Pricing group membership*

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Abstract

We consider a model where agents differ in their 'types' which determines their voluntary contribution towards a public good. We analyze what the equilibrium composition of groups are under centralized and centralized choice. We show that there exists a top-down sorting equilibrium i.e. an equilibrium where there exists a set of prices which leads to groups that can be ordered by level of types, with the first k types in the group with the highest price and so on. This exists both under decentralized and centralized choosing. We also analyze the model with endogenous group size and examine under what conditions is top-down sorting socially efficient. We illustrate when integration (i.e. mixing types so that each group's average type if the same) is socially better than top-down sorting. Finally, we show that top down sorting is efficient even when groups compete among themselves.

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1 Introduction

In this paper we analyze whether groups can 'price' group membership to screen who becomes a member. Joining many groups and clubs is not costless. Many of them require monetary fees, or non-monetary 'sacrifices' to become a member Winslow (1999). Examples abound, new members of a criminal gang or a terrorist organization have to undergo hazing rituals (Vigil, 1996). Similarly, new recruits into many military units and college fraternities also have to go through painful or shameful activities to be accepted into them (Ostvik and Rudmin 2001, Mercuro, Merritt and Fiumefreddo 2014, De Klerk 2013, Groves, Griggs and Leflay 2012, Keating et al 2005). Even religious groups have elaborate initiation rites (Berman 2000, Iannacone 1992). Exclusive clubs, or elite schools require the payment of very expensive fees to become members (Jenkins, Micklewright and Schnepf 2008). The question of interest in this paper is the rationale for these practices, the equilibrium value of such fees, and their welfare implications.

In all those groups, the value of being a member depends on the extent to which other participants contribute to the common cause or public good they provide. At the same time, individuals differ in their inclination and ability to dedicate themselves to the cause, i.e. to contribute to the public good provided by the group. For this reason, the utility obtained by any member in belonging to a group will depend on the types of all the other members. As a consequence, the group will be interested in making sure that those who gain entry as members are of the right type. But these types are not necessarily observable, and the entry fee is a way to select members in an incentive compatible way.

Our model assumes there are a number of participants in a game which has two stages. The first stage develops as follows. A set of 'entrepreneurs', one for each of a fixed number of groups, posts a price for belonging to their group. The participants then decide, independently and simultaneously, whether to pay the price to one of those groups, or none at all. In the second stage, the participants who are in a group decide on their level of effort/contribution to a public good for the group to which they belong.

The participants differ in the way they benefit from the public good. Each individual is characterized by a type that is a multiple of the amount of the public good. This heterogeneity can be variously interpreted as either a different intrinsic personal enjoyment of the good, or a difference in the degree of altruism (i.e. some individuals internalize the benefit of the public good on others to a larger extent). One important implication of this assumption is that the group members do not care care directly about the types of others, or about the group size. This is because we assume there is no direct externality caused by others' types or the number of individuals in the group. Individuals do care about the actions taken by other group members, because those actions affect the amount of public good provided which they enjoy, and contributions to the public good are indeed affected by their types. This assumption distinguishes our model from others in the literature of club games (starting with Buchanan, 1965) and congestion games (Oakland 1972, Baumol and Oates 1988).

Clearly, since the groups provide a public good, we would in general have an under-provision of the public good in the group if the provision is voluntary and decided on an individual basis. But many of the groups that we have used to motivate our model have the possibility of imposing a contribution level within the group. We can think for example of the tightly hierarchical organization of most armies, and even gangs. For this reason, we also study group formation when contributions within groups are decided by a planner within the group.

Whether the individual or the group leader (social planner) decides the contribution level, our first result is that in both cases we can establish the existence of a *top-down sorting equilibrium.* This equilibrium has the following characteristic. The 'entrepreneurs' announce a list of distinct prices, which thus can be ordered from highest to lowest. The participants sort themselves into groups by their types. A set composed of the highest types chooses to belong to the group with the highest price. Another set of types just below the first chooses to belong to the group with the second highes price. This continues until the last set of types, who belong to a group with price zero.

We analyze the equilibrium under both centralized and decentralized decision-making. One difference between the equilibrium under centralized and decentralized choice of effort is that under decentralized effort choice every individual gains whenever the average type of the group increases (although relatively higher types gain more, which is the basis for segregation). On the other hand, with centralized effort choice within the group an increase in average type improves the utility of above average types within the group, but decreases for below average types. This happens because the group planner does not know the individual types within the group, and so she requires the same level of contribution from every one.

We next turn our attention to some welfare properties of the decentralized process for group formation. For this we need to take into account both group composition and group size. In terms of group composition, we compare the welfare of the *top-down sorting equilibrium* with the one arising from an equilibrium in which there is no sorting, and all groups have equal average types. It turns out that the answer depends on the curvature of the output function. For example, when the concavity is such the risk aversion parameter in the

CRRA function is bigger than one (i.e. $\alpha > 1$), then the *top-down sorting equilibrium* is less socially beneficial than one where the population is sorted accross groups so that all groups have the same average type. We call this 'integration'. A corollary of this result is that in this case, the segregation occurring in equilibrium is socially inefficient. On the other hand, when $\alpha < 1$, then the *top-down sorting equilibrium* has a larger total welfare than the one achieved when groups are 'integrated' and have the same average types.

