, p)$ where $p = H(t)$.

Consider any standard auction with a reserve price $r \in [0, \omega]$. A *symmetric (Bayes perfect) entry equilibrium* is a threshold $t \in [\underline{c}, \bar{c}]$ such that for all $z \in [\underline{c}, \bar{c}]$: $U(r, H(t)) > z$ implies $t > z$, and $U(r, H(t)) < z$ implies $t < z$; i.e., in a symmetric entry equilibrium t a bidder enters if her expected utility to entering is above her entry cost, and does not enter if it is below.

We now define a mapping that will describe the symmetric entry equilibrium threshold of a standard auction for every reserve price $r \in [0, \omega]$. This mapping, $t^* : [0, \omega] \rightarrow [\underline{c}, \bar{c}]$, is given by $t^*(r) = \underline{c}$ if $U(r, 0) \leq \underline{c}$, and by the unique solution to the equation

$$t = U(r, H(t)) \tag{4}$$

if $U(r, 0) > \underline{c}$. The mapping $t^*(\cdot)$ is a continuous function on $[0, \omega]$. Denoting by $\hat{r}$ the unique solution to

$$U(r, 0) = \underline{c},$$

then $t^*(\cdot)$ is decreasing on $[0, \hat{r}]$ and it is equal to $\underline{c}$ for $r \in [\hat{r}, \omega]$ – see Lemma 1 in the Appendix.

Proposition 2 establishes that a standard auction with a reserve price has a unique symmetric entry equilibrium.[8]

Proposition 2. *A standard auction with a reserve price $r \in [0, \omega]$ has a unique symmetric entry equilibrium, given by $t = t^*(r)$.*

Assume that each bidder enters when her entry cost is less than $t \in [\underline{c}, \bar{c}]$. Then the social surplus generated in a standard auction with a reserve price $r \in [0, \omega]$ is

$$\hat{W}(r, t) = \sum_{n=1}^{N} p_n^N(H(t)) s(r, n) - N c(t), \tag{5}$$

[7] In general, entry decisions are described by a mapping from $[\underline{c}, \bar{c}]$ into $[0, 1]$ indicating for each entry cost the probability with which the buyer enters the auction. When H is atomless, however, it is without loss of generality to restrict attention to entry strategies described by a threshold.

[8] Tan and Yilankaya (2006) obtain an analogous result in Samuelson's model.

where

$$c(t) = \int_{\underline{c}}^{t} z dH(z)$$

is the expected entry cost incurred by each bidder. Also the maximum social surplus that can be achieved by any mechanism is $\hat{W}(0,t)$. Write $\hat{W}^* = \max_{t\in[\underline{c},\bar{c}]} \hat{W}(0,t)$ for the "constrained" maximum social surplus; i.e., $\hat{W}^*$ is the maximum surplus if we restrict attention to symmetric entry rules.

Recall that a standard auction with a reserve price of zero maximizes social surplus when entry costs are homogeneous. Proposition 3 establishes that a standard auction with a reserve price of zero also maximizes social surplus when entry costs are heterogeneous. In particular, the symmetric entry equilibrium threshold $t^*(0)$ induces the socially optimal entry; that is, $\hat{W}(0, t^*(0)) = \hat{W}^*$.

Proposition 3. *A standard auction with a reserve price equal to zero maximizes social surplus, i.e.,* $\hat{W}(0, t^*(0)) = \hat{W}^*$.

In a standard auction with reserve price r the expected surplus of a bidder is

$$\int_{\underline{c}}^{t^*(r)} (t^*(r) - z) dH(z).$$

Hence if the reserve price r is below $\hat{r}$, so that $t^*(r) > \underline{c}$, then each bidder's expected surplus is positive, which we state as the following result.

Proposition 4. *In a standard auction with a reserve price $r \in [0, \hat{r})$ bidders capture a positive surplus. Hence seller revenue is less than the social surplus.*

In contrast, when entry costs are homogenous, bidder surplus is zero for any reserve price set by the seller. This difference between the homogeneous and heterogenous entry cost cases has important implications for the seller's optimal reserve, as we see shall shortly.

When bidders have heterogeneous entry costs, seller revenue in a standard auction with a reserve price $r \in [0, \omega]$ is $\Pi(r, H(t^*(r)))$. An *optimal reserve price* r^* satisfies $r^* \in \arg\max_r \Pi(r, H(t^*(r)))$. It is well known that when the number of bidders is exogenously given, then the optimal reserve price $\bar{r}$ is positive, and is the solution to the equation

$$r = \frac{1 - F(r)}{f(r)}, \tag{6}$$

independently of the number of bidders present in the auction – see Riley and Samuelson (1981) and Myerson (1981).

Proposition 5 establishes that an optimal reserve price r^* is strictly below $\bar{r}$, since the seller has an incentive to induce additional entry through a lower reserve price. Unlike in the homogeneous entry cost case where the optimal reserve price is zero, when entry costs are heterogeneous the optimal reserve price may be positive. This is the case if bidders' values are uniformly distributed, for arbitrary distributions of entry costs H.

Proposition 5. *In a standard auction an optimal reserve price r^* satisfies $0 \leq r^* < \bar{r}$. Moreover, if values are uniformly distributed, then $0 < r^*$.*

It is worth discussing why the optimal reserve price may be positive. With homogenous entry costs, the seller captures the entire social surplus and hence optimally sets the reserve to zero in order to maximize social surplus. While a zero reserve also maximizes social surplus when entry costs are heterogenous (Proposition 3), the seller no longer captures the entire surplus. Hence, although setting a positive reserve price reduces social surplus, the seller is better off if it generates a distribution of the social surplus sufficiently more favorable to him.

Entry fees

Assume that the seller can set both an anonymous entry fee (or subsidy) as well as a reserve price.[9] Proposition 6 establishes that an entry fee enables the seller to obtain more revenue than he can obtain with a reserve price alone. In fact, when the seller can set both an entry fee and a reserve price, then the optimal reserve price is zero (the seller's value). Thus, when bidders have heterogenous entry costs, an entry fee is a more effective instrument to increase seller revenue than a reserve price. In contrast, it is well-known that when the number of bidders is exogenous, reserve

[9] Of course, often it is not possible for the seller to charge an entry fee. For example, none of the Internet auction websites allow the seller to charge an entry fee.

prices are equivalent to entry fees. And, as established earlier, when the number of bidders is endogenous but entry costs are homogeneous, the optimal entry fee and reserve price are both zero.

