Doubts and Equilibria*

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Abstract

In real life strategic interactions decision-makers are likely to entertain doubts about the degree of optimality of their play. To capture this feature of real choice-making, we present a model based on the doubts felt by an agent about how well is playing a game. The doubts are coupled with (and mutually reinforced by) imperfect discrimination capacity, which we model by means of similarity relations. These cognitive features, together with an adaptive learning process guiding agents' choice behavior leads to doubt-based selection dynamic systems. We introduce the concept of Mixed Strategy Doubt Equilibrium and show its theoretical and empirical relevance.

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1 Introduction

In real life, introspection before taking a decision (business venture, cancer treatment) is often pervaded with doubts about the consequences associated to the alternatives. Even experienced decision-makers face doubts in their domain of expertise. There is a vast amount of literature on decision-making under risk (when the probabilities are known) and uncertainty (when the probabilities are unknown), both theoretical and experimental, but, to our knowledge, doubt, as a cognitive mechanism influencing choice, has attracted little attention in decision theory, behavioural economics and other fields. The present work introduces doubts in relation to choices in strategic environments in an attempt at formalizing that feeling to become theoretically instrumental. Our purpose is to bring into sharper focus the role of doubts and its influence in the final outcome of an interactive decision-making process.

Doubts are commonly related to uncertainty, but doubts and uncertainty should be distinguished. It may be said that doubts appear mainly as a *consequence* of uncertainty. While uncertainty is a characteristic of some states of nature, doubts belong to the human intellect, they are the manner in which uncertainty is perceived and interpreted by the human mind. Doubts, in the Ramsey-Savage tradition, were taken into account with the purpose of building a model of rational choice under uncertainty: *"if he* (the subject) *were doubtful his choice would not be decided so simply. I propose to lay down axioms and definitions concerning the principles governing choices of this kind."* Ramsey (1928). Both the notion of doubt used here and the notion of measurement of belief coincide in that they are a basis for action. They differ in that for Ramsey-Savage the degrees of certainty (or doubts) in one's beliefs about the consequences of a certain action are measured by the odds that one will just take. While here, we assume that the agents measure their doubts by just observing the choices made by their fellow agents: the level of doubts that I feel is sensitive to the proportion of people who have adopted the same strategy as mine; the higher this proportion the smaller are my doubts (we shall come back to this below). Our purpose thus is not normative. We deal with doubtful agents who cannot take into consideration gambles outside the game they are compelled to play. The economic agents have to put up with their doubts, and make choices for real, not hypothetical. Furthermore, we can safely say that the essence of a choice behaviour by doubtful agents is that, if given the opportunity, they switch actions. Some times they switch quite often, and with no apparent reason, so that each of the available action has almost the same probability to be chosen. The reason that could explain the switching behaviour is that it is the level of doubts what determines the probability of retaining the agent's current action: the higher the level of doubts about how good the chosen action is, the higher the probability of switching to a new one. Then, to understand and measure doubts, the least that it is required is that the (experimental) subjects have the opportunity to repeat the act of taking a decision from the same choice set (in a setting where some parameters, representing the environment, take different values). For this reason, we think that neither the normative approach nor the psychological (or descriptive) approach to choice under risk and uncertainty (such as prospect theory, cumulative prospect theory and support theory, Khaneman and Tversky 1979, Tversky and Kahneman 1992, Tversky and Kohler 1994) have isolated completely the influence of doubts on choices.

Contrary to the feeling of regret or rejoice, the feeling of doubts appear from introspection without information feedback. We here treat *doubts* as a cognitive reaction to making choices with uncertain consequences. Decision-making with doubts requires a more intense use of the cognitive resources of the human mind (such as those based on personal experience, knowledge and capabilities, memory, reasoning, discrimination capacity and learning) to tackle uncertainty and ambiguity. It is probably safe to say that strategic environments are more cognitive demanding and raise more doubts (in part because strategic uncertainty is typically of a more ambiguous nature). Thus, the strategic environment requires that we should model doubtful agents as if they were in a state of heightened watchfulness, adopting cautious, step-by-step, procedures for decision-making, and, as well, ready to revise the choices made in the past.

Economists have understood for a long time that imitation of "common" behavior is a widespread human decision-making strategy (see e.g. Alchian 1950, Smallwood and Conlisk 1979 or Nelson and Winter 1982). Individuals and organizations learn from direct experience and from the experience of others, obtaining sources of information that might be used to reduce uncertainty (see Levitt and March, 1988;for empirical work about imitation by organizations, see Henisz and Delios 2003, and for theoretical work on imitation by individuals see Schlag 1998). Thus, imitation, trial and error processes, traditional rules of thumb, conventions and bayesian updating, are some of the strategies used to deal with uncertainty by real economic agents, generating an adaptive dynamics which can be modelled by means of several selections dynamics of a different nature (deterministic or stochastic). Our view is that one way of knowing some of the consequences of doubts on choices is by studying the long-run outcome of the dynamics generated by the (action) switching behaviour of the doubtful agents.

To this end, we relate doubts with strategic decisions in a dynamic model in which doubtful agents interact frequently. Let us think of a game continuously played by two player populations, each with a set of pure strategies to choose from. A player population is composed of many (doubtful) agents, each playing a pure strategy. Our purpose is to characterize the long-run outcome of the choice behavior of this kind of agents.

A central element in the architecture of decision-making of doubtful agents is a measure of their personal doubts. We call this measure "the doubt function". When players make choices continuously, they obtain information about payoffs, but also about the fraction of fellow agents playing each strategy. A hesitant agent tends to reinforce his decision procedure by taking into account whatever information might help to elicit his preferences. In our crude approach to this complex scenario, agents calibrate their doubts by means of information about what other agents in the same situation are doing. Each agent accumulates over time, as a participant in the game, a certain amount of playing experience, information and knowledge. Furthermore, people have a tendency to rely on the intelligence or experience of others, and thus imitate them. Then, if the number of agents who play a strategy increase, it could be taken as the signal of an aggregate improvement in the experience of that strategy's users. This, on the one hand, increases its relative popularity in the eyes of those who are currently playing it. Also, it will call the attention of those who are not using it. Hence, if an agent observes that many others play a given strategy, it is natural for him to entertain less doubts about whether that strategy is a "good" option (we shall see that, in our model, this is not necessarily a conformity behaviour). Thus, doubts about a strategy which decrease with the number of people using it are the most natural ones[1] and a great deal of the current paper is dedicated to them. But note that we are not saying that this *is* the way that personal doubts are measured in real life choice situations. Probably there are several ways of measuring doubts. We are just assuming that agents do measure them in that way and we study the consequences of doing so.

A natural feature of the decision procedure used by the doubtful agents is that doubts give rise to similarity thresholds for expected payoffs and strategy frequencies.[2] It is known that

[1]Even if they are not the only possible type, as we shall see below.

[2]The work of Kahneman and Tversky has plenty of examples about how the human cognitive system copes

similarity judgments are part of observed decision procedures (see Tversky 1977, Rubinstein 1988 and Arieli et al. 2009). Hence, it seems natural to assume that the doubtful agent would build procedures based on similarity relations to elicit whether to continue playing with his current strategy or switch to a different one. This adjusting behaviour, with which two opposing population of agents choose over time the strategies in a game, gives rise to what we call the *doubt-based selection dynamic model.*

We explore the long run properties of this selection dynamics for constant-sum $2{\times}2$ games with a unique equilibrium in mixed strategies. The rest point of the *doubt-based selection dynamics* is called Mixed Strategy Doubt Equilibrium (MSDE). Some of the results of the paper depend on two limiting cases of doubtful behaviour which could be thought of as simple heuristics (Gigerenzer and Todd 1999)*:* the *doubt-less mode of play* and the *doubt-full mode of play.* In the former mode, the agent trusts so intensely in the opinion of the others, -"the wisdom of crowd"-, that no matter the proportion of his fellow agents whose choice coincides with the one he has made, the agent has extremely low level of doubts. In the latter mode, the agent has extremely high level of doubts, no matter the proportion of people who have chosen like him. He does not believe in the "wisdom of crowd" and we call this attitude as *Cartesian skeptical.* Notice that, in both modes of play, the agent could said that it is not sensitive to the popularity of the choices made by his fellow agents, and, hence, it cannot be said that there is herding or conformity.

The results we obtain happen to agree with ordinary ideas, such as skepticism is a good guide for action or too much trust in the wisdom of crowd is not a good strategy and neither is conformity. More specifically, we study the relationship between the MSDE and the Mixed Strategy Nash Equilibrium (MSNE). An MSNE is, under some conditions, a rest point for the *doubt-based selection dynamic system.* Now, suppose first the situation in which all agents operate under the *doubt-full mode* of play (i.e. all agents are *Cartesian skeptical*). We show that the system converges to population frequencies close to the Mixed Strategy Nash Equilibrium when all agents are Cartesian skeptical. A different situation is when agents have very small level doubts (even if they still decrease in the frequency of play); this is the *doubt-less* mode of play. Dynamics in this case are different as any perturbation, however small, sends the system away from the equilibrium.