In terms of size, for $\alpha > 1$, the optimal size of the group is N, the whole population, since individual payoff increases with the size of the coalition. For $\alpha < 1$, the optimal size of the group has to trade off the benefits from segregation (which are highest if average group quality increases, say by dropping the lowest types in the group), with those arising from having larger groups, because group size affects equilibrium contribution levels.

Finally, up to now we have assumed that there is no inter-group conflict. However, in some of our applications the groups compete with one another after they are formed (again, it is easier to think of armies and gangs, but even educational institutions compete ex post for top-paying jobs). We show that our *top-down sorting equilibrium* also exists in this context, so our conclusions are robust to this kind of setting.

Our paper has important implications for policy. There are relevant environments ($\alpha > 1$) where the social planner would like an 'integrated' society, which might not arise in equilibrium. She wants to achieve this purely on social surplus maximization grounds. This provides a new rationale for integration in various social domains, like education and housing, without having to resort to preference about equality. We will discuss this in more depth in the conclusion.

1.1 Literature

Our paper develops a novel theory of group formation. Clearly, this paper links to the classical literature on club theory (Buchanan 1965, Berglas 1976). This literature does not consider differences in information about preferences, and therefore it does not provide a rationale for undertaking actions to signal preferences. Somewhat closer to our work is Ben Porath and Dekel (1992), who use the potential for self sacrifice as a way to signal future intentions. However, they do it because of equilibrium selection problems, and the self sacrifice does not actually occur in equilibrium. Helsley and Strange (1991), in turn, model club formation in a context with homogeneous tastes and costs. In their model fees and prices are used to obtain second best usage when congestion within a club is a problem.

In terms of theory, two close papers to our's are Jaramillo and Moizeau (2002), and

Cornes and Silva (2013). Jaramillo and Moizeau (2003) study a model where individuals differ in income, and higher income people desire a different level of the local public good. Since information about income is private, individuals use costly signalling to join a club with others who have similar income (and, hence, preferences for the local public good). Different from us, the group formation is uncoordinated (there is no entrepreneur creating the clubs), there are only two types, and they do not contemplate the possibility of internal coordination of contributions. Also, in their model there is no reason for individuals of different types to group together in the social optimum. In Cornes and Silva (2013) some people love prestige (they have higher utility from contributing relative to the average) and others are purists. Fees serve to sort them into clubs. The model, unlike ours has both positive and negative externalities, but the competitive equilibrium is inefficient. Competition yields co-ordination benefits with the formation of prestige clubs.

Our paper also has some connection to a literature that is related to group formation in contexts were types are differentiated horizontally. For example, Levy and Razin (2012) analyze explicit displays of religious beliefs and cooperation within the religious group. Baccara and Yariv (2013), also analyze group formation and contribution to horizontally differentiated tasks.

Another relevant literature for us relates to assortative matching. Durlauf and Seshadri (2003) examines when assortative matching is efficient. Legros and Newman (2007) study sufficient conditions for assortative matching in equilibrium. Hoppe, Moldovanu and Sela (2008) analyze when costly signaling is necessary for assortative matching under incomplete information. Importantly, they check when gains form assortative matching are offset by signalling costs. There is also a large literature studying sorting into schools, to take advantage of peer effects (Epple Romano 1998, Cullen, Jacob, and Levitt 2003, Hsieh and Urquiola 2006 or MacLeod and Urquiola 2015).

There is a body of work in the experimental literature exploring how endogenous sorting into groups can help solve the problem of free riding. For example, costly rituals have been shown to promote greater co-operation (Sosis 2004, Ruffle and Sosis 2007). Page, Putterman and Unel (2005) show that endogenous segregation helps avoid free-riding and provides some support for the efficiency of top down sorting. Cimino (2011) does a survey based experiment where participants are asked about an initiation activity for a group, which itself provides benefits to members. Those participants randomly allocated to a group providing a good with a more public component were significantly more likely to choose a stressful initiation task. This sort of hazing can be considered as an entry fee to prevent free riding. This would be very much in the spirit of our model. Aimone et al. (2013) show that something akin to our endogenous segregation equilibrium happens in the lab. In their experiment participants can choose to participate in a voluntary contribution public good game in groups having different rates of return. People that are more prone to contribute to public goods join those groups with lower rates of return, thus signalling their type. As in our model, a costly choice (in this case, choosing a less efficient technology) is a credible signal of a type wishing to contribute more to the public good.

In addition to experimental results, there is a significant ethnographic evidence that is connected to our issues. For example, Vigil (1996) analyzes gang initiation. This study is consistent with our model, i.e. initiation rites are used to screen potential members. Sosis, Kress and Boster (2007) show the importance of such costly male rituals in signalling commitment and promoting solidarity among men who need to organise themselves for warfare. Soler (2012) rationalises the existence of wasteful religious rituals. It analyzes the practice of Candomble, an Afro Brazilian ritual. It shows that participation is correlated with higher contributions to public goods, which is consistent with our model. Cleaver (2004) shows that entry fees are used in clubs to exclude the "non elite" .

2 Model: signaling and coalition formation

This model has two stages. There are a large and finite number of agents playing the game $\mathcal{N}$. Each agent $j \in \mathcal{N}$ is characterized by a level of preference for the public good ξ_j. In the first stage the agents form coalitions of a fixed number of players N, in a way we will describe in section 2.1. In the second stage, once a coalition is formed, every agent simultaneously decides how much to contribute towards a public good, x_j taking as given the coalition structure.