Proposition 6. *In a standard auction, a zero reserve price and an optimal entry fee yields a greater seller revenue than a positive reserve price and no entry fee.*

The intuition for this result is simple: if the reserve price is positive then the seller can reduce the reserve price to zero and at the same time raise the entry fee so that the expected utility to a bidder to entering the auction is unchanged. This entry fee (combined with the zero reserve) induces the same entry by bidders without incurring the ex-post inefficiencies of a positive reserve price. Seller revenue rises since social surplus rises, while bidder surplus is unchanged. Propositions 5 and 6 imply the following.

Proposition 7. *In a standard auction, if values are uniformly distributed and H is arbitrary, then a zero reserve price with a positive entry fee is optimal.*

It's easy to see that a result analogous to Proposition 3 holds for a standard auction with an entry fee and a reserve price; namely, that social surplus is maximized when both the entry fee and the reserve price are both zero. Although the outcome with a positive entry fee and zero reserve is ex-post efficient, a positive entry fee induces less entry than would be socially optimal.

Market Thickness

In this section we study the impact on seller revenue and social surplus of an increase in the number of bidders. When entry costs are homogeneous, Levin and Smith (1994) show that seller revenue and social surplus (which in this case coincide) decrease as the number of bidders increases when symmetric entry equilibrium is in mixed strategies. Simple examples show that a direct extension of the result of LS to the case of heterogeneous entry costs does not hold: whether seller revenue and social surplus increase or decrease with the number of bidders depends on the distribution of entry costs and values. For example, if values are uniformly distributed on $[0, 1]$, then as the number of bidders increases from $N = 1$ to $N = 2$ both the social surplus and seller revenue increase when the distribution of entry costs is uniform on $[0, 1]$, but decrease when it is uniform on $[.49, .5]$.

We show that as the number of bidders N grows large, the asymptotic properties of equilibrium in LS are closely related to the asymptotic properties of equilibrium with heterogeneous entry costs. In particular, when bidders have a homogeneous entry cost, $c > 0$, asymptotic seller revenue and asymptotic social surplus are the same as when bidders have heterogenous entry costs and the lower bound of the support of entry costs is $\underline{c} = c$. Consequently, when entry cost are heterogeneous, asymptotic seller revenue is invariant to changes in the distribution of entry costs that preserve the lower bound of its support. Further, asymptotic seller revenue equals asymptotic social surplus, and hence asymptotic bidder surplus is zero. In addition, a zero reserve is *asymptotically optimal* (i.e., seller revenue with a zero reserve is asymptotically equal to seller revenue with an optimal reserve). These results are established in Proposition 8 below.

For each N, denote by W_N^* the maximum constrained social surplus when all bidders have the same entry cost $c > 0$, and write Π_N^* for the seller revenue in a standard auction with an optimal reserve price. Recall that $W_N^* = \Pi_N^*$ for each N – Levin and Smith (1994). Likewise, for each N denote by $\hat{W}_N^*$ the maximum constrained social surplus when the bidders' (heterogenous) entry cost are independent draws from a *c.d.f.* H, and write $\hat{\Pi}_N^*$ for seller revenue in a standard auction with an optimal reserve price. Recall that $\hat{W}_N^* > \hat{\Pi}_N^*$ by Proposition 4.

Proposition 8. *If* $c = \underline{c} > 0$, *then asymptotic seller revenue and asymptotic social surplus are positive and the same, whether bidders' entry costs are heterogeneous or homogeneous; i.e.,* $\lim_{N\to\infty} W_N^* = \lim_{N\to\infty} \Pi_N^* = \lim_{N\to\infty} \hat{W}_N^* = \lim_{N\to\infty} \hat{\Pi}_N^*$. *Further, a zero reserve price is asymptotically optimal.*

An interesting case not covered by Proposition 8 is when the lower bound of the support of entry costs is zero, i.e., $\underline{c} = 0$. Proposition 9 below establishes that if values are uniformly distributed, then both asymptotic seller revenue and asymptotic social surplus equal ω (the upper bound of the support of values). An immediate implication of this result is that the total entry costs incurred by bidders, as well as total bidder surplus, are both asymptotically zero. As when $\underline{c} > 0$, a zero reserve is asymptotically optimal.

Proposition 9. *If the lower bound of the support of entry costs is zero, i.e.,* $\underline{c} = 0$*, and values are distributed uniformly on* $[0, \omega]$*, then asymptotic seller revenue (and asymptotic social surplus) is* ω*, i.e.,* $\lim_{N\to\infty} \hat{\Pi}_N^* = \lim_{N\to\infty} \hat{W}_N^* = \omega$*. Further, a zero reserve price is asymptotically optimal.*

As mentioned above, for homogeneous entry costs LS show that seller revenue decreases with the number of bidders. As illustrated in Figure 1 below, when entry costs are heterogeneous one can find examples where seller revenue increases with the number of bidders. Assume that values are distributed uniformly on $[0, 1]$. The top curve shows seller revenue as a function of the number of bidders when all bidders have an entry cost of $\frac{1}{8}$. Consistent with the LS result, seller revenue decreases with the number of bidders. The bottom curve is the graph of seller revenue as a function of the number of bidders when entry costs are distributed uniformly on $[\frac{1}{8}, \frac{1}{2}]$. It shows that seller revenue increases with the number of bidders. The two curves approach each other as the number of bidders becomes large – see Proposition 8.

Figure 1 goes here.

5 Conclusions

The conclusions obtained when entry costs are homogeneous, namely that (i) the optimal reserve price is zero, (ii) social surplus is maximized at the optimal reserve, and (iii) the seller captures the entire social surplus, are not robust to the introduction of heterogeneity in entry costs. In the generic case of heterogeneous entry costs, we rather find that (I) the optimal reserve price may be positive – e.g., if values follow a uniform distribution; (II) the social surplus may be below the (constrained) maximum surplus – because a positive reserve price both induces less entry than would be socially optimal and generates ex-post inefficient outcomes with positive probability; and (III) seller revenue is less than the social surplus – heterogeneity of entry costs generates informational rents, allowing bidders to capture a positive share of the social surplus. While auctions are of greatest interest for small number of bidders, as the number of bidders grows large, asymptotic seller revenue depends only on the lower bound of entry costs $\underline{c}$ and is the same as when entry costs are homogeneous and equal to $\underline{c}$.