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with such situations of limited capacity for discrimination. See, for instance, Tversky (1977), Kahneman and Tversky (1979), Kahneman (2003) and the references therein.

There are also quite interesting intermediate cases with strictly decreasing doubts that are less extreme than the previous cases, in between the *doubt-full* and *doubt-less* modes of play. The equilibrium of the doubt-based dynamic system, the MSDE, is not a Nash equilibrium and has the following feature: the most popular strategy has smaller (expected) payoffs. This is a *general* characteristic of equilibria with decreasing doubt functions. But in the *doubt-full* mode of play it is not so evident since the equilibrium is close to being Nash and we have asymptotic stability, whereas in the *doubt-less* mode, we get unstable dynamics. We believe that this feature of the equilibria of the *doubt-based selection dynamics system* is a relevant and testable implication of our model, and we provide some preliminary evidence to support it.

Finally, we should mention as well the case of *constant* doubts. This means that each agent's hesitations and feelings of uncertainty are not affected by the fraction of fellow agents from his population playing the same strategy. Thus, society does not have any direct influence on this type of agent. Then we show that the adjusting behavior would lead us to a doubt-based selection dynamics that is closely related to the replicator dynamics.[3]

To conclude, we think that this paper, by insisting on doubts related with imperfect perception, highlights the need of more evidence from fuzzier, that is, more realistic, experimental environments. Thus, our next step should be to test the theory presented in this paper. A different line of research would be in the field of choice theory; that is the study of choice behaviour under doubts to tackle the problem of choice under uncertainty.

2 A model of doubt-based selection dynamics

2.1 Notation

Consider a noncooperative finite game G in normal form, with $K = \{1, 2, ..., n\}$ denoting the set of players. For each player $k \in K$, let $S_k = \{1, 2, ..., m_k\}$ be her finite set of pure strategies, for some integer $m_k \geqslant 2$.

Imagine that there exist n large populations, one for each of the n player positions in the game. Members of the n populations chosen at random -one member from each player

[3]This result is yet another rationalization for the replicator dynamics. Other foundations for this dynamical system can be found in Binmore, Gale and Samuelson (1995), Weibull (1995), Cabrales (2000) and Schlag (1998), among others.

population- are repeatedly matched to play the game. In what follows, we shall speak of *players* when referring to the game G and we shall speak of *agents* when referring to the members of the populations. Each agent is characterized by a pure strategy. From now on, we shall refer to the agent ki as a member of the player population $k \in K$ who plays pure strategy $i \in S_k$. Let $f_{ki}(t) \in F_{ki} = [0,1]$ be the relative frequency of ki agents at time t, with $f(t)$ being the vector collecting such probabilities. Time index suppressed, $\pi_{ki}(f)$ will denote agent ki's expected payoff given the population state f. Without loss of generality, we may assume that payoffs are strictly positive and smaller than one; that is, $\pi_{ki}(f) \in \Pi_{ki} = [m, M]$, $m > 0$ and $M < 1$. Finally, $\overline{\pi}_k(f) = \sum_{i=1}^{m_k} f_{ki}(f)\ \pi_{ki}(f)$ is the average payoff in player population $k \in K$. To simplify notation, we shall denote $\pi_{ki}(f)$ as π_{ki}.

Doubtful behaviour We assume that the game is played by boundedly rational players who have doubts about how well they are playing. More precisely, every agent of each player population is endowed with a (primitive) function that we call the "doubt function". This function, denoted d_{ki}, measures the doubts felt by agent ki about how good is his current strategy $i \in S_k$, available to player population $k \in K = \{1, 2, ..., n\}$, as a response to the strategies that the remaining players are using. Each agent ki relates his doubts to $f_{ki} \in F_{ki}$, the proportion of individuals, in his player population, who are equally using his current strategy $i \in S_k$.

We shall assume that the agents are endowed with a strictly decreasing doubt function. That is, an agent's doubts about how well is playing gradually decrease when he observes (or is informed of) a gradual increase in the number of agents from his player population playing the same strategy as the one he is currently using. The underlying logic of this assumption is a belief on the part of agents about the collective *wisdom of crowds*, combined with the cognitive ease of trusting others relative to thinking through the decision problem.[4]

We may distinguish different degrees of trust on the *wisdom of crowds* to calibrate one's doubts. We shall classify them into two broad groups, each with a type of doubtful behaviour.

a) The *Herding doubts agent* (or, in short, the *Herding agent*): a typical agent in this group believes in "the wisdom of crowd" and so his doubts are very sensitive to the level of popularity, $f_{ki} \in (0, 1)$, of his current strategy $i \in S_k$.

b) The *Skeptical agent*: this type of agent is suspicious about "the wisdom of crowd" and

[4] A similar argument is made in Smallwood and Conlisk (1979).

has high level of doubts for any $f_{ki} \in (0, 1)$.

The doubts of the *Skeptical agent* are not as those mentioned for the *Herding agent*, a feeling that arises context depending that is sensitive to the proportion of individuals choosing the same action. Now doubts depend less than before on any specific decision problem. We may think that individuals might have life experience built-in doubts. The individual is conscious about them, and doubts are systematically used as a method for reasoning and learning or as a procedure for decision-making. Those methodological doubts could also be reinforced by philosophical principles, as advised, for instance, by Hume (2007) and Descartes (2008, and http://plato.stanford.edu/entries/descartes-epistemology/). This kind of doubts can be seen as a model of behaviour or heuristics (Gigerenzer and Todd, 1999) used by the agents. Hume asks *"What is meant by a sceptic? And how far it is possible to push these philosophical principles of doubt and uncertainty?"* Scepticism, answers Hume, must be understood *"as a sovereign preservative against error and precipitate judgment"*. But Hume is not in favour of excessive skepticism. He refuses the radical skepticism of Cartesian doubts on practical rather than theoretical grounds. The *Cartesian skeptical agent* thus, might be viewed as a kind of limiting case of *Skeptical agents*. This skeptical agent behaves as if guided by the method of doubt described in the theory of knowledge of Descartes (1996). The aim of the method is to build knowledge based on solid principles and obtain certainty as in mathematics. Skeptic doubts would play the role of epistemic demolition, to clear the ground of false beliefs and find those that are free of error that would be candidates for the foundations of knowledge.

The aim of Descartes' method of doubt applied to the context of strategic interaction would be to find a set of strategies that is optimal for each player. The strategic uncertainty would then be the principal reason for doubting. And the way the *Cartesian skeptical agents* proceed would be based on Descartes' advice: that doubts must be *hyperbolic* and *universal.* In the quest of building knowledge based on unshakable principles, Descartes' advice is that the epistemic demolition must be carried out using heavy duty tools, the more hyperbolic the doubts the better. A *Cartesian skeptical* agent is thus endowed with *hyperbolic doubts.* For us, this would mean to have nearly the maximum level of doubts allowed by the model described below.

On the other hand, a *skeptical* agent with *universal doubts* would think of the set of strategies like Descartes' basket of apples (apples as the analog of beliefs in Descartes), where some of them are rotten and thus there exists a risk of rot spreading. Which ones would be chosen? and how would the choice process be? The advice is that the best way to accomplish the separation of the rotten apples from the sound ones and get a rot-free basket is to reject all of them:

"They then attempt to separate the false beliefs from the others, so as to prevent their contaminating the rest and making the whole lot uncertain. Now the best way they can accomplish this is to reject all their beliefs together in one go, as if they were all uncertain and false. They can then go over each belief in turn and re-adopt only those which they recognize to be true and indubitable" (see http://plato.stanford.edu/entries/descartes-epistemology/ ; in particular, see section 2. Methods: Foundationalism and Doubt).

Hence, in our context, *universal doubts* would mean that no strategy could be trusted. All of them are under doubt, and each one of them should be scrutinized and experimented. A *Cartesian skeptical* agent, endowed with *hyperbolic* and *universal* doubts, will not perceive differences when comparing pairs of strategies: each one of them rises the same level of doubts: the maximum possible. And this would mean that the doubts felt by a *Cartesian skeptical* agent should not be sensitive to the popularity of his current strategy. Since strategic uncertainty would always be present, the reason for doubting would persist and the *Cartesian skeptical* agent would then be always highly dissatisfied. As a consequence, this agent will be endlessly switching strategies and experimenting with all the available strategies.

We formalize these types of doubtful behaviour as follows.