The total amount of the public good is given by $V\left(\sum_{i=1}^{N} x_i\right)$, with $V(.)$ being a strictly concave twice continuously differentiable function. The personal cost of the contribution x_j is given by $x_j^2/2$. The utility of every agent after the coalition is formed is

$$U_j = \xi_j V\left(\sum_{i=1}^{N} x_i\right) - \frac{1}{2}x_j^2$$

The parameter ξ_j indicates that the individual may care for more than only his own utility but she internalizes the benefits of other players. A player with a $\xi_j > 1$ is thought to be "altruistic" whereas $\xi_j = 1$ is a selfish player. The value of ξ_j is obviously relevant for all players in any coalition, as it increases the marginal value of contributions of their owners.

It is also the private information of the players, which can be (partially) solved using a pre-game costly signaling exercise which we now describe.

2.1 The signaling game

Consider a finite set of coalitions $l \in \{1, \ldots, L\}$, each with N slots.[1] Let $\bar{\xi}_l = \sum_{i \in l} \xi_i$. Assume as well that each N is large enough so that the compositional impact of changing one member's type on the $\bar{\xi}_l$ of coalition l is negligible. The coalition formation game is as follows. A set of 'entrepreneurs' post a set of prices p_l for $l \in \{1, \ldots, L\}$. We think of prices that need not be 'monetary'. Any costly action whose effect in utility is separable would work. Then all individuals in the population decide which coalition to join (and thus pay the price of joining). If a coalition is oversubscribed (it has $M > N$ candidates applying to it), then a fair lottery decides which $M - N$ candidates get allocated to not being in a coalition, which is free and obviously has enough capacity for the full group.

Order arbitrarily the available coalitions. We denote by *top-down sorting* the following assignment of members into coalitions according to their type. Coalition 1 gets assigned the N highest type members, coalition 2 the N highest type members among the remaining ones, and so on until all members are assigned to one (and only one) coalition. The top-down sorting leads to a coalition structure with types stratified from higher to lower. Namely, given two coalitions $l > k$ and two members i, j that are assigned to either coalition by top-down sorting, then, $\xi_i > \xi_j$ and $\bar{\xi}_l \geq \bar{\xi}_k$. To ensure that this inequality is strict for at least one pair of players in two different coalitions, we assume that two successive coalitions cannot be fully occupied by players of the same type. As mentioned above to join a coalition l, they must choose an action with cost p_l. The last group is the option not being in a coalition. We say that an assignment of members to coalitions and a vector of entry costs forms an equilibrium when, given the costs, no individual prefers to change coalitions and either a coalition is full or its associated cost is zero.

Denote by

$$\frac{\partial U_j}{\partial \bar{\xi}_l}$$

the derivative of the equilibrium utility of individual j with respect to changes in the average type $\bar{\xi}_l$ of the coalition to which they belong. The differential sensitivity of different types of potential coalition members to the composition of coalitions has implications for coalition formation that we now analyze.

[1] We assume for simplicity that all coalitions are equally sized. Nothing substantial changes if they are still exogenous but differently sized. Later on we address endogenously sized coalitions.

Proposition 1 *There exists an assignment equilibrium with top-down sorting if whenever i and j are such that $\xi_i > \xi_j$ we have that*

$$\frac{\partial U_i}{\partial \bar{\xi}_l} - \frac{\partial U_j}{\partial \bar{\xi}_l} > 0 \tag{1}$$

Letting $\xi_{i^*(l)}$ be the type of the lowest member in coalition l, the fee for a full coalition l is defined recursively as:

$$p_l = \int_{\bar{\xi}_{l+1}}^{\bar{\xi}_l} \frac{\partial U_{i^*(l)}}{\partial \bar{\xi}_j} d\bar{\xi}_j + p_{l+1}, l = 1, \ldots, L-1, \tag{2}$$

with $p_L = 0$.

Proof. See Appendix. ■

This condition provides a test for the existence of endogenous sorting.

2.2 Trust and group contributions

Given the coalitions that have formed, we now analyze the equilbria in the subgames where the coalitions are present. Remember that

$$U_j = \xi_j V\left(\sum_{i=1}^{N} x_i\right) - \frac{1}{2}x_j^2$$

The FOC of the second stage problem are

$$\frac{\partial U_j}{\partial x_j} = \xi_j V'\left(\sum_{i=1}^{N} x_i\right) - x_j = 0$$

This implies that for all i, j

$$\frac{x_j}{\xi_j} = \frac{x_i}{\xi_i}$$

So normalizing $\xi_1 = 1$

$$\xi_i x_1 = x_i$$

and hence in equilibrium

$$V'\left(x_1 \sum_{i=1}^{N} \xi_i\right) = x_1 \tag{3}$$

which yields a unique equilibrium given the composition of the coalition. Let a particular coalition A composed of a group of people with qualities $(\xi_1, \xi_2, \ldots, \xi_n)$ with

$$\bar{\xi}_l = \frac{1}{N}\sum_{i=1}^{N} \xi_i$$

Totally differentiating 3 with respect to $\bar{\xi}_l$ we get;