6 Appendix

Proof of Proposition 1: For $n > 1$, by interchanging the order of integration we obtain

$$\begin{aligned} u(0,n) &= \int_0^\omega \left(\int_0^y F(x)^{n-1} dx \right) f(y) dy \\ &= \int_0^\omega \left(\int_x^\omega f(y) dy \right) F(x)^{n-1} dx \\ &= \int_0^\omega (1 - F(x)) F(x)^{n-1} dx. \end{aligned}$$

Integrating by parts we get

$$\begin{aligned} \int_0^\omega F(x)^n dx &= xF^n(x)|_0^\omega - \int_0^\omega nxF(x)^{n-1} f(x) dx \\ &= \omega - E\left(Y_1^{(n)}\right). \end{aligned}$$

Hence

$$\begin{aligned} u(0,n) &= \int_0^\omega F(x)^{n-1} dx - \int_0^\omega F(x)^n dx \\ &= \left(\omega - E\left(Y_1^{(n-1)}\right)\right) - \left(\omega - E\left(Y_1^{(n)}\right)\right) \\ &= s(0,n) - s(0,n-1). \end{aligned}$$

For $n = 1$ we have

$$u(0,1) = \int_0^\omega y f(y) dy = E(Y^{(1)}) = s(0,1) = s(0,1) - s(0,0). \square$$

Henceforth assume that entry costs are heterogeneous. In order to prove Proposition 2 we begin by establishing some properties of the mapping t^*.

Lemma 1: *The mapping t^* is a continuous function on $[0, \omega]$. Further, it is decreasing and satisfies $\bar{c} > t^*(r) > \underline{c}$ on $[0, \hat{r})$, where $\hat{r} \in (0, \omega]$ is the unique solution to the equation $U(r, 0) = \underline{c}$.*

Proof: Let $r \in [0, \omega]$; we have

$$U(r, 0) = \sum_{n=0}^{N-1} p_n^{N-1}(0) u(r, n+1) = u(r, 1).$$

(Note that $p_0^{N-1}(0) = 1$, and $p_n^{N-1}(0) = 0$ for $n \in \{1, \ldots, N-1\}$.) Since $u(0, 1) > \underline{c}$ by assumption, $u(\omega, 1) = 0 \leq \underline{c}$, and $U(\cdot, 0) \equiv u(\cdot, 1)$ is continuous and decreasing on $[0, \omega]$, the equation $U(r, 0) = \underline{c}$ has a unique solution, $\hat{r} \in (0, \omega]$.

For $r \in [0, \hat{r})$ we have $U(r, 0) > \underline{c}$, and

$$U(r, 1) = \sum_{n=0}^{N-1} p_n^{N-1}(1) u(r, n+1) = u(r, N) \leq u(0, N) < \bar{c}.$$

(Note that $p_n^{N-1}(1) = 0$ for $n \in \{0, 1, \ldots, N-2\}$ and $p_{N-1}^{N-1}(1) = 1$.) Hence, since $U(r, H(\cdot))$ is continuous (because H is absolutely continuous) the equation

$$t = U(r, H(t))$$

has a solution on $[\underline{c}, \bar{c}]$; and since $U(r, H(\cdot))$ is decreasing on $[\underline{c}, \bar{c}]$ (because $U(r, p)$ is decreasing in p and H is increasing), there is a unique solution. Therefore the function $t^*(\cdot)$ is well defined, and since $U(r, H(\cdot))$ is continuous (because each $u(\cdot, n)$ for $n \in \{1, ..., n\}$ is continuous), then $t^*(\cdot)$ is also continuous. We show that $t^*(\cdot)$ is decreasing on $[0, \hat{r})$. Let $r', r'' \in [0, \hat{r})$ be such that $r'' > r'$. Write $t^*(r') = t'$, and $t^*(r'') = t''$. Suppose by way of contradiction that $t'' \geq t'$. Since $U(r, H(t))$ is decreasing in both r and t, then we have

$$t' = U(r', H(t')) > U(r'', H(t'')) = t'',$$

which is a contradiction.

Let $r \in [0, \hat{r})$. We show that $t^*(r) > \underline{c}$. Suppose that $t^*(r) = \underline{c}$. Then

$$\underline{c} = t^*(r) = U(r, H(t^*(r))) = U(r, H(\underline{c})) = U(r, 0) = u(r, 1) > \underline{c},$$

which is a contradiction. We show that $t^*(r) < \bar{c}$. Suppose that $t^*(r) = \bar{c}$. Then

$$\bar{c} = t^*(r) = U(r, H(t^*(r))) = U(r, H(\bar{c})) = U(r, 1) = u(r, N) \leq u(0, N) < \bar{c},$$

which is a contradiction. □

Proof of Proposition 2: Consider a standard auction with a reserve price $r \in [0, \omega]$. We show that $t^*(r)$ is the unique symmetric entry equilibrium. If $r \in [\hat{r}, \omega]$, clearly $t^*(r) = \underline{c}$ is the unique symmetric entry equilibrium. If $r \in [0, \hat{r})$, then $t^*(r)$ is a symmetric entry equilibrium. We show that no other symmetric entry equilibrium exists. By Lemma 1 $\underline{c} < t^*(r) < \bar{c}$. Let $\bar{t} \in [\underline{c}, t^*(r))$. We show that $\bar{t}$ is not a symmetric entry equilibrium. Since $U(r, H(\cdot))$ is decreasing we have

$$U(r, H(\bar{t})) > U(r, H(t^*(r))) = t^*(r).$$

Therefore for $\bar{t} < z < t^*(r)$ we have $z < U(r, H(\bar{t}))$. Hence $\bar{t}$ is not a symmetric entry equilibrium. An analogous argument establishes that no $\bar{t} \in (t^*(r), \bar{c}]$ is a symmetric entry equilibrium either. $\square$