The doubt functions Formally, let us consider the following set of strictly decreasing and differentiable doubt functions:

$$D = \left\{ d_{ki} : F_{ki} \to [0,1] : \widehat{f}_{ki} > \widetilde{f}_{ki} \Rightarrow d_{ki}(\widehat{f}_{ki}) < d_{ki}(\widetilde{f}_{ki}) \right\}$$

When an element $d_{ki} \in D$ is interpreted as a doubt function, $d_{ki}(f_{ki})$, for some $f_{ki} \in F_{ki}$ known by the ki agent, measures the doubts (about how well is playing the game) felt by the agent ki when the proportion of agents in player population k playing strategy $i \in S_k$ at time t is $f_{ki} \in F_{ki}$.

Let $m < M$, with both m and M in $(0,1)$. We will be working with the following types of agents:

- *Herding agents:* they are endowed with doubt functions in the set $D_m \subset D$ such that $d_{ki}(f_{ki}) < m$ for all $f_{ki} \in [0, 1]$.

- *Skeptical agents*: they are endowed with doubt functions in the set $D_M \subset D$, such that $d_{ki}(f_{ki}) > M$, for all $f_{ki} \in [0, 1]$.

Now, let $\delta \in (0, m)$ be small enough so that $1 - \delta \in (M, 1)$. Inside these two types of agents, we should note the following:

1. The *Doubt-less agent:* this agent is endowed with a function in the class $D^{\delta} \subset D_m$, such that $d_{ki}(f_{ki}) < \delta$ for all $f_{ki} \in [0, 1]$. When δ is sufficiently small, we say that the agent ki is in the *doubt-less mode* because, whatever the level of popularity of his current strategy, $f_{ki} \in (0, 1)$, his doubts are almost zero. The agent endowed with such a function strongly believes in his current strategy.

2. The *Cartesian skeptical agent* (or the *Doubt-full agent*): this agent is endowed with a doubt function in the class $D^{1-\delta} \subset D_M$, such that $d_{ki}(f_{ki}) > 1 - \delta$ for all $f_{ki} \in [0, 1]$. Whatever the level of popularity of his current strategy, f_{ki}, his doubts are almost one; that is, the agent has *hyperbolic doubts.* We say that a Cartesian skeptical agent plays in the *doubt-full mode.* Thus, this type of agent is very suspicious of "the wisdom of crowd" to trust in his current strategy.

An Index for dissatisfied agents Our adaptive agents are current users of some strategy and, very likely, past and future user of some others. Inside a player population its members are likely to share their experience and information about the game. This naturally leads to imitation processes which give rise to observational learning, herding and other forms of convergent behavior. Thus, it is reasonable to assume that adaptive agents have access to information about the relative popularity of each strategy available to them, as well as to their payoffs. The flows of agents among the strategies derive from the level of satisfaction felt with their current strategy. To avoid the use of different parameters determining the level of doubts, we will be working with just one type of doubtful agents: either they are all *Herding agents* or they are all *Skeptical agents.*

Let

$$\alpha_{ki} = \alpha_{ki}(\pi_{ki}, f_{ki}, \pi_{kj}, f_{kj}),\ i \neq j$$

denote the proportion of ki strategists who feel dissatisfied with strategy i at time t. In the Appendix A we justify and microfound the following choice of this function via a model of (correlated) similarities relations[5] :

$$\alpha_{ki} = \frac{\lambda_{ki}}{\Sigma_{i=1}^{m_k} \lambda_{ki}} = \frac{\lambda_{ki}}{\lambda_k}$$

Where the function $\lambda_{ki}(\pi_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - d_{ki}(f_{ki})}$, - given some $f_{ki} \in [0,1]$-, is used to build correlated similarities on the frequency space F_{ki} (a detailed account of λ_{ki} and α_{ki} is given in Appendix A). This function determines the size of the similarity interval in F_{ki}. And the doubt level, $d_{ki}(f_{ki})$, determines the size of the correlated similarity interval on Π_{ki}. Thus, both $d_{ki}(f_{ki})$ and $\lambda_{ki}(\pi_{ki})$ are the thresholds of their corresponding similarity interval. Note that no matter the type of agents, - *Herding* or *Skeptical* -, the sign of α_{ki} is positive.

In Appendix A, we show how the agent ki builds a procedural preference on $F_{ki} \times \Pi_{ki}$ compatible with this pair of correlated similarity relations (in the same spirit of Rubinstein 1988, Aizpurúa et al. 1993 and Uriarte 1999). Given a vector $(\pi_{ki}, f_{ki}) \in F_{ki} \times \Pi_{ki}$ attached to strategy i, the thickness of its corresponding indifference set is sensitive to both $d_{ki}(f_{ki})$ and π_{ki}. The higher the doubts and/or the smaller the payoffs, the thicker the indifference set will be; hence, the higher is the distance from (π_{ki}, f_{ki}) to its preferred set and so the more dissatisfied the agent ki will feel. It can be seen that the variations of this distance are captured by the properties of λ_{ki} (that is, the variations of λ_{ki} due to changes in π_{ki} and f_{ki}). Hence the λ_{ki} function could be taken as a measure of the degree of dissatisfaction of the agent ki with respect to his current strategy $i \in S_k$; $\Sigma_{i=1}^{m_k} \lambda_{ki} = \lambda_k$ will the be the total dissatisfaction level in population $k \in K$.

The limit case of the *herding agents*, the *doubt-less agent,* would be highly satisfied with his current strategy because his doubts are almost zero and hence the indifference set will be almost a singleton. On the other hand, it can be seen (in Appendix A) that the *skeptical agent* has indifference sets covering the whole choice space $F_{ki} \times \Pi_{ki}$ and thus will feel highly dissatisfied. And the *Cartesian skeptical* agent endowed with hyperbolic and *universal* doubts, as describe above, will be continuously switching and experimenting new strategies.

[5]For the definition of similarity relation, see Rubinstein (1988).

The Doubt-Based Selection Dynamics We assume that time is divided into discrete periods of length τ. In every period, $1-\tau$ is the probability that the agent does retain his current strategy; thus, τ is the probability that each agent does not retain his current strategy. We make now the following assumption to build a selection dynamic model[6]

ASSUMPTION **1** *When an agent feels dissatisfied with his current strategy, she will choose a new strategy with a probability that is equal to the proportion of agents playing that strategy. From Assumption 1,* $\tau \frac{\lambda_{ki}}{\lambda_k} f_{ki}$ *will denote the proportion of ki strategists who will choose a new strategy (the outflow), and, since a particular strategy is chosen with a probability that is equal to the proportion of agents playing that strategy, then* $\tau \sum_{j=1}^{m_k} \frac{\lambda_{kj}}{\lambda_k} f_{kj} f_{ki} = \tau \frac{\overline{\lambda}_k}{\lambda_k} f_{ki}$ *is the proportion of agents who will choose strategy i (the inflow), where* $\overline{\lambda}_k = \sum_{j=1}^{m_k} \lambda_{kj} f_{kj}$.

Therefore,

$$f_{ki}(t+\tau) = f_{ki}(t) - \tau \frac{\lambda_{ki}}{\lambda_k} f_{ki} + \tau \frac{\overline{\lambda}_k}{\lambda_k} f_{ki}$$

As $\tau \to 0$, in the limit we get the *doubt-based* selection dynamic equation:

$$\overset{\bullet}{f}_{ki} = f_{ki} \left[\frac{\overline{\lambda}_k - \lambda_{ki}}{\lambda_k} \right] \tag{1}$$

To gain some intuition, let us now look at equation (1) in a less compact way. Let G be a two-population constant-sum game with $S_I = \{U, D\}$ and $S_{II} = \{L, R\}$ denoting player I and player II's strategy sets, respectively. Let x denote the probability of playing U, y the probability of playing L and $I = [(x^*, 1-x^*), (y^*, 1-y^*)]$ the Mixed Strategy Nash Equilibrium, with $x^* > 0$ and $y^* > 0$.

We denote the four doubt functions $d_i \in D$ (where $i = U, D, L, R$). From (1), the *doubt-based* selection dynamics for G is represented by the following system:

$$\dot{x} = \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)} (\pi_U d_D - \pi_D d_U) \equiv G_1(x,y) F_1(x,y) \tag{2}$$

$$\dot{y} = \frac{y(1-y)}{\pi_L(\pi_R - d_R) + \pi_R(\pi_L - d_L)} (\pi_L d_R - \pi_R d_L) \equiv G_2(x,y) F_2(x,y) \tag{3}$$

[6] For a justification see, for example, Binmore et al. (1995).

Clearly, a stationary point for the *doubt-based* system (2)-(3), with $x^* > 0$ and $y^* > 0$, requires $\pi_U d_D = \pi_D d_U$ and $\pi_L d_R = \pi_R d_L$. We call this point the Mixed Strategy Doubt Equilibrium (MSDE).

2.2 Mixed Strategy Nash Equilibrium (MSNE) and Mixed Strategy Doubt Equilibrium (MSDE)

We should distinguish between the Mixed Strategy Nash Equilibrium (MSNE) and the Mixed Strategy Doubt Equilibrium (MSDE) for the doubt-based dynamic system (1).