$$V''\left(Nx_1\bar{\xi}_l\right)\frac{\partial x_1}{\partial\bar{\xi}_l}N\bar{\xi}_l + V''\left(Nx_1\bar{\xi}_l\right)Nx_1 = \frac{\partial x_1}{\partial\bar{\xi}_l}$$

$$\frac{\partial x_1}{\partial\bar{\xi}_l} = \frac{V''\left(Nx_1\bar{\xi}_l\right)Nx_1}{1 - V''\left(Nx_1\bar{\xi}_l\right)N\bar{\xi}_l} < 0$$

so contributions decrease with average quality. Nevertheless equilibrium utility

$$U_j = \xi_j V\left(Nx_1\bar{\xi}_l\right) - \frac{\xi_j^2}{2}x_1^2$$

so that

$$\begin{aligned}\frac{\partial U_j}{\partial\bar{\xi}_l} &= \xi_j V'\left(Nx_1\bar{\xi}_l\right)N\bar{\xi}_l\frac{\partial x_1}{\partial\bar{\xi}_l} + \xi_j N x_1 V'\left(Nx_1\bar{\xi}_l\right) - \xi_j^2 x_1\frac{\partial x_1}{\partial\bar{\xi}_l} \\ &= \xi_j N x_1^2\left(\frac{1 - V''\left(Nx_1\bar{\xi}_l\right)\xi_j}{1 - V''\left(Nx_1\bar{\xi}_l\right)N\bar{\xi}_l}\right) > 0\end{aligned}$$

Furthermore, from inspection it is clear that if $\bar{\xi}_l$ changes little with ξ_j, say because every individual is negligible, then

$$\frac{\partial^2 U_j}{\partial\bar{\xi}_l\partial\xi_j} > 0 \tag{4}$$

From equation (4) it is immediate that

PROPOSITION **2** *There exists an assignment equilibrium with top-down sorting under decentralized effort choice.*

2.3 The public good game: centralized choosing

Suppose instead that the level of contributions in a coalition are decided centrally by a utilitarian social planner.

$$U_j = \xi_j V\left(\sum_{i=1}^{N} x_i\right) - \frac{1}{2}x_j^2$$

$$\mathcal{U} = \sum_{j=1}^{N} U_j = \sum_{j=1}^{N}\left(\xi_j V\left(\sum_{i=1}^{N} x_i\right) - \frac{1}{2}x_j^2\right) = N\bar{\xi}_l V\left(\sum_{i=1}^{N} x_i\right) - \frac{1}{2}\sum_{j=1}^{N} x_j^2$$

The FOC of the second stage problem are now

$$\frac{\partial\mathcal{U}}{\partial x_j} = N\bar{\xi}_l V'\left(\sum_{i=1}^{N} x_i\right) - x_j = 0$$

so for all i, j within a coalition

$$x_i = x_j$$

$$N\bar{\xi}_l V'(Nx_1) - x_1 = 0 \tag{5}$$

Totally differentiating 5 with respect to $\bar{\xi}_l$ we get;

$$N\frac{\partial x_1}{\partial \bar{\xi}_l} V''(Nx_1) N\bar{\xi}_l + NV'(Nx_1) = \frac{\partial x_1}{\partial \bar{\xi}_l}$$

$$\frac{\partial x_1}{\partial \bar{\xi}_l} = \frac{V'\left(Nx_1\bar{\xi}_l\right) N}{1 - V''(Nx_1) N^2\bar{\xi}_l} > 0$$

so contributions increase with average quality. Hence equilibrium utility

$$U_j = \xi_j V(Nx_1) - \frac{x_j^2}{2}$$

remember that $N\bar{\xi}_l V'(Nx_1) - x_1 = 0$, so that

$$\begin{aligned}
\frac{\partial U_j}{\partial \bar{\xi}_l} &= \xi_j V'(Nx_1) N\frac{\partial x_1}{\partial \bar{\xi}_l} - x_1\frac{\partial x_1}{\partial \bar{\xi}_l} \\
&= \xi_j \frac{x_1}{\bar{\xi}_l}\frac{\partial x_1}{\partial \bar{\xi}_l} - x_1\frac{\partial x_1}{\partial \bar{\xi}_l} = \left(\frac{\xi_j - \bar{\xi}_l}{\bar{\xi}_l}\right) x_1 \frac{\partial x_1}{\partial \bar{\xi}_l} \\
&= \left(\frac{\xi_j - \bar{\xi}_l}{\bar{\xi}_l}\right) x_1 \frac{V'\left(Nx_1\bar{\xi}_l\right) N}{1 - V''(Nx_1) N^2\bar{\xi}_l}
\end{aligned}$$

where the sign is positive for j with $\xi_j - \bar{\xi}_l > 0$ and negative otherwise.

Remark **1** *It is clear that with respect to the decentralized equilibrium some types of players, i.e. those with a higher than average type within a coalition, have a higher utility while others, i.e. those with lower than average type, have a lower utility.*

Furthermore, from inspection it is clear that if changing a single ξ_j does not change much the average type of a coalition, then

$$\frac{\partial^2 U_j}{\partial \bar{\xi}_l \partial \xi_j} > 0 \tag{6}$$

From equation (6) it is immediate that

Proposition **3** *There exists an assignment equilibrium with top-down sorting under centralized effort choice.*

3 Endogenous size and social optima

In the previous sections, the size of coalitions has been exogenously fixed at N. But given the environment considered, it would natural to consider the equilibrium when the coalition size is also endogenous. This could have important implications both for the equilibrium contributions, and for the efficient composition and size of the groups. One main tradeoff is the following. In a larger group, the free-riding can become more problematic. On the other hand, in a larger group, the marginal benefits of an action positively affect a larger set of people. What is optimal will likely depend on specific features of the technology.