Proof of Proposition 3: Differentiating $\hat{W}(0, t)$ yields

$$\hat{W}'(0, t) = \sum_{n=1}^{N} \frac{dp_n^N(H(t))}{dt} s(0, n) - Nth(t).$$

For $n \leq N - 1$ we have

$$\frac{dp_n^N(H(t))}{dt} = N(p_{n-1}^{N-1} - p_n^{N-1})h(t),$$

and

$$\frac{dp_N^N(H(t))}{dt} = Np_{N-1}^{N-1}h(t).$$

(All binomial probabilities are calculated for $p = H(t)$.) Substituting these expressions and using Proposition 1, we have

$$\begin{aligned} \hat{W}'(0, t) &= Nh(t) \left(p_{N-1}^{N-1} s(0, N) + \sum_{n=1}^{N-1} (p_{n-1}^{N-1} - p_n^{N-1}) s(0, n) - t \right) \\ &= Nh(t) \left(\sum_{n=0}^{N-1} p_n^N u(0, n+1) - t \right) \\ &= Nh(t) \left(U(0, H(t)) - t \right). \end{aligned}$$

Since $U(0, H(t^*(0))) = t^*(0)$ by Lemma 1, we have

$$\hat{W}'(0, t^*(0)) = 0.$$

Moreover, since $h(t) > 0$ and $U(0, H(\cdot))$ is decreasing on $[\underline{c}, \bar{c}]$, then $\hat{W}'(0,t) > 0$ for $t < t^*(0)$, and $W'(t) < 0$ for $t > t^*(0)$. Hence $t = t^*(0)$ uniquely maximizes $\hat{W}(0,t)$ on $[\underline{c}, \bar{c}]$. Clearly $\hat{W}(0,t) > \hat{W}(r,t)$ for $r > 0$. Hence $\hat{W}(0, t^*(0)) \geq \hat{W}(0,t) \geq \hat{W}(r,t)$ for all (r,t), where the first inequality is strict if $t \neq t^*(0)$ and the second inequality is strict if $r > 0$. $\square$

The following lemmas are useful in the proof of Proposition 5. Recall that $\bar{r}$, the solution to the equation $r = (1 - F(r))/f(r)$, uniquely maximizes $\pi(\cdot, n)$ on $[0, \omega]$ for all $n \in \{1, ..., N\}$ – see Riley and Samuelson (1981) and Myerson (1981).

Lemma 2. *If* $\bar{r} < \hat{r}$, *then* $\Pi(\bar{r}, H(t^*(\bar{r}))) > \Pi(r, H(t^*(r)))$ *for* $r \in (\bar{r}, \omega]$.

Proof: Assume that $\bar{r} < \hat{r}$, and let $r \in (\bar{r}, \omega]$. Since $t^*(\bar{r}) > t^*(r)$ Lemma 1, the *c.d.f.* of the binomial $B(N, p(H(t^*(\bar{r}))))$ first order stochastically dominates the *c.d.f.* of the binomial $B(N, p(H(t^*(r))))$. Thus, because π is strictly increasing with respect to n, and $\pi(\bar{r}, n) > \pi(r, n)$ for all $n \in \{1, ..., N\}$, we have

$$\begin{aligned}
\Pi(\bar{r}, H(t^*(\bar{r}))) &= \sum_{n=1}^{N} p_n^N(H(t^*(\bar{r})))\pi(\bar{r}, n) \\
&> \sum_{n=1}^{N} p_n^N(H(t^*(r)))\pi(\bar{r}, n) \\
&\geq \sum_{n=1}^{N} p_n^N(H(t^*(r)))\pi(r, n) \\
&= \Pi(r, H(t^*(r))).\square
\end{aligned}$$

Lemma 3. *If* $\bar{r} < \hat{r}$, *then* $\left.\frac{d\Pi(r, t^*(r))}{dr}\right|_{r=\bar{r}} < 0$.

Proof: Since H is differentiable, then both $t^*(\cdot)$ and $\Pi(\cdot, H(t^*(\cdot)))$ are differentiable on $(0, \hat{r})$. We have

$$\left.\frac{d\Pi(r, H(t^*(r)))}{dr}\right|_{r=\bar{r}} = \sum_{n=1}^{N} \left(\left.\frac{dp_n^N(H(t^*(r)))}{dr}\right|_{r=\bar{r}} \pi(\bar{r}, n) + p_n^N(H(t^*(\bar{r}))) \left.\frac{d\pi(r,n)}{dr}\right|_{r=\bar{r}} \right).$$

Since $\bar{r}$ maximizes $u(\cdot, n) \in [0, \omega]$ for all $n \in \{1, ...N\}$ – see Riley and Samuelson (1981) and Myerson (1981) – we have

$$\left.\frac{d\pi(r,n)}{dr}\right|_{r=\bar{r}} = 0$$

for all $n \in \{1, ...N\}$. Denote by $p^* = p(H(t^*(\bar{r}))) = H(t^*(\bar{r}))$ the binomial probability at $t^*(\bar{r})$. Hence

$$\begin{aligned}
\left.\frac{d\Pi(r, H(t^*(r)))}{dr}\right|_{r=\bar{r}} &= \sum_{n=1}^{N} \left.\frac{dp_n^N(H(t^*(r)))}{dr}\right|_{r=\bar{r}} \pi(\bar{r}, n) \\
&= \sum_{n=1}^{N} \left.\frac{dp_n^N(p)}{dp}\right|_{p=p^*} \left.\frac{dp(H(t))}{dt}\right|_{t=t^*(\bar{r})} \left.\frac{dt^*(r)}{dr}\right|_{r=\bar{r}} \pi(\bar{r}, n) \\
&= h(t^*(\bar{r}))\frac{dt^*(\bar{r})}{dr} \left(\sum_{n=1}^{N} \left.\frac{dp_n^N(p)}{dp}\right|_{p=p^*} \pi(\bar{r}, n) \right).
\end{aligned}$$

In this expression, $h(t^*(\bar{r})) > 0$, and $\frac{dt^*(\bar{r})}{dr} < 0$ by Proposition 1. The last term,

$$\sum_{n=1}^{N} \left.\frac{dp_n^N(p)}{dp}\right|_{p=p^*} \pi(\bar{r}, n),$$

measures the effect of a marginal variation of the binomial probability around p^* on the seller revenue. This term positive: an increase in the binomial probability induces a new binomial distribution whose *c.d.f.* first order stochastically dominates the *c.d.f.* of $B(N, p^*)$ which, because π is increasing with respect to n, increases the seller revenue. Therefore

$$\left.\frac{d\Pi(r, H(t^*(r)))}{dr}\right|_{r=\bar{r}} < 0. \ \square$$

Lemma 4. *If values are distributed uniformly on* $[0, \omega]$, *then*

$$\left.\frac{d\Pi(r, H(t^*(r)))}{dr}\right|_{r=0} > 0.$$

Proof: Normalize $\omega = 1$. We have

$$\begin{aligned}
\left.\frac{d\Pi(r, H(t^*(r)))}{dr}\right|_{r=0} &= \left.\frac{\partial\Pi(r, H(t^*(r)))}{\partial r}\right|_{r=0} + \left.\frac{dt^*(r)}{dr}\right|_{r=0} \left.\frac{\partial\Pi(r, H(t^*(r)))}{\partial t}\right|_{r=0} \\
&= \sum_{n=1}^{N} p_n^N(H(t^*(0))) \left.\frac{\partial\pi(0, n)}{\partial r}\right|_{r=0} \\
&\quad + \left.\frac{dt^*(r)}{dr}\right|_{r=0} \sum_{n=1}^{N} \left.\frac{dp_n^N(H(t))}{dt}\right|_{t=t^*(0)} \pi(0, n).
\end{aligned}$$