1. In a MSNE the requirement is that all strategies in the support of the equilibrium have equal payoffs; that is:

$$\pi_{ki}(f^*) = \pi_{kj}(f^*) \text{ for all } i, j \text{ with } f_i^* > 0 \text{ and } f_j^* > 0 \text{ and all } k.$$

2. From (1) we deduce that for a MSDE the requirement is (recall the assumption $d_{ki} = d \in D$):

$$\frac{\pi_{ki}(f^*)}{d_{ki}(f_i^*)} = \frac{\pi_{kj}(f^*)}{d_{kj}\left(f_j^*\right)} \text{for all } i, j \text{ with } f_i^* > 0 \text{ and } f_j^* > 0 \text{ and all } k$$

Note that in this case, the expected payoffs to the strategies in the support of the equilibrium need not be equal, as it is required in the MSNE. We have the following result:

Proposition 1 *Suppose that all the agents are endowed with a doubt function* $d_{ki} = d$ *. Then for all* k *and all* i, j*, with* $0 < f_{kj}^* < f_{ki}^* < 1$*, since the doubt functions are strictly decreasing,* $d(f_{ki}^*) < d\left(f_{kj}^*\right)$*; thus, in order to satisfy the Mixed Strategy Doubt Equilibrium condition, we must have* $\pi_{ki}(f^*) < \pi_{kj}(f^*)$*.*

Proof: Direct from the Mixed Strategy Doubt Equilibrium (MSDE) condition.

In words, the more frequent strategies in a MSDE should have lower expected payoffs. This situation is clearly distinct from a Nash equilibrium and is a general feature of the (decreasing) doubt-based dynamic system.

3 Doubt-based selection dynamics in constant sum games

In this section we shall explore the relationship between the MSNE and MSDE for different levels of doubts.

3.1 Relationship between a MSNE and a MSDE

Let us recall what game theorists say about a MSNE:

"The point of randomizing is to keep the other player(s) just indifferent between the strategies that the other player is randomizing among. One randomizes to keep one's rivals guessing and not because of any direct benefit to oneself." (Kreps 1990, p 408).

We shall see below that the doubt-based model seems to capture that state of players' mutual guessing that characterizes a MSNE. Assume that we are dealing with 2×2 constant sum games having a unique mixed equilibrium with full support. Consider Player I; how would this player interpret different values of (his own probability) x, say 0.2 and 0.6? A rational Player I knows that Player II is randomizing to keep him indifferent between the strategies he is randomizing among. Therefore, in terms of our model of doubts, $x = 0.2$ and $x = 0.6$ would induce in the Player I's rational mind the same level of doubts as to which is the best probability distribution, because both of them get the same expected payoff. But, for the same reason, Player I's equilibrium strategy in the game will induce the same level of doubts as 0.2 or 0.6. In other words, Player I does not see, in a preference sense, any real difference between different probability distributions in the open unit interval (0,1). As a consequence, he must have (nearly) equal level of doubts at any x in (0,1). The same will happen to Player II.

The above suggests that we should ask first, *which are the level of doubts embedded in the players' mutual guessing that characterizes steady states very close to the MSNE.* This is answered in Proposition 2 below, where we show that, if all agents are playing in the *doubt-full mode,* any interior MSNE coincides with an MSDE ; that is, an MSNE is a Mixed Strategy Doubt-Full Equilibrium (MSDFE).

The second issue to deal with is the following: how is the MSNE reached? or,which is the equilibrating process that may lead to the MSNE? This will be answered in Propositions 4 and 5 below.

Let G be a two-population, two-strategy, constant-sum game with $I = [(x^*, 1 - x^*), (y^*, 1 - y^*)]$,

$x^* > 0$, $y^* > 0$, denoting its MSNE.

Proposition 2

1. The (Euclidean) distance between an MSDE and MSNE converges to zero as δ goes to zero if every agent plays with a doubt function in the $D^{1-\delta}$ class; that is, for *Cartesian skeptical* agents. Hence, an MSNE $(x^*, y^*) \in (0,1) \times (0,1)$ is an MSDFE.

2. For any interior point of the simplex $A = [(x', 1-x'), (y', 1-y')]$ (i.e. with $0 < x' < 1$ and $0 < y' < 1$) there is a sequence of functions $d^\delta \in D^\delta$ such that the (Euclidean) distance between an MSDE and A converges to zero as δ goes to zero. That is, if every agent plays in a *doubt-less mode*, any interior point of the simplex can be a MSDLE for some kind of doubt-less behavior.

Proof: See appendix C

This means that if $I = [(x^*, 1-x^*), (y^*, 1-y^*)]$ is the MSNE of G, then it is compatible (in the sense of Proposition 1) with agents playing in any of the two modes of play, *doubt-full* or *doubt-less*.

3.2 Learning to Play a Mixed Strategy Nash Equilibrium (MSNE)

We have seen that an MSNE and an MSDE satisfy different equilibrium properties and therefore, in general, they do not coincide. However, from Proposition 2, we know that an MSNE could be converted into an MSDE when all agents are *Cartesian skeptical* or play in the *doubt-full mode*. In other words, an MSNE could be converted into a rest point of the doubt-based dynamic system (2)-(3). Hence, now we are ready to answer the question: how do the boundedly rational player populations learn to coordinate in the MSNE? Proposition 4, below, shows that an introspective element, such as doubts, could be crucial for learning to play optimally.

We know that a fully rational player must avoid being guessed by the opponents and that to achieve this he will behave in such a way so as to create a random sequence of choices. This suggests that a *doubt-less* mode of playing -that implies almost no strategy switching behavior- would be far from being an adjusting process leading to the Nash equilibrium. It seems that, in an equilibrating process, what makes more sense is that players be very skeptical; that is , that they should behave in the *doubt-full* mode. In our deterministic dynamic model, the *Cartesian skeptical* agents will have a tendency to keep trying new strategies and, thus, generating not a truly random sequences of choices, but individual processes of trial-and-error adjustments which could find their way to the MSNE. In Proposition 3 below we show that this is the case: if every agent behaves as if he were constantly with hyperbolic doubts, the agents' adjusting behavior would lead them to the MSNE and endow the equilibrium with a strong stability property. Proposition 4 shows that the *doubt-less* mode of play has just the opposite consequence.

Proposition 3 *Let G be a two-population, two-strategy, constant-sum game with* $I^* \equiv [(x^*, 1-x^*), (y^*, 1-y^*)]$, $x^* > 0$ *and* $y^* > 0$, *denoting its MSNE. Then a point close to* I^* *is asymptotically*

stable for the doubt-based dynamic system (2)-(3) if every agent plays in the doubt-full mode of play (that is, they are all Cartesian skeptical).

Proof: See appendix C

Proposition 4 *Let G be a game as in Proposition 3. For any interior point of the simplex* $A = [(x', 1-x'), (y', 1-y')]$ *(i.e. with* $0 < x' < 1$ *and* $0 < y' < 1$*). If every agent is in the doubt-less mode of play and if the initial conditions of the doubt-based dynamic system (2)-(3) are different from A, there is a sequence of functions* $d^\delta \in D^\delta$ *such that the system diverges to a corner of the simplex. That is, if every agent plays in a doubt-less mode, any interior point of the simplex can be a source for some kind of doubt-less behavior.*

Proof: See appendix C

One may then ask about how to explain the modes of play of Proposition 3 and 4 would arise. Needless to say, doubts are a subjective feeling and hence it is difficult to ascertain the precise reason why they may arise in each particular case. Proposition 3 suggests that the origin of high level of doubts (i.e of being skeptical) lies in the fact that every agent seems to be aware that the proportion with which each available strategy is being played and the sequence that the agents, as a player population, are producing is not random. *Cartesian skeptical* agents have developed *a priori* a theory that make them to be aware and adapted to face this setting. Thus, the *hyperbolic* and *universal* doubts felt by every member of each player population would be what the context demands. If not by a theory, smart agents would develop high doubts from the fear of being guessed and exploited by the opponent. As a consequence, since agents are very unhappy with their current strategies a high proportion of agents will experiment with new strategies in the next period. The fear and the doubts of the agents will continue to be high and, joint with the choices that exploit the variations both in the payoffs and in the strategy proportions, the adjusting behavior would lead the system to the Mixed Strategy Nash Equilibrium. Once in the equilibrium, payoffs are equalized across strategies and the doubt levels continue to be very high and equal across strategies too. Thus, the *doubt-full mode* of play advised by the Cartesian theory of doubts endow the MSNE with strong stability properties.

An interpretation of Proposition 4 is that the extreme sensitivity to the "opinions" of others, leads play to a situation where players imitate, whenever doubtful, the current most fashionable action. This creates a tendency to diverge in population behavior. In addition, the doubt-less agents are quite satisfied with their current strategies and do not feel the need to experiment with new strategies to exploit the differences in payoffs and strategy proportions. Hence, a low level of imitation and strategy adjustment takes place, and the populations diverges very slowly to a situation where initially popular strategies dominate.