In this section we show that when the group size can be centrally chosen a utilitarian social planner would sometimes prefer top-down sorting (we call this ‘segregation’) while under other conditions she would prefer types to mix so that all groups have the same average type (we call this ‘integration’).

We now use a CRRA $V(.)$ function to analyze this problem.

$$V\left(\sum_{i=1}^{N} x_i\right) = \frac{\left(\sum_{i=1}^{N} x_i\right)^{1-a} - 1}{1-a}$$

In order for this $V(\cdot)$ function to make sense as a production function, we assume $x_i \geq 1$. Note that there is no risk in this problem, so we use the CRRA function as a convenient way to parameterize concavity.

Then, for a given N the FOC for coalition efforts in this case are

$$\begin{aligned} N\bar{\xi}_l\left((Nx_1)^{-a}\right) - x_1 = & \quad 0 \\ \bar{\xi}_l N^{1-a} = & \quad x_1^{1+a} \end{aligned}$$

Remember that in coalition all members choose the same value under centralized decision-making and so $x_i = x_j = x_1$ within the coalition. Thus the total utility within the coalition is

$$\mathcal{U}_l = N\bar{\xi}_l \frac{\left(N\left(\bar{\xi}_l N^{1-a}\right)^{\frac{1}{1+a}}\right)^{1-a} - 1}{1-a} - \frac{1}{2} N \left(\left(\bar{\xi}_l N^{1-a}\right)^{\frac{1}{1+a}}\right)^2$$

so that

$$\mathcal{U}_l = \left(\frac{1}{1-a} - \frac{1}{2}\right) \bar{\xi}_l^{\frac{2}{1+a}} N^{\frac{3-a}{1+a}} - \frac{N\bar{\xi}_l}{1-a}$$

and society welfare is

$$\mathcal{U} = \sum_{l=1}^{L} \mathcal{U}_l = \sum_{l=1}^{L} \left(\left(\frac{1}{1-a} - \frac{1}{2}\right) \bar{\xi}_l^{\frac{2}{1+a}} N^{\frac{3-a}{1+a}} - \frac{N\bar{\xi}_l}{1-a}\right) \tag{7}$$

Define $\mathcal{U}_{TD}$ as the total utility obtained in society when coalitions are formed with *top-down sorting* and $V\left(\sum_{i=1}^{N} x_i\right) = \left(\left(\sum_{i=1}^{N} x_i\right)^{1-a} - 1\right) / (1-\alpha)$.

Suppose that it is possible to organize coalitions so that all coalitions have the same mean $\bar{\xi}_l$ and define $\mathcal{U}_I$ as the total utility obtained in society when all coalitions have the same mean $\bar{\xi}_l^* = \sum_{l=1}^{L} \bar{\xi}_l / L$ where $\bar{\xi}_l$ is the group l mean under top-down sorting. Because all the groups have the same average type, the optimally chosen contributions are the same in all of them and thus no one has an incentive to choose the group to which they belong.

Remember that the central planner does not know the types of any of the players. Thus, we assume groups are sufficiently large that simply allocating individuals randomly to groups generates groups with equal average types in expectation.

PROPOSITION 4 *Suppose coalitions size is exogenously set to N. Then $\mathcal{U}_{TD} > \mathcal{U}_I$ if and only if $a < 1$.*

Proof. This follows from (7) since

$$\mathcal{U}_{TD} = \sum_{l=1}^{L}\left(\left(\frac{1}{1-a}-\frac{1}{2}\right)\bar{\xi}_l^{*\frac{2}{1+a}} N^{\frac{3-a}{1+a}} - \frac{N\bar{\xi}_l^*}{1-a}\right) = \left(\frac{1}{1-a}-\frac{1}{2}\right) N^{\frac{3-a}{1+a}} L\bar{\xi}_l^{*\frac{2}{1+a}} - \frac{LN\bar{\xi}_l^*}{1-a}$$

$$= \left(\frac{1}{1-a}-\frac{1}{2}\right) N^{\frac{3-a}{1+a}} L\left(\frac{\sum_{l=1}^{L}\bar{\xi}_l}{L}\right)^{\frac{2}{1+a}} - \frac{N\sum_{l=1}^{L}\bar{\xi}_l}{1-a}$$

$$\mathcal{U}_I = \sum_{l=1}^{L}\left(\left(\frac{1}{1-a}-\frac{1}{2}\right)\bar{\xi}_l^{\frac{2}{1+a}} N^{\frac{3-a}{1+a}} - \frac{N\bar{\xi}_l}{1-a}\right) = \left(\frac{1}{1-a}-\frac{1}{2}\right) N^{\frac{3-a}{1+a}} \sum_{l=1}^{L}\bar{\xi}_l^{\frac{2}{1+a}} - \frac{N\sum_{l=1}^{L}\bar{\xi}_l}{1-a}$$

so that by applying Jensen's inequality, we have that

$$\mathcal{U}_{TD} > \mathcal{U}_I$$

■

$$\frac{\mathcal{U}_l}{N} = \frac{1+a}{2(1-a)}\bar{\xi}_l^{\frac{2}{1+a}} N^{\frac{2-2a}{1+a}} - \frac{\bar{\xi}_l}{1-a} \tag{8}$$

$$\frac{\partial\left(\frac{\mathcal{U}_l}{N}\right)}{\partial N} = \bar{\xi}_l^{\frac{2}{1+a}} N^{\frac{2-2a}{1+a}} > 0$$

This implies that for a given $\bar{\xi}_l$ the average payoff in a group increases in N independently of the concavity of the individual functional form within the CRRA class.