Since bidders values are distributed uniformly on $[0,1]$, direct calculation yields

$$\pi(0,n) = \frac{n-1}{n+1}$$

for $n \in \{1,...,N\}$, and

$$\left.\frac{\partial \pi(r,n)}{\partial r}\right|_{r=0} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Hence

$$\left.\frac{d\Pi(r, H(t^*(r)))}{dr}\right|_{r=0} = p_1^N(H(t^*(0))) + \left.\frac{dt^*(r)}{dr}\right|_{r=0} \sum_{n=1}^{N} \left.\frac{dp_n^N(H(t))}{dt}\right|_{t=t^*(0)} \pi(0,n).$$

Now

$$\frac{dt^*(r)}{dr} = \frac{\frac{\partial U(r,H(t))}{\partial r}}{1 - \frac{\partial U(r,H(t))}{\partial t}},$$

where

$$\frac{\partial U(r,H(t))}{\partial r} = \sum_{n=0}^{N-1} p_n^{N-1}(H(t)) \frac{\partial u(r,n+1)}{\partial r},$$

and

$$\frac{\partial U(r,H(t))}{\partial t} = \sum_{n=0}^{N-1} \frac{dp_n^{N-1}(H(t))}{dt} u(r,n+1).$$

Since values are uniformly distributed on $[0,1]$, direct calculation yields

$$u(0,n) = \frac{1}{n\,(n+1)}$$

for $n \in \{1,...,N\}$, and

$$\left.\frac{\partial u(r,n)}{\partial r}\right|_{r=0} = \begin{cases} -1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Thus

$$\left.\frac{\partial U(r,H(t))}{\partial r}\right|_{r=0} = p_0^{N-1}(H(t)) \frac{\partial u(r,1)}{\partial r} = -\left(1 - H(t)\right)^{N-1}.$$

Substituting and simplifying notation by writing $p = H(t^*(0))$ and $\frac{dp_n^{N-1}}{dt} = \frac{p_n^{N-1}(H(t))}{dt}$, we get

$$\begin{aligned}
\left.\frac{\Pi(r, H(t^*(r)))}{dr}\right|_{r=0} &= Np\,(1-p)^{N-1} \\
&\quad - (1-p)^{N-1}\left(1 - \sum_{n=0}^{N-1} \frac{dp_n^{N-1}}{dt} u(0, n+1)\right)^{-1} \sum_{n=1}^{N} \frac{dp_n^N}{dt} \pi(0, n) \\
&= (1-p)^{N-1}\left(1 - \sum_{n=0}^{N-1} \frac{dp_n^{N-1}}{dt} u(0, n+1)\right)^{-1} \Delta_N,
\end{aligned}$$

where

$$\Delta_N = Np - Np \sum_{n=0}^{N-1} \frac{dp_n^{N-1}}{dt} u(0, n+1) - \sum_{n=1}^{N} \frac{dp_n^N}{dt} \pi(0, n)$$

Note that $\frac{dt^*(r)}{dr} < 0$ and $\frac{\partial U(r, H(t))}{\partial r} < 0$ imply $1 - \frac{\partial U(r, H(t))}{\partial t} > 0$. Hence

$$1 - \left.\frac{\partial U(r, t)}{\partial t}\right|_{r=0} = 1 - \sum_{n=0}^{N-1} \frac{dp_n^{N-1}(t)}{dt} u(0, n+1) > 0.$$

Since $t^*(0) \in (0, c)$ by Lemma 1, and since H is increasing, we have $p = H(t^*(0)) \in (0, 1)$. We prove that

$$\left.\frac{\Pi(r, H(t^*(r)))}{dr}\right|_{r=0} > 0$$

by showing that

$$\Delta_N = Np > 0.$$

We have

$$\sum_{n=0}^{N-1} \frac{dp_n^{N-1}}{dt} u(0, n+1) = \sum_{n=1}^{N} \frac{dp_{n-1}^{N-1}}{dt} u(0, n),$$

and therefore

$$\Delta_N = Np - Np \sum_{n=1}^{N} \frac{dp_{n-1}^{N-1}}{dt} u(0, n) - \sum_{n=1}^{N} \frac{dp_n^N}{dt} \pi(0, n).$$

Since

$$\frac{dp_n^N}{dt} = h(t) \frac{dp_n^N}{dp},$$

and $u(0, n) = \frac{1}{n(n+1)}$ and $\pi(0, n) = \frac{n-1}{n+1}$, we have

$$\Delta_N = Np - h(t^*(0))\,(Q_N + R_N)\,,$$

where

$$Q_N = Np\sum_{n=1}^{N}\frac{dp_{n-1}^{N-1}}{dp}\frac{1}{n\,(n+1)},$$

and

$$R_N = \sum_{n=1}^{N}\frac{dp_n^N}{dp}\frac{n-1}{n+1}.$$

Now

$$\begin{aligned} Q_N &= Np\sum_{n=1}^{N}\frac{1}{(n+1)n}\frac{(N-1)!}{(n-1)!(N-n)!} \\ &\quad\times[(n-1)p^{n-2}(1-p)^{N-n}-(N-n)p^{n-1}(1-p)^{N-n-1}] \\ &= N!p\sum_{n=1}^{N}\frac{(n-1)p^{n-2}(1-p)^{N-n}-(N-n)p^{n-1}(1-p)^{N-n-1}}{(n+1)!(N-n)!}. \end{aligned}$$