4 Example

In the numerical example we shall assume, without loss of generality, that the doubt function takes the following form: $d_{ki}(f_{ki}) = (1 - f_{ki})^{\alpha}$. Assuming that $\alpha \in (0, \infty)$, we would obtain a large enough subclass of doubt functions in the set D. Note, in particular, that this class contains the two extreme types of doubt functions mentioned above: when α is very small, near zero, the doubt parameter characterizing agent ki, denoted as $H = \frac{1}{\alpha}$, is very high for any $f_{ki} \in (0, 1)$. Then the function will have a graph looking like the one of figure 2, and we shall say that the agent is *Cartesian skeptical* or is in the *doubt-full mode of play*. When α is very high, the graph of d_{ki} is close to the axes, as in figure 1, and so the doubt parameter, $H = \frac{1}{\alpha}$, is very small, for any $f_{ki} \in (0, 1)$. This is the agent in the *doubt-less mode of play* .

As in Binmore et al. (1995), we approach equation

$$\overset{\bullet}{f}_{ki} = f_{ki}\left[\frac{\overline{\lambda}_k - \lambda_{ki}}{\lambda_k}\right] \tag{4}$$

by means of the equation

$$f_{ki}(t+\tau) - f_{ki}(t) = \tau f_{ki}\left[\frac{\overline{\lambda}_k(t) - \lambda_{ki}(t)}{\lambda_k(t)}\right] \tag{5}$$

where the step size $\tau = 0.01$. We shall consider, like Binmore et al.(1995), that the system has converged on a point when the first 15 decimals are unchanging.

The Penalty Kick Game

Palacios-Huerta (2003) found that the equilibrium theory predictions are observed in the professional players' behavior: (i) their choices follow a random process and (ii) that the probability that a goal will be scored must be the same across each player's strategies and equal to the equilibrium scoring probability (that is, in the Mixed Strategy Nash Equilibrium each player is indifferent among the available strategies). Palacios-Huerta and Volij (2007) extend this result by observing that professional players are capable of transferring their skills from the field to the laboratory, a completely unknown setting for them, and yet behave in a way that is significantly near the Nash equilibrium.

Palacios-Huerta and Volij (2007), from a sample of 2,717 penalty kicks collected from European first division football (soccer) leagues during the period 1995-2004, built the following two player (Player I: the kicker and Player II: goal keeper) two strategy (Left, Right) game.

	(y) L	R
(x)L	0.60, 0.40	0.95 , 0.05
R	0.90, 0.10	0.70, 0.30

where $\pi_I(i,j)$ denotes the kicker's probability of scoring when he chooses i and the goalkeeper chooses j, for $i, j \in \{L, R\}$ .The Mixed Strategy Nash Equilibrium of this game is: $x^* = 0.363\,64, y^* = 0.454\,55$.

Football matches are continuously played and players' game is based on the study of the opponents in the field and watching their play on TV and videotapes, so that their behavior in the penalty kicks is collected and analyzed. Thus, there is a history of play of each player and, hence, an interactive learning process. Thus, a natural issue is to investigate the type of dynamic process that may lead to the result found by Palacios-Huerta (2003). The *doubt-based* model seems to be a suitable model for this task.

The *doubt-based* selection dynamic system (2)-(3) corresponding to this game is the following:

$$
\begin{aligned}
\dot{x} &= \frac{x(1-x)((0.95-0.35y)x^{\alpha}-(0.2y+0.7)(1-x)^{\alpha})}{2(0.95-0.35y)(0.2y+0.7)-(0.95-0.35y)x^{\alpha}-(0.2y+0.7)(1-x)^{\alpha}} \\
\dot{y} &= \frac{y(1-y)((0.1+0.3x)y^{\alpha}-(0.3-0.25x)(1-y)^{\alpha})}{2(0.1+0.3x)(0.3-0.25x)-(0.1+0.3x)y^{\alpha}-(0.3-0.25x)(1-y)^{\alpha}}
\end{aligned}
$$

The vector field defining (2)-(3) is

$$
\begin{aligned}
F(x,y) &= \Big(\frac{x(1-x)((0.95-0.35y)x^{\alpha}-(0.2y+0.7)(1-x)^{\alpha})}{2(0.95-0.35y)(0.2y+0.7)-(0.95-0.35y)x^{\alpha}-(0.2y+0.7)(1-x)^{\alpha}}, \\
&\quad \frac{y(1-y)((0.1+0.3x)y^{\alpha}-(0.3-0.25x)(1-y)^{\alpha})}{2(0.1+0.3x)(0.3-0.25x)-(0.1+0.3x)y^{\alpha}-(0.3-0.25x)(1-y)^{\alpha}}\Big)
\end{aligned}
$$

We compute first the derivative $DF(x,y)$ and then evaluate $DF(x,y)$ at $(0.363\,64, 0.454\,55)$ to get the following Jacobian matrix:

$$
DF(0.363\,64, 0.454\,55) = \begin{bmatrix} \frac{\alpha}{1.\,581\,8-2\times 0.363\,64^{\alpha}} & \frac{0.146\,29(-0.2\times 0.636\,36^{\alpha}-0.35\times 0.363\,64^{\alpha})}{0.790\,91-0.363\,64^{\alpha}} \\ 0.592\,88\frac{0.25\times 0.545\,45^{\alpha}+0.3\times 0.454\,55^{\alpha}}{0.209\,09-0.454\,55^{\alpha}} & \frac{\alpha}{0.418\,18-2\times 0.454\,55^{\alpha}} \end{bmatrix}
$$

It is easy to see that for values of $\alpha \in (0, 0.231\,88)$, all the eigenvalues of $DF(0.363\,64, 0.454\,55)$ have negative real parts and the associated determinants are all positive. Thus, the equilibrium $(0.363\,64, 0.454\,55)$ is a *spiral sink,* for those values of α, and, therefore, it is asymptotically stable.

5 The Empirical Relevancy of the Mixed Strategy Doubt Equilibrium (MSDE)

A Mixed Strategy Doubt Equilibrium (MSDE), the requirement is that for all i, j with $f^*_{ki} > 0$ and $f^*_{kj} > 0$,

$$
\frac{\pi_{ki}\left(f^*\right)}{d\left(f^*_{ki}\right)} = \frac{\pi_{kj}\left(f^*\right)}{d\left(\ f^*_{kj}\right)}
$$

So that when $f_{ki}^* > f_{kj}^*$ we should have $\pi_{ki}(f^*) < \pi_{kj}(f^*)$.

We have seen (in Proposition 3) that this startling result gets smoothed when the agents play like *Cartesian skeptical*; then the differences in popularity have almost no implications on expected payoffs and the MSDFE can be said to coincide with the MSNE. But this doubtfull mode is not the behaviour we generally observe in real choice situations because learning from others´ decisions is a commonly used procedure and the markets are moved in a great part by publicity shaped behaviour.

Notice that the condition for the MSDE applies as well to a pure decision problem. So a supportive piece of evidence for our equilibrium condition could come from consumer choice situations. In this section we show that the car market in the UK could be an empirical example that illustrates the situation described in our theoretical finding, as it is hinted in the following news:

"*Warranty Direct, an independent supplier of insurance-based car warranties, has analyzed claims made over a 12-month period covering more than 450,000 vehicles (between three and nine years of age) driven in both the UK and the US* [....][The study shows that the] *reliability of the UK's five top-selling makes does not necessarily reflect their status amongst the car-buying populace. The nation's number one choice, Ford, was 14th, followed by Vauxhall, 19th, Volkswagen, 23rd, Renault, 29th. Peugeot was the best of the bunch in 13th place."* (Miles Brignall, *The Guardian*, February 3, 2007: *http://www.guardian.co.uk/theguardian*)

Based on data about the claims made on Warranty Direct policies, the insurance company has built the UK car Reliability Index (RI). The index covers not only the overall reliability of a car, but also the reliability of its parts (air conditioning, axle and suspension, braking system, transmission, cooling and heating system, electrical system, ..). The index shows how much, on average, you can expect to pay for repairs, how long each car typically spends off the road, and the frequency of failures. Cars must have an average age of not under 4.50 years and average mileage of no less than 30,000. Thus, we are dealing with real cars and the data are updated daily. This valuable information is free (see *www.reliabilityindex.co.uk*), and can be used to decide which car to buy. The index RI_X (for car make X or for a particular model of X) is calculated as the cost of repair and the frequency of failure against the number of policies held. Separately to this figure, we may also look at the average cost of repairs for a particular make or model; a car with a good reliability index and a high average cost would imply that the frequency of failure is low, however when it does fail the bill will be a lot more than the average.