Given that proposition 4 establishes that average payoff in the coalition is concave in average type for $a > 1$ and thus you want to form heterogeneous coalitions, it follows immediately that

Proposition 5 *For $a > 1$ the social planner would like a single group of the maximal size.*

When, on the other hand $a < 1$, there is a tradeoff for the "high" quality groups. On the one hand, they would prefer to have total segregation to increase payoff, since average payoff increases with average type as in equation 8 we see that it increases in $\bar{\xi}_l^{\frac{2}{1+a}}$. On the other hand, a bigger size increases payoff within the group, as in equation 8 we see that it increases in $N^{\frac{2-2a}{1+a}}$. Clearly this tradeoff pushes for at least some degree of mixing.

4 Inter-coalition competition

So far, we have studied the problem of coalition formation and activities as if the activities of those coalitions did not interact with one another. But in our motivation we discussed the evidence that group formation often occurs in contexts where the groups compete, such as gangs (Vigil 1996) or warfare (Sosis, Kress and Boster 2007). For this reason we will now study the case where, after the groups form, they compete. The main insight of this section, is that our earlier results also apply in this case. In particular, let us assume that the coalitions compete, after having formed, in a contest. We will show that the incentives for coalition formation that allow for a top-down sorting assignment (as in Proposition 2) still hold in this case. To be more precise, take two coalitions with respective sizes M and N but otherwise identical utility functions.

Then assume that the payoffs are given by[2]

$$U_j = \frac{\xi_j}{N} \frac{V\left(\sum_{i=1}^N x_i\right)}{V\left(\sum_{i=1}^N x_i\right) + V\left(\sum_{i=1}^M y_i\right)} - \frac{1}{2}x_j^2$$

and

$$V\left(\sum_{i=1}^N x_i\right) = \left(\sum_{i=1}^N x_i\right)^b ; V\left(\sum_{i=1}^M y_i\right) = \left(\sum_{i=1}^M y_i\right)^b$$

and $b < 1$ which as shown in Proposition 4 is the case favorable to segregation into more than one coalition.

Proposition 6 *There exists an assignment equilibrium with top-down sorting under decentralized effort choice when coalitions compete.*

Proof. See Appendix. ■

[2]We assume that payoffs are the outcomes of a contest, and given by a contest success function, as in Skaperdas (1996).

5 Conclusion

We have studied a model of public good provision within groups. The group members have heterogeneous preferences for the public good, and high types contribute more towards its provision. Thus, all individuals prefer to be in groups with higher average types. We allow for the existence of 'entrepreneurs' who create the groups and demand a (possibly non-monetary) 'fee' to enter the group. We study the case where the 'entrepreneurs' can enforce a contribution level within the group and also when once in the group, contributions are voluntary.

We first show that both under centralized and decentralized provision there is *top-down sorting equilibrium* in which groups are organized assortatively by type. Under centralized contributions every participant is better off if average types increase. When contributions are centralized that is not true and only above average types are better off.

There are interesting results in terms of welfare. Under more concave utility functions, utilitarian welfare is higher when average types are the same across groups ('integration') than under *top-down sorting* ('segregation'). With less concave utilities, the opposite is true. Concavity also favors large groups. Under less concave functions there is a trade-off, as very homogeneous groups are good, but large groups are also good, so one could be willing to sacrifice in terms of homogeneity to increase contributions because of group size. The results are robust to environments in which groups compete among themselves.

While our results provide important insights, there are of course limitations to our analysis. For example, we have not studied repeated interactions, which, through reciprocity, could lead to different results. We conjecture that in a repeated environment we would be more likely to observe the results we posit under 'centralized' group management. We think the reasons for assortative matching will survive in that case.

We have studied an environment where all the agents are 'symmetric' even if they have different types. However, there are important applications where matching is bilateral, like the marriage market. We think that some of our insights will carry over in those applications. Indeed Greenwood et al. (2014) document an increase in marriage market assortativity as an important source of inequality. However, it is not immediately obvious that our welfare results on sorting will carry over in that context. Another intriguing question in that environment has to do with the fact that in many species, it is only one of the sides of the marriage market that engages in costly signaling before mating (Jiang, Bolnick and Kirkpatrick 2014).

Our paper also provides interesting insights for public policy. Most extant foundations for 'integration' policies, either in housing or school settings have to do either with concerns to reduce inequality (Ananat 2011, Reardon 2016) or with direct spillovers from higher ability individual on other individuals (Duflo, Dupas, Kremer 2011, Graham, Imbens, Ridder 2014). We provide a new foundation for integration that is based on indirect spillovers. Individuals do not care directly about the type of others', they care because high types provide higher effort towards public good provision.