Similarly,

$$\begin{aligned} R_N &= \sum_{n=1}^{N}\frac{n-1}{(n+1)}\frac{N!}{n!(N-n)!}[np^{n-1}(1-p)^{N-n}-(N-n)p^n(1-p)^{N-n-1}] \\ &= N!p\sum_{n=1}^{N}\frac{(n-1)\left[np^{n-2}(1-p)^{N-n}-(N-n)p^{n-1}(1-p)^{N-n-1}\right]}{(n+1)!(N-n)!}. \end{aligned}$$

Hence

$$Q_N+R_N = N!p\sum_{n=1}^{N}\frac{(n+1)(n-1)p^{n-2}(1-p)^{N-n}-n(N-n)p^{n-1}(1-p)^{N-n-1}}{(n+1)!(N-n)!}.$$

We have

$$\begin{aligned} \sum_{n=1}^{N}\frac{(n+1)(n-1)p^{n-2}(1-p)^{N-n}}{(n+1)!(N-n)!} &= \sum_{n=2}^{N}\frac{(n+1)(n-1)p^{n-2}(1-p)^{N-n}}{(n+1)!(N-n)!} \\ &= \sum_{n=1}^{N-1}\frac{np^{n-1}(1-p)^{N-n-1}}{(n+1)!(N-n-1)!}, \end{aligned}$$

and

$$\begin{aligned} \sum_{n=1}^{N}\frac{n(N-n)p^{n-1}(1-p)^{N-n-1}}{(n+1)!(N-n)!} &= \sum_{n=1}^{N-1}\frac{n(N-n)p^{n-1}(1-p)^{N-n-1}}{(n+1)!(N-n)!} \\ &= \sum_{n=1}^{N-1}\frac{np^{n-1}(1-p)^{N-n-1}}{(n+1)!(N-n-1)!}. \end{aligned}$$

Hence

$$Q_N + R_N = 0,$$

and therefore

$$\Delta_N = Np. \ \Box$$

Proof of Proposition 5: We show that $r^* \leq \bar{r}$. If $\bar{r} \geq \hat{r}$, then $r \geq \bar{r}$ implies $t^*(r) = \underline{c}$; hence $H(t^*(r)) = 0$, and therefore

$$\Pi(r, H(t^*(r))) = 0.$$

Since $\Pi(r, H(t^*(r))) > 0$ for $0 < r < \hat{r}$, we have $r^* < \bar{r}$. If $\bar{r} < \hat{r}$, then $\Pi(r, H(t^*(r))) < \Pi(\bar{r}, H(t^*(\bar{r})))$ for all $r > \bar{r}$ by Lemma 2, and therefore $r^* \leq \bar{r}$. Hence Lemma 3 implies $r^* < \bar{r}$. Finally, if values are uniformly distributed, then $r^* > 0$ by Lemma 4. $\Box$

Proof of Proposition 6: Consider a standard auction with an entry fee $\phi \in \mathbb{R}$ and a reserve price $r \in [0, \omega]$. An entry strategy for a bidder is described by a threshold $t \in [\underline{c}, \bar{c}]$. Given (ϕ, r) if all bidders follow the same entry strategy $t \in [\underline{c}, \bar{c}]$, then the expected utility of an entering bidder is

$$\tilde{U}(\phi, r, H(t)) = U(r, H(t)) - \phi,$$

and seller revenue is

$$\tilde{\Pi}(\phi, r, H(t)) = \Pi(r, H(t)) + NH(t)\phi.$$

A *symmetric (Bayes perfect) entry equilibrium* is a threshold $t \in [\underline{c}, \bar{c}]$ such that for all $z \in [\underline{c}, \bar{c}]$: $\tilde{U}(\phi, r, H(t)) > z$ implies $t > z$ and $\tilde{U}(\phi, r, H(t)) < z$ implies $t < z$. For $\phi \in \mathbb{R}$ and $r \in [0, \omega]$ let $\tilde{t}^*(\phi, r) = \underline{c}$ if $\tilde{U}(\phi, r, 0) < \underline{c}$, and otherwise let $\tilde{t}^*(\phi, r)$ be the solution to the equation

$$t = \tilde{U}(\phi, r, H(t)). \tag{7}$$

The mapping $\tilde{t}^*$ can be shown to have properties analogous to those of the mapping t^*. It is easy to see that an analog of Proposition 2 holds; i.e., *any standard auction with an entry fee ϕ and a reserve price r has a unique symmetric entry equilibrium $t = \tilde{t}^*(\phi, r) \in [\underline{c}, \bar{c}]$.*

Let $r \in (0, \omega]$. We establish Proposition 6 by showing that there is $\phi \geq 0$ such that

$$\tilde{\Pi}(\phi, 0, H(\tilde{t}^*(\phi, 0))) > \Pi(r, H(t^*(r))).$$

If r is so large that $t^*(r) = \underline{c}$, and therefore $H(t^*(r)) = \Pi(r, H(t^*(r))) = 0$, then for $\phi = 0$, our assumption that $\underline{c} < u(0, 1)$ implies

$$\tilde{\Pi}(\phi, 0, H(\tilde{t}^*(\phi, 0))) > 0 = \Pi(r, H(t^*(r))).$$

Hence assume that $t^*(r) > \underline{c}$ and hence $H(t^*(r)) > 0$. Define ϕ by the equation

$$\tilde{t}^*(\phi, 0) = t^*(r).$$

Note that $\phi > 0$. Thus

$$\tilde{U}(\phi, 0, H(\tilde{t}^*(\phi, 0))) = \tilde{t}^*(\phi, 0) = t^*(r) = U(r, H(t^*(r))).$$

Since the gross surplus is distributed between the seller and bidders, we have

$$\tilde{\Pi}(\phi, 0, H(\tilde{t}^*(\phi, 0))) + NH(\tilde{t}^*(\phi, 0))\tilde{U}(\phi, 0, H(\tilde{t}^*(\phi, 0))) = \sum_{n=1}^{N} p_n^N(H(\tilde{t}^*(\phi, 0)))s(0, n),$$

and

$$\Pi(r, H(t^*(r))) + NH(t^*(r))\tilde{U}(r, H(t^*(r))) = \sum_{n=1}^{N} p_n^N(H(t^*(r)))s(r, n).$$