The choice of a car make has some consequences in terms of expected costs and rate of failure, which are the data measured by the Reliability Index. Thus, we may take the index of a make as a proxi of the payoffs that the owner of a car of that make could expect. Let π_X denote the expected payoffs from the make X. The share of the market would be then be the popularity of the car manufacturer, f_X, as reflected in the Tables 1, 2 below. Our model suggests that reliability (as a proxi for payoffs) is lower for brands with higher sales/market share. In our words, when $f_X > f_Y$ we should observe $R_X > R_Y$ and hence $\pi_X < \pi_Y$.

News about car reliability, such as the one above, often refer to the gap between Japanese car makers as compared to their European (or American) counterparts. That the Japanese manufacturers are at the top of reliability ranking is not new in the UK. In a list of the Top 100 most reliable used cars over the period 1995-2005, compiled by Warranty Direct, every car in the Top 10 and 16 out of the Top 20, were Japanese models.

Tables 1 and 2 gather data from two groups: the most reliable makes for the year 2010 and the top-selling makes in the UK market during 2009. The brands in the first group are Suzuki, Honda, Mazda, Toyota, Skoda, Smart, Citroen and Hyunday (all together they have a market share of 25.39 %) and those in the second are Ford, Vauxhall, Volkswagen and Peugeot (with a total market share of 41%). Notice these are popular makes, with non significant differences in prices, which year after year are positioned in those groups (with the exception of the newcomers Skoda, Smart, Citroen and Hyunday in the first group). They produce models that compete mostly in the popular segments, Mini, Supermini, Lower Medium (or Small Family Cars) and Upper Medium (or Family Cars).[7] The registrations of new cars from these popular segments add up to an average of more than 80% of the market share during the 1999-2009 period (see the document Motor Industry Facts 2010 in www.smmt.co.uk). A segment is composed of models with power and size in some given range; say, the Supermini segment consists of cars normally between 1.0-1.4 CC and a length not exceeding 3745 mm . There are no significant differences in the range of prices for the cars in a segment, so that consumers could be said to be indifferent in terms of prices. The same can be said at the level of makes.

In most of the cases, a binary comparison, one brand from each of the two groups,

[7] Executive and Luxury cars are the segment were makes like Mercedes, BMW and Audi are the best sellers.

would satisfy the condition that a higher market share is accompanied with less expected payoffs for the consumer. See figure 1. For example Mazda's figures are (2.40 %, 53.50 RI) and Volkswagen's (8.08%, 97.47 RI); thus, Mazda has a smaller share of the market, which means that it is less popular than Volskwagen, but has a smaller rate of failure, meaning higher expected payoffs to the owner of a Mazda car. Had we used the data of the same makes in a segment, say, the Supermini, we would obtain, essentially, the same result.

Using the data of Tables 1 and 2, we can say that the market share variable is statistically relevant for the model: $R^2 = 25.32\%$ and, at 0.10 significance level, $[t = 1.93, P - value = 0.08]$; $RI = 53.1297 + 1.855408MS$, where RI stands for reliability index and MS for market share. This is, of course, far from an empirical proof of our result since it cannot be extended to the whole car industry. We are dealing with the extreme cases of popularity and reliability, -the much commented issue of the gap between Japanese car reliability and the European car makes-, where the condition of a mixed strategy doubt equilibrium is satisfied. We predict that a similar example can be obtained with data from the USA car makes; see the *Consumer Reports 2008 and 2009 Annual Car Reliability Survey* in www.consumerreports.org/cro/cars/.

We think that a rational decision-maker should follow the data contained in the car reliability index. But we can see that the market share ranking and the reliability ranking diverge. The data show that reliability is not the main reason that consumers take into account when deciding which make to buy. Our theoretical findings would say that, very likely, the existence of herding doubts (nurtured by advertising, aesthetic tastes, nationalistic views, ..) must have induced a cascade of bad choices ending up in a situation where, at similar prices, popularity is accompanied with lower expected payoffs, as shown by the above data from the UK car market. We suspect that this situation is far from being stable. Empirically, this would mean that the owner of a non top-reliable car, when time comes to buy another car, will not repeat the brand and, instead of measuring the goodness of a choice by its popularity level, will become more skeptical and make a better informed decision. Bad cars will exit the market and, in the medium run, we predict that the reliability and the market share rankings will converge as a consequence of both better choices from the buyers and the efforts from the car manufacturers to increase the reliability of their products.

Table 1. UK Top-10 Reliability Index

Car Make	Market Share (2009)	Reliability Index (2010)
Suzuki	1.43	31.13
Honda	3.75	33.17
Mazda	2.40	53.50
Toyota	5.14	56.22
Skoda	1.87	58.87
Smart	0.42	66.40
Citroen	3.63	66.61
Huynday	2.84	66.97
Nissan	3.91	70.02
Ford	15.86	70.71

Table 2: UK Top-Selling Makes

Car Make	Market Share (2009)	Reliability Index (2010)
Ford	15.86	70.71
Peugeot	5.14	74.02
Vauxhall	11.92	80.77
Volkswagen	8.08	84.75

Source: *The Society of Motor Manufacturer and Traders (www.smmt.co.uk) and www.reliabilityindex.co.uk.*

6 Constant doubt-based selection dynamics

The individual choice model that we are going to use in this section is derived from a choice procedure introduced by Aizpurúa, Ichiishi, Nieto and Uriarte (1993) in the space of simple lotteries. We consider now the case when the level of doubts felt is constant, for any value of $f_{ki} \in F_{ki}$. This means that society has no influence upon the doubt level of the agents. Formally,

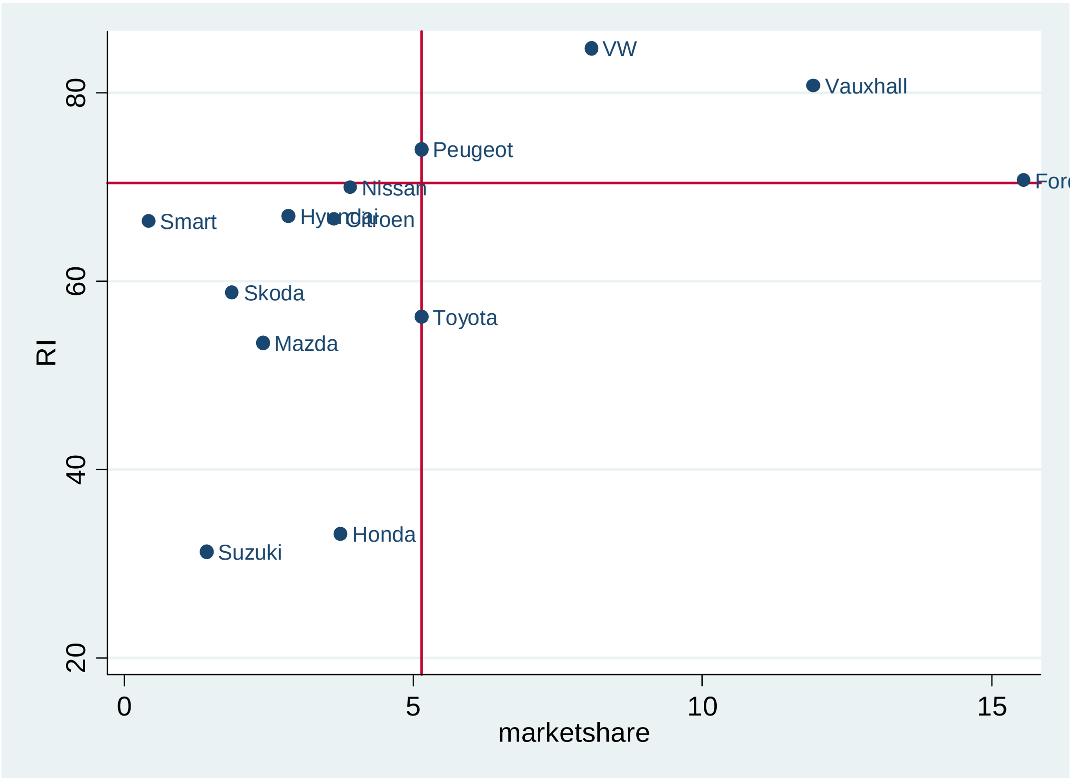

Figure 1: Figure 1. The Top-selling versus the Top-reliable makes in the UK car market.

Assumption 2 *The Constant Doubt Function. For all $k \in K$, $i \in S_k$ and $f_{ki} \in F_{ki}$, the function $d_{ki} : F_{ki} \to [0,1]$ is constant; i.e.*

$$d_{ki}(f_{ki}) = \epsilon_k \in (0,1)$$

We assume that the constant level of doubts ϵ_k felt by agent ki induces *threshold levels* in both expected payoffs and strategy frequencies and that these threshold levels are described by means of similarity relations.

As in the previous case, it is by means of Assumption 2 about the doubt function that we may define a similarity relation on $\Pi_{ki} = (0,1]$ and correlated similarity relations on $F_{ki} = [0,1]$. Suppose that (π_{ki}, f_{ki}) is the vector of expected payoff-strategy proportion attached to strategy i at time t.