It is worth commenting that the integration result in our model comes from the objective of social surplus maximization rather than equity considerations. Nonetheless, it has equity implications as low types benefit from the higher provision of public goods in their group. Thus, the efficient distribution is also equitable and may provide a new rationale for integration efforts in different societies. Our model may also provide a different rationale to whether the attainment gap of children (of lower socio-economic status) in segregated neighborhoods (see for instance Ananat, 2011) may have something to do with the lower provision of public good in such neighborhoods. If so, this may have positive implications, particularly across generations, which may further strengthen case for integration. We leave this interesting issue for future work.

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Appendix A

Proof of Proposition 1

A member of coalition l with type ξ_i does not want to move to coalition $l+1$ provided that:

$$\begin{aligned} U_i\left(\bar{\xi}_l\right) - p_l \geq & \quad U_i\left(\bar{\xi}_{l+1}\right) - p_{l+1} \\ U_i\left(\bar{\xi}_l\right) - U_i\left(\bar{\xi}_{l+1}\right) \geq & \quad p_l - p_{l+1}. \end{aligned}$$

Such person will have a type such that $\xi_{i^*(l-1)} \geq \xi_i \geq \xi_{i^*(l)}$. Then we have that:

$$\begin{aligned} U_i\left(\bar{\xi}_l\right) - U_i\left(\bar{\xi}_{l+1}\right) = & \int\limits_{\bar{\xi}_{l+1}}^{\bar{\xi}_l} \frac{\partial U_i}{\partial \bar{\xi}_j} d\bar{\xi}_j \\ \geq & \int\limits_{\bar{\xi}_{l+1}}^{\bar{\xi}_l} \frac{\partial U_{i^*(l)}}{\partial \bar{\xi}_j} d\bar{\xi}_j \\ = & \quad p_l - p_{l+1}, \end{aligned}$$

where the inequality is true by (1). Similarly a member of coalition l with type ξ_i does not want to move to coalition $l-1$ provided that:

$$\begin{aligned} U_i\left(\bar{\xi}_l\right) - p_l \geq & \quad U_i\left(\bar{\xi}_{l-1}\right) - p_{l-1} \\ p_{l-1} - p_l \geq & \quad U_i\left(\bar{\xi}_{l-1}\right) - U_i\left(\bar{\xi}_l\right) \end{aligned}$$

Remember that $\xi_{i^*(l-1)} \geq \xi_i \geq \xi_{i^*(l)}$. Thus:

$$\begin{aligned} p_{l-1} - p_l = & \int\limits_{\bar{\xi}_l}^{\bar{\xi}_{l-1}} \frac{\partial U_{i^*(l-1)}}{\partial \bar{\xi}_j} d\bar{\xi}_j \\ \geq & \int\limits_{\bar{\xi}_l}^{\bar{\xi}_{l-1}} \frac{\partial U_i}{\partial \bar{\xi}_j} d\bar{\xi}_j \\ = & \quad U_i\left(\bar{\xi}_{l-1}\right) - U_i\left(\bar{\xi}_l\right), \end{aligned}$$

where, again, the inequality is true by (1). ■

Proof of Proposition 6

The FOC for the different players are now

$$\frac{\xi_j}{N}\frac{b\left(\sum_{i=1}^{M} y_i\right)^b\left(\sum_{i=1}^{N} x_i\right)^{b-1}}{\left(\left(\sum_{i=1}^{N} x_i\right)^b+\left(\sum_{i=1}^{M} y_i\right)^b\right)^2}=x_j$$

$$\frac{1}{N}\frac{b\left(\sum_{i=1}^{M} y_i\right)^b\left(\sum_{i=1}^{N} x_i\right)^{b-1}}{\left(\left(\sum_{i=1}^{N} x_i\right)^b+\left(\sum_{i=1}^{M} y_i\right)^b\right)^2}=\frac{x_j}{\xi_j}=x_1$$

$$\frac{1}{N}\frac{b\left(y_1\sum_{i=1}^{M} \xi_i\right)^b\left(x_1\sum_{i=1}^{N} \xi_i\right)^{b-1}}{\left(\left(x_1\sum_{i=1}^{N} \xi_i\right)^b+\left(y_1\sum_{i=1}^{M} \xi_i\right)^b\right)^2}=x_1$$

$$\frac{1}{N}\frac{b\left(y_1M\bar{\xi}_k\right)^b\left(x_1N\bar{\xi}_l\right)^{b-1}}{\left(\left(x_1N\bar{\xi}_l\right)^b+\left(y_1M\bar{\xi}_k\right)^b\right)^2}=x_1 \tag{9}$$

$$\frac{1}{M}\frac{b\left(y_1M\bar{\xi}_k\right)^{b-1}\left(x_1N\bar{\xi}_l\right)^{b}}{\left(\left(x_1N\bar{\xi}_l\right)^b+\left(y_1M\bar{\xi}_k\right)^b\right)^2}=y_1$$

$$\frac{\bar{\xi}_kM^2y_1}{\bar{\xi}_lN^2x_1}=\frac{x_1}{y_1}$$

$$\sqrt{\bar{\xi}_k}My_1=\sqrt{\bar{\xi}_l}Nx_1$$

$$\bar{\xi}_kMy_1=\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}Nx_1 \tag{10}$$

and substituting (10) into (9) we get

$$x_1=\frac{1}{N}\frac{b\left(y_1M\bar{\xi}_k\right)^b\left(x_1N\bar{\xi}_l\right)^{b-1}}{\left(\left(x_1N\bar{\xi}_l\right)^b+\left(y_1M\bar{\xi}_k\right)^b\right)^2}=\frac{1}{x_1}\frac{1}{N}\frac{b\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}N\right)^b\left(N\bar{\xi}_l\right)^{b-1}}{\left(\left(N\bar{\xi}_l\right)^b+\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}N\right)^b\right)^2}$$

$$x_1^2=\frac{1}{N^2}\frac{b\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}\right)^b\left(\bar{\xi}_l\right)^{b-1}}{\left(\left(\bar{\xi}_l\right)^b+\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}\right)^b\right)^2}=\frac{1}{N^2}\frac{1}{\bar{\xi}_l}\frac{b\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}\right)^b}{\left(\left(\sqrt{\bar{\xi}_l}\right)^b+\left(\sqrt{\bar{\xi}_k}\right)^b\right)^2}$$

$$y_1^2=\frac{\bar{\xi}_k\bar{\xi}_l\left(Nx_1\right)^2}{\left(\bar{\xi}_kM\right)^2}=\frac{1}{M^2}\frac{1}{\bar{\xi}_k}\frac{b\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}\right)^b}{\left(\left(\sqrt{\bar{\xi}_l}\right)^b+\left(\sqrt{\bar{\xi}_k}\right)^b\right)^2}$$

$$
\begin{aligned}
U_j &= \frac{\xi_j}{N}\left(1-\frac{\left(y_1 M\bar{\xi}_k\right)^b}{\left(x_1 N\bar{\xi}_l\right)^b+\left(y_1 M\bar{\xi}_k\right)^b}\right)-\frac{1}{2}x_j^2 \\
&= \frac{\xi_j}{N}\left(1-\frac{\left(\sqrt{\bar{\xi}_k}\right)^b}{\left(\sqrt{\bar{\xi}_l}\right)^b+\left(\sqrt{\bar{\xi}_k}\right)^b}\right)-\frac{1}{2}\xi_j^2\frac{1}{N^2}\frac{1}{\bar{\xi}_l}\frac{b\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}\right)^b}{\left(\left(\sqrt{\bar{\xi}_l}\right)^b+\left(\sqrt{\bar{\xi}_k}\right)^b\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial U_j}{\partial \bar{\xi}_l} &= \frac{1}{2N^2}\frac{b}{\bar{\xi}_l^2}\frac{\xi_j^2}{\left(\left(\sqrt{\bar{\xi}_l}\right)^b+\left(\sqrt{\bar{\xi}_k}\right)^b\right)^2}\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}\right)^b \\
&+\frac{1}{2N}\frac{b}{\sqrt{\bar{\xi}_l}}\frac{\xi_j}{\left(\left(\sqrt{\bar{\xi}_l}\right)^b+\left(\sqrt{\bar{\xi}_k}\right)^b\right)^3}\left(\sqrt{\bar{\xi}_k}\right)^b\left(\sqrt{\bar{\xi}_l}\right)^{b-1} \\
&+\frac{1}{2N^2}\frac{b^2}{\bar{\xi}_l^{\frac{3}{2}}}\frac{\xi_j^2}{\left(\left(\sqrt{\bar{\xi}_k}\right)^b+\left(\sqrt{\bar{\xi}_l}\right)^b\right)^3}\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}\right)^b\left(\sqrt{\bar{\xi}_l}\right)^{b-1} \\
&-\frac{1}{4N^2}\frac{b^2}{\bar{\xi}_l^{\frac{3}{2}}}\left(\sqrt{\bar{\xi}_k}\right)\frac{\xi_j^2}{\left(\left(\sqrt{\bar{\xi}_l}\right)^b+\left(\sqrt{\bar{\xi}_k}\right)^b\right)^2}\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}\right)^{b-1}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\frac{\partial^2 U_j^b}{\partial \bar{\xi}_B \partial \xi_j} &= \frac{1}{2N^2}\frac{b}{\bar{\xi}_l^2}\frac{2\xi_j}{\left(\left(\sqrt{\bar{\xi}_l}\right)^b+\left(\sqrt{\bar{\xi}_k}\right)^b\right)^2}\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}\right)^b \\
&+\frac{1}{2N^2}\frac{b^2}{\bar{\xi}_l^{\frac{3}{2}}}\frac{2\xi_j}{\left(\left(\sqrt{\bar{\xi}_k}\right)^b+\left(\sqrt{\bar{\xi}_l}\right)^b\right)^3}\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}\right)^b\left(\sqrt{\bar{\xi}_l}\right)^{b-1} \\
&+\frac{1}{4N^2}\frac{b^2}{\bar{\xi}_l^{\frac{3}{2}}}2\xi_j\frac{\left(\sqrt{\bar{\xi}_l}\right)^b-\left(\sqrt{\bar{\xi}_k}\right)^b}{\left(\left(\sqrt{\bar{\xi}_l}\right)^b+\left(\sqrt{\bar{\xi}_k}\right)^b\right)^2}\left(\sqrt{\bar{\xi}_k}\sqrt{\bar{\xi}_l}\right)^b
\end{aligned}
$$

and thus the condition 7 is satisfied so that we can have an analog of Proposition 2. ■