Hence $s(0, n) > s(r, n)$ for each n and $H(\tilde{t}^*(\phi, 0)) = H(t^*(r)) > 0$ imply

$$\begin{aligned}
\tilde{\Pi}(\phi, 0, H(\tilde{t}^*(\phi, 0))) &= \left(\sum_{n=1}^{N} p_n^N(H(\tilde{t}^*(\phi, 0)))s(0, n)\right) - NH(\tilde{t}^*(\phi, 0))\tilde{U}(\phi, 0, H(\tilde{t}^*(\phi, 0))) \\
&= \left(\sum_{n=1}^{N} p_n^N(H(t^*(r)))\left(s(r, n) - s(r, n) + s(0, n)\right)\right) \\
&\quad - NH(t^*(r))U(r, H(t^*(r))) \\
&= \Pi(r, H(t^*(r))) + \sum_{n=1}^{N} p_n^N(H(t^*(r)))\left(s(0, n) - s(r, n)\right) \\
&> \Pi(r, H(t^*(r))). \quad \square
\end{aligned}$$

Proof of Proposition 8. Assume $c = \underline{c} > 0$. Using (3) we can calculate the social surplus for each p and N, $W_N(p)$. As established in Section 2, a standard auction with a zero reserve price generates the maximum "constrained" social surplus that can be achieved by any mechanism, – see also Levin and Smith (1994), Proposition 6; i.e.,

$$W_N^* \equiv W_N(p_N^*),$$

where p_N^* is the equilibrium probability of entry. Further, the sequence $\{W_N^*(p_N^*)\} \subset [0, \omega]$ is decreasing by Proposition 9 in Levin and Smith (1994), and hence has a limit, which we denote by W.

For each N, denote by $\hat{W}_N(r,t)$ the social surplus generated in a standard auction with a reserve $r \in [0, \omega]$ when bidders have heterogeneous entry costs and use the entry threshold $t \in [\underline{c}, \bar{c}]$ – this surplus can be calculated using (5). Also for each N and $r \in [0, \omega]$ denote by $t_N^*(r)$ the equilibrium entry threshold – the mapping $t_N^*(\cdot)$ is well defined by Lemma 1. By Proposition 3, a standard auction with a zero reserve price generates the maximum "constrained" social surplus that can be achieved by any mechanism; i.e.,

$$\hat{W}_N^* \equiv \hat{W}_N(0, t_N^*(0)).$$

We first show that

$$W_N^* \geq \hat{W}_N(r, t_N^*(r))$$

for each N and $r \in [0, \omega]$; i.e., equilibrium social surplus is greater when entry costs are homogeneous than when they are heterogeneous. When entry costs are heterogeneous and the reserve price is r, then the expected cost of each entrant is

$$E[z | \underline{c} \leq z \leq t_N^*(r)] > \underline{c},$$

whereas it is only $c = \underline{c}$ with homogeneous costs. Hence writing $\hat{p}_N = H(t_N^*(r))$ we have

$$\begin{aligned}
W_N^* &\geq W_N(\hat{p}_N) \\
&= \sum_{n=1}^{N} p_n^N(\hat{p}_N)s(0,n) - N\hat{p}_N c \\
&\geq \sum_{n=1}^{N} p_n^N(\hat{p}_N)s(0,n) - N\hat{p}_N E(z \mid z \leq t_N^*(r)) \\
&\geq \sum_{n=1}^{N} p_n^N(\hat{p}_N)s(r,n) - N\hat{p}_N E(z \mid z \leq t_N^*(r)) \\
&= \hat{W}_N(r, t_N^*(r)).
\end{aligned}$$

The above inequalities imply

$$W_N^* \geq \hat{W}_N(0, t_N^*(0)) \geq \hat{W}_N(r_N^*, t_N^*(r_N^*)) \geq \Pi_N(r_N^*, t_N^*(r_N^*)) \geq \Pi_N(0, t_N^*(0)).$$

We show $\lim_{N\to\infty} \hat{W}_N(0, t_N^*(0)) = W$. For each N let $\hat{t}_N \in [\underline{c}, \bar{c}]$ be such that $H(\hat{t}_N) = p_N^*$. Then

$$\hat{W}_N(0, \hat{t}_N) = \sum_{n=1}^{N} p_n^N(p_N^*)s(0,n) - Np_N^* E(z \mid z \leq \hat{t}_N).$$

Since $W_N^* \geq 0$, then $0 \leq Np_N^* \leq \frac{\omega}{c}$ for each N, and hence $\lim_{N\to\infty} p_N^* = \lim_{N\to\infty} H(\hat{t}_N) = 0$. Therefore $\lim_{N\to\infty} \hat{t}_N = \underline{c} = \lim_{N\to\infty} E(z \mid z \leq \hat{t}_N)$. Since

$$0 \leq W_N^* - \hat{W}_N(0, \hat{t}_N) = Np_N^*(\underline{c} - E(z \mid z \leq \hat{t}_N)),$$

and $\{Np_N^*\}$ is a bounded sequence, then $\lim_{N\to\infty}(W_N^* - W_N(0, \hat{t}_N)) = 0$, and therefore

$$W = \lim_{N\to\infty} W_N^* - \lim_{N\to\infty}(W_N^* - \hat{W}_N(0, \hat{t}_N)) = \lim_{N\to\infty} \hat{W}_N(0, \hat{t}_N).$$

By Proposition 3 and the inequality above we have

$$\hat{W}_N(0, \hat{t}_N) \leq \hat{W}_N(0, t_N^*(0)) \leq W_N^*$$

for all N. Hence

$$\lim_{N\to\infty} \hat{W}_N(0, \hat{t}_N) = \lim_{N\to\infty} W_N^* = W,$$

implies

$$\lim_{N\to\infty} \hat{W}_N(0, t_N^*(0)) = W.$$

Next we show that $\lim_{N\to\infty} \Pi_N(0, t_N^*(0)) = W$. For each N, write

$$U_N(0,p) = \sum_{n=0}^{N-1} p_n^{N-1}(p)u(0,n+1).$$

Since $u(0,n)$ is decreasing in n, then $U_N(0,p)$ is decreasing in p. Hence the equilibrium entry threshold when entry costs are heterogeneous, $t_N^*(0) \in [\underline{c}, \bar{c}]$, and the equilibrium entry probability when the bidders have homogeneous entry costs, p_N^*, satisfy

$$U_N(0, H(t_N^*(0))) = t_N^*(0) \geq \underline{c} = U_N(0, p_N^*).$$

Hence $0 \leq H(t_N^*(0)) \leq p_N^*$ for all N. Therefore $\lim_{N\to\infty} p_N^* = 0$ implies $\lim_{N\to\infty} H(t_N^*(0)) = 0$, $\lim_{N\to\infty} t_N^*(0) = \underline{c}$, and $\lim_{N\to\infty} E(z \mid z \leq t_N^*(0)) = \underline{c}$. Hence if the seller sets $r = 0$, the asymptotic total bidder surplus is

$$\lim_{N\to\infty} NH(t_N^*(0))[t_N^*(0) - E(z \mid z \leq t_N^*(0))] = 0,$$

and thus the asymptotic seller revenue is

$$\lim_{N\to\infty} \Pi_N(0, t_N^*(0)) = \lim_{N\to\infty} \hat{W}_N(0, t_N^*(0)) = W.$$