The similarity relation on Π_{ki}, denoted $S\Pi_{ki}$, is assumed to be of the difference type and it is defined as follows

$$\pi_{ki} S\Pi_{ki} \pi'_{ki} \Leftrightarrow |\pi_{ki} - \pi'_{ki}| \leq \epsilon_k$$

On F_{ki}, we define now the correlated similarity relations as follows. First, for all $\pi_{ki} > \varepsilon_k > 0$ we build the function $\phi_{ki} : \Pi_{ki} \to (1, \infty]$ as follows,

$$\phi_{ki}(\pi_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - \epsilon_k} > 1$$

Then, we can establish the following similarity relation (of the ratio-type) between f_{ki} and other frequencies in F_{ki}, such as f'_{ki}, given π_{ki}.

$$f_{ki} SF_{ki}(\pi_{ki}) f'_{ki} \Leftrightarrow \frac{1}{\phi_{ki}(\pi_{ki})} \leqq \frac{f_{ki}}{f'_{ki}} \leqq \phi_{ki}(\pi_{ki})$$

We call $SF_{ki}(\pi_{ki})$ a **correlated** similarity relation because the similarity on F_{ki} depends on the level of expected payoff π_{ki} at period t. For values of $\pi_{ki} \leqq \epsilon_k$ the function ϕ_{ki} is not defined and we assume that in that case that $SF_{ki}(\pi_{ki})$ is the degenerate similarity relation (see Rubinstein (1988)).

Remark 1 *The threshold level in the frequency space is inversely related to expected payoffs: $\frac{\partial \phi_{ki}(\pi_{ki})}{\partial \pi_{ki}} < 0$. This means that as the expected payoffs at stake increases, the discrimination on the frequency space F_{ki} increases.*

Assumption 3 *Every agent in a given player position is able to observe the relative frequency of every strategy available to that position. When an agent feels dissatisfied with his current strategy, he will choose a new strategy with a probability that is equal to the proportion of agents playing that strategy.*

We proceed as in the previous case (for simplicity we shall write ϕ_{ki} instead of $\phi_{ki}(\pi_{ki})$). Let the ratio

$$\frac{\phi_{ki}}{\Sigma_{i=1}^{m_k}\phi_{ki}} = \frac{\phi_{ki}}{\phi_k}$$

denote the proportion of ki strategists who feel dissatisfied with strategy i. Note that, everything equal, this function increases with ϕ_{ki}. Hence, an increase in ϕ_{ki}, due to a decrease in the expected payoffs π_{ki}, will increase the proportion of dissatisfied ki strategists.

As before, $\tau\frac{(\phi_{ki}-1)}{\phi_k}f_{ki}$ denotes the proportion of ki strategists who will choose a new strategy at time t (the *outflow*). Since a particular strategy is chosen with a probability that is equal to the proportion of agents playing that strategy, then $\tau\sum_{j=1}^{m_k}\frac{\phi_{kj}}{\phi_k}f_{kj}f_{ki} = \tau\frac{\overline{\phi}_k}{\phi_k}f_{ki}$ denotes the proportion of agents who choose strategy i ; i.e. the *inflow* (where $\overline{\phi}_k = \sum_{j=1}^{m_k}\phi_{kj}f_{kj}$ is the average perception in player population k at time t).

Therefore

$$f_{ki}(t+\tau) = f_{ki}(t) - \tau\frac{\phi_{ki}}{\phi_k}f_{ki} + \tau\frac{\overline{\phi}_k}{\phi_k}f_{ki}. \qquad (6)$$

Proposition 5 *As* $\tau \to 0$, *equation (6) becomes*

$$\overset{\bullet}{f}_{ki} = f_{ki}\left[\frac{\overline{\phi}_k - \phi_{ki}}{\phi_k}\right] \qquad (7)$$

1. If for all player position $k \in K = \{1, 2, ..., n\}$, the strategy set S_k consists of two elements, i.e. if $m_k = 2$ then, equation (7) is just the standard Replicator Dynamics (RD) multiplied by a positive function (i.e. is aggregate monotonic).

2. If $m_k > 2$, then we obtain a selection dynamics that approximates the RD, but preserves only the positive sign of the RD (i.e. is weakly payoff positive).

Proof: See appendix C

7 Concluding remarks

In 2×2 games with Mixed Strategy Nash Equilibria, the introduction of agents with doubts coupled with (and mutually reinforced by) imperfect discrimination capacity, permits a departure from the long-run behavior of traditional selection dynamic systems. For instance, if we assume that the feeling of doubts is sensitive to the popularity of a pure strategy, then we obtain doubt-based selection dynamics that are not payoff monotonic. The main feature of the doubt-based system is that its equilibrium does not require expected payoffs to be equalized across strategies. Nevertheless, the curvature of the decreasing doubt functions has strong implications on the long run behavior of the system. If agents do not beleive in the wisdom of crowd, are very *skeptical* and thus play in the *doubt-full mode*,- i.e. agents are endowed with an extremely concave doubt function-, a Mixed Strategy Nash Equilibrium is a Mixed Strategy Doubt-Full Equilibrium and it is shown to be asymptotically stable. But stability is lost when agents have *herding doubts;* that is, doubts that are influenced by the relative popularity of each of the pure strategies available to his player role. We have shown this result when agents are in the *doubt-less mode*, i.e. when they are endowed with an extremely convex doubt function. Herding doubts lead to the *Mixed Strategy Doubt Equilibrium* in which the most popular strategies receive lower expected payoffs. We present some preliminary data whose qualitative features are along the lines of this theoretical result.

The present behaviorally-based theoretical work could be applicable to experimental research. Our view is that experiments should be designed to capture the decision schemes that are actually used by subjects. Two features of bounded rationality are, in our view, embedded in those schemes: doubts and imperfect perception. It is known that similarity judgments are part of observed decision procedures (see Tversky (1977), Rubinstein (1988) and Arieli et al. (2009)). We think that feedback on the popularity of different strategies would be important to consider, as well as less sharply defined payoffs. However, subjects in experiments usually do not have information about the proportion of people using each strategy. For example, the only experiment from those surveyed in chapter 3 of Camerer (2003) in which agents are given that information is the one carried out by Tang (2001). We suspect, though, that the highly precise (and, we would argue, unnatural) form of the feedback given to subjects eliminates the "doubt" considerations that are important in the build-up of our model. It would be unrealistic to assume that the agents get the correct numbers. We believe that more evidence, and hopefully, from "fuzzier" (more realistic) environments would be useful to confront some predictions made in this work. Hence, a translation of our theoretical model into an experimental design should be our next task.

References

[1] Aizpurua, J.M., Ichiishi, T., Nieto, J. and Uriarte, J.R. (1993) "Similarity and Preferences in the Space of Simple Lotteries", *Journal of Risk and Uncertainty* 6, 289-297.

[2] Alchian, A. (1950) "Uncertainty, Evolution, and Economic Theory", Journal of Political Economy, Vol 58, No. 3, 211-221.

[3] Arieli, A., Ben-Ami, B. and Rubinstein, A. (2009) "Fairness motivations and procedures of choice between lotteries as revealed through eye movements" School of Economics, Tel Aviv University.

[4] Binmore. K., Gale, J., and Samuelson, L. (1995) "Learning to be Imperfect: the Ultimatum Game," *Games and Economic Behavior* 8, 56-90.

[5] Binmore, K. and Samuelson, L. (1999), "Evolutionary Drift and Equilibrium Selection," *Review of Economic Studies* 66. 363-393.

[6] Binmore, K. Swierzbinski, J. and Proulx, C. (2001) Does Minimax Work? An Experimental Study", *The Economic Journal* 111: 445-464.

[7] Cabrales, A. (2000) "Stochastic Replicator Dynamics,"*International Economic Review* 41: 451-481.

[8] Camerer, C. (2003) *Behavioral Game Theory*, Princeton (NJ): Princeton University Press.

[9] Delios, A. and Henisz, W.J. (2003) "Political hazards, experience, and sequential entry strategies: the international expansion of Japanese firms 1980-1998", International Management Journal 24: 1153-1164.

[10] Descartes, R. (2008), "Meditations on First Philosophy: with selections from the Objections and Replies", Oxford: O.U.P.

http://plato.stanford.edu/entries/descartes-epistemology/

[11] Gigerenzer, G., Todd (1998) "Fast and Frugal Heuristic: How Ignorance Makes Us Smart" in Gigerenzer, G., Todd, P. M. and the ABC Research Group (1999) "Simple Heuristic that Make us Smart" New York: Oxford University Press.

[12] Hume, D. (2007) "An Enquiry Concerning Human Understanding" Peter Millican (ed). Oxford: Oxford University Press

[13] Kahneman, D., and Tversky, A. (1979), "Prospect Theory: An Analysis of Decisions under Risk," *Econometrica* 47, 313-327.

[14] Kahneman, D. (2003), "A Perspective on Judgment and Choice: Mapping Bounded Rationality," *American Psychologist* 58, 697-720.

[15] Kreps, D. M. *A Course in Microeconomic Theory*, Princeton, N.J.: Princeton University Press.

[16] Levit, B. and March, J.G. (1988) "Organizational Learning" Annual Review of Sociology 14: 319-340.

[17] Nelson, Richard R.; Winter, Sidney G. (1982), *An evolutionary theory of economy change*, Cambridge. The Belknap Press

[18] Palacios-Huerta, I. (2003), "Professionals Play Minimax," *Review of Economic Studies* 70, 395-415.

[19] Palacios-Huerta, I. and Volij, O. (2007), "Experientia Docet: Professionals Play Minimax in Laboratory Experiments," mimeo, Brown University.

[20] Ramsey, F.P.(1926) "Truth and Probability" in Ramsey , 1931 *The Foundation of Mathematics and other Logical Essays*, Ch. VII p.156-198, edited by R.B. Braithwaite, London: Kegan, Paul, Trench & Co., New York: Harcourt, Brace and Company.

[21] Rubinstein, A.(1988) "Similarity and Decision-Making under Risk(Is there a Utility Theory Resolution to the Allais Paradox?), *Journal of Economic Theory* 46, 145-153.

[22] Schlag, K. (1998) "Why Imitate, and if so, How? A Boundedly Rational Approach to Multi-Armed Bandits", *Journal of Economic Theory* 78, 130–156.

[23] Smallwood, D., and J. Conlisk (1979), "Product Quality in Markets Where Consumers are Imperfectly Informed," *Quarterly Journal of Economics* 93, 1–23.

[24] Tang, F. F. (2001) "Anticipatory Learning in Two-Person Games: Some Experimental Results" *Journal of Economic Behavior and Organization* 44, 221-232.

[25] Tversky, A. (1977) "Features of Similarity ", *Psychological Review* 84, 327-352.

[26] Tversky, A. and Kahneman, D. (1992), "Advances in Prospect Theory: Cumulative Representation of Uncertainty", *Journal of Risk and Uncertainty 5, 297-323.*

[27] Tversky, A. and Kohler, D.J. (1994) "Support Theory: A Nonextensional Representation of Subjective Probability", Psychological Review 101, 547-567.

[28] Uriarte, J.R. (1999) "Decision-making under Risk: Editing Procedures based on Correlated Similarities, and Preference Overdetermination", *Review of Economic Design* 4, 1-12.

[29] Uriarte, J.R. (2007) "A Behavioral Foundation for Models of Evolutionary Drift" *Journal of Economic Behavior and Organization* 63, 497-513.

[30] Weibull. J.W.(1995) *Evolutionary Game Theory*, Cambridge, MA: MIT Press.

Appendices

A Satisficing Procedural Preferences based on Similarity Judgements

In the present model, doubts are closely related to imperfect discrimination capacity (of real numbers, such as strategy frequencies and expected payoffs). An environment shaped by uncertainty and doubts about the correctness of the choices made is effort demanding for the cognitive system of decision-makers. One way subjects cope with the ambiguous nature of this situation is by simplifying its complexity; for instance, by grouping numbers in intervals of similarity. Inside those intervals, whose size depend on threshold levels that change, values - of, say, expected payoffs and strategy frequencies -, are not distinguished. We, thus, model subjects' limited capacities and imperfect discrimination by means of similarity relations[8]. Then they would build a procedural preference relation compatible with those similarity relations, as in Rubinstein (1988).

To be more specific, let $(\pi_{ki},\ f_{ki})$ be the vector of expected payoff-proportion of agents of player population k attached to strategy $i \in S_k$ at time t with $f_{ki} \in (0,1)$.

Correlated Similaritites on Π_{ki} and F_{ki}

The doubt function serves to build correlated similarity relations on both Π_{ki} **and** F_{ki}. Let (π_{ki}, f_{ki}) and $(\overline{\pi}_{ki}, \overline{f}_{ki})$ be two vectors in $\Pi_{ki} \times F_{ki}$, with $\overline{f}_{ki}$, $f_{ki} \in (0,1)$.

(A) On the space of expected payoffs, Π_{ki}:

The doubt function d_{ki} defines correlated similarities of the difference-type as follows: given f_{ki} we say that $\overline{\pi}_{ki}$ is similar to π_{ki}, (formally written as $\overline{\pi}_{ki} S\Pi[f_{ki}]\pi_{ki}$), if and only if $|\overline{\pi}_{ki} - \pi_{ki}| \leqq d_{ki}(f_{ki})$, where $|.|$ stands for absolute value. Thus, there is one similarity relation on Π_{ki}, for each $\overline{f}_{ki} \in (0,1)$

Then the similarity interval of π_{ki}, given f_{ki} is:

$$[\pi_{ki} - d_{ki}(f_{ki}), \pi_{ki} + d_{ki}(f_{ki})]$$

Note that $d_{ki}(f_{ki})$, the doubt level felt by $\sum$ agent ki given the proportion f_{ki}, becomes the threshold level in the definition of this type of similarity relation. If f_{ki} increases, the threshold, $d_{ki}(f_{ki})$, decreases and so the similarity intervals of π_{ki} shrink (giving rise to the vertical cone-shaped form in figure 2).This means that when f_{ki} increases, the discrimination capacity on the space of expected payoffs to strategy i increases (probably because the accumulated experience with strategy i has increased due to the increased number of agents from population k currently playing strategy i). When f_{ki} is such that $\pi_{ki} - d_{ki}(f_{ki}) \leq m$ and $\pi_{ki} + d_{ki}(f_{ki}) \geq M$, the whole set $\Pi_{ki} = [m, N]$ is similar to π_{ki} and when $f_{ki} = 1$ only π_{ki} is similar to itself. This variations in perception induces a vertical wedge type form, as it can be seen in figure 2.

Notice that for a *Cartesian skeptical agent*, the similarity interval is

$$[\pi_{ki} - d_{ki}(f_{ki}), \pi_{ki} + d_{ki}(f_{ki})] = [m, M]$$

[8] They are, in fact, correlated similarities: an extension of the similarity relations defined by Rubinstein (1988). Rather than being constant, correlated similarities depend on the value of some relevant parameter. For more details, see Aizpurua et al. (1993) and Uriarte (1999).

That is, since in this case $d_{ki} \in D_M$, then $d_{ki}(f_{ki}) > M$ for all $f_{ki} \in (0,1)$, thus m is similar to M, and hence the similarity on Π_{ki} is degenerate (see Rubistein 1988).

(B) On the strategy frequency space, F_{ki}:

The doubt function d_{ki} defines correlated similarity relations of the ratio-type by means of the function $\lambda_{ki} : \Pi_{ki} \to R$, which is defined as follows: for a given $f_{ki} \in (0,1)$ and d_{ki},

$$\lambda_{ki}(\pi_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - d_{ki}(f_{ki})}$$

Note that:

1. For the *herding agents*: $d_{ki} \in D_m$ and the function $\lambda_{ki} > 1$ is then used to define on F_{ki} correlated similarity relations of the ratio-type whose similarity interval for that f_{ki} is:

$$[f_{ki}/\lambda_{ki}(.),\ \ f_{ki}.\lambda_{ki}(.)]$$

Thus, for a given level of π_{ki} and a given f_{ki}, $\lambda_{ki}(\pi_{ki})$ determines the threshold of this similarity interval of f_{ki} : Note that if expected payoffs increase the perception effort increases, thus the threshold $\lambda_{ki}(\pi_{ki})$ decreases, inducing a horizontal wedge type form (see figure 2 below).

2. For the *skeptical agents*: $d_{ki} \in D_M$, then $\lambda_{ki} < 0$ will define a *degenerate* similarity relation (see Rubistein 1988). Thus, when doubts are of a skeptical nature, the similarity relations on both Π_{ki} and F_{ki} are *degenerate*. Hence, on F_{ki}, given a $f_{ki} \in (0,1)$, the correlated similarity relation $SF_{ki}[\pi_{ki},\ f_{ki}]$ will induce the following similarity intervals for f_{ki}:

$$[f_{ki}/\lambda_{ki}(.),\ \ f_{ki}/\lambda_{ki}(.)] = [0,1]$$

The size of this degenerate similarity interval does not change with π_{ki}; it remains constant for any value of π_{ki}.

Procedural Preference on $\Pi_{ki} \times F_{ki}$:

Based on a model developed in Uriarte (1999), we show now how the above two correlated similarity relations build a (non-complete and non-transitive) preference-indifference relation defined on the space of expected payoffs and frequencies, $\Pi_{ki} \times F_{ki}$, attached to pure strategy $i \in S_k$. Let us assume that each agent ki compares pairs of alternatives in $\Pi_{ki} \times F_{ki}$ with the aid of a pair of correlated similarity