Since

$$\hat{W}_N(0, t_N^*(0)) \geq \hat{W}_N(r_N^*, t_N^*(r_N^*)) \geq \Pi_N(r_N^*, t_N^*(r_N^*)) \geq \Pi_N(0, t_N^*(0))$$

for all N, and $\lim_{N\to\infty} \hat{W}_N(0, t_N^*(0)) = \lim_{N\to\infty} \Pi_N(0, t_N^*(0)) = W$, we have

$$\lim_{N\to\infty} \hat{W}_N(r_N^*, t_N^*(r_N^*)) = \lim_{N\to\infty} \Pi_N(r_N^*, t_N^*(r_N^*)) = W. \ \Box$$

Proof of Proposition 9. Without loss of generality, assume that $\omega = 1$. We first establish that $\lim_{N\to\infty} \hat{W}_N^* = 1$ by showing that for every $\varepsilon > 0$ there is $\bar{N}$ sufficiently large that $\hat{W}_N^* > 1 - \varepsilon$ for all $N \geq \bar{N}$.

Let λ be such that $1 - \frac{1}{\lambda}(1 - e^{-\lambda}) > 1 - \varepsilon$, i.e., $\frac{1}{\lambda}(1 - e^{-\lambda}) < \varepsilon$. Such a λ exists since $\lim_{\lambda\to\infty} \frac{1}{\lambda}(1 - e^{-\lambda}) = 0$. For each $N > \lambda$, let $t_N \in [0, \bar{c}]$ be such that $H(t_N) = \frac{\lambda}{N}$. Note t_N exists since H is increasing. Also note that for $\varepsilon > 0$, we have $H(\varepsilon) > 0$; hence there is $\tilde{N}$ sufficiently large that for all $N > \tilde{N}$, $NH(\varepsilon) > \lambda$; hence $t_N < \varepsilon$ for $N > \tilde{N}$, and therefore $\lim_{N\to\infty} t_N = 0$.

We have

$$\hat{W}_N(0, t_N) = \sum_{n=0}^{N} p_n^N(H(t_N))\frac{n}{n+1} - NH(t_N)\int_0^{t_N} zdH(z).$$

Since $NH(t_N) = \lambda$ for all N and $\lim_{N\to\infty} t_N = 0$, we have

$$\lim_{N\to\infty} NH(t_N)\int_0^{t_N} zdH(z) = \lambda \lim_{N\to\infty}\int_0^{t_N} zdH(z) = 0.$$

Since the limit of a binomial distribution as N goes to infinity, holding $NH(t_N) = \lambda$ fixed, is the Poisson distribution, we have

$$\begin{aligned}
\lim_{N\to\infty}\sum_{n=0}^{N} p_n^N(H(t_N))\frac{n}{n+1} &= \sum_{n=0}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!}\frac{n}{n+1} \\
&= \sum_{n=0}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!}(1-\frac{1}{n+1}) \\
&= 1-\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{e^{-\lambda}\lambda^{n+1}}{n!}\frac{1}{n+1}.
\end{aligned}$$

Letting $k = n+1$, i.e., $n = k-1$ we have

$$1-\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{e^{-\lambda}\lambda^{n+1}}{n!}\frac{1}{n+1} = 1-\frac{1}{\lambda}(-e^{-\lambda}+\sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}) = 1-\frac{1}{\lambda}(-e^{-\lambda}+1).$$

Let $\bar{N}$ sufficiently large that for all $N > \bar{N}$

$$\left|\hat{W}_N(0, t_N) - \left(1-\frac{1}{\lambda}(1-e^{-\lambda})\right)\right| < \delta,$$

where $0 < \delta < \varepsilon - \frac{1}{\lambda}(1-e^{-\lambda})$. Then for each $N > \bar{N}$ we have

$$\hat{W}_N^* \geq \hat{W}_N(0, t_N) \geq 1-\frac{1}{\lambda}(1-e^{-\lambda}) - \delta > 1-\varepsilon.$$

Now, since $\hat{W}_N^* = \hat{W}_N(0, t_N^*(0))$ for all N by Proposition 3, we have

$$\hat{W}_N^* = \sum_{n=0}^{N} p_n^N(H(t_N^*(0)))\frac{n}{n+1} - \lim_{N\to\infty} NH(t_N^*(0))\int_0^{t_N^*(0)} zdH(z).$$

Since $\left\{\sum_{n=0}^{N} p_n^N(H(t_N^*(0)))\frac{n}{n+1}\right\} \subset [0,1]$ and $NH(t_N^*(0))\int_0^{t_N^*(0)} zdH(z) > 0$ for all N, then $\lim_{N\to\infty}\hat{W}_N^* = 1$ implies

$$\lim_{N\to\infty}\sum_{n=0}^{N} p_n^N(H(t_N^*(0)))\frac{n}{n+1} = 1,$$

and therefore

$$\lim_{N\to\infty} NH(t_N^*(0))\int_0^{t_N^*(0)} zdH(z) = \lim_{N\to\infty}\sum_{n=0}^{N} p_n^N(H(t_N^*(0)))\frac{n}{n+1} - \lim_{N\to\infty}\hat{W}_N^* = 0.$$

It is easy to see that this implies

$$\lim_{N\to\infty} t_N^*(0) = 0,$$

which in turn implies that the asymptotic total surplus captured by bidders is

$$\lim_{N\to\infty} N\int_0^{t_N^*(0)} (t_N^*(0) - z)dH(z) = 0.$$

And since in the limit bidders capture no surplus, all the surplus is captured by the seller; i.e.,

$$\lim_{N\to\infty}\hat{\Pi}_N^* = \lim_{N\to\infty}\hat{W}_N^* = 1. \ \square$$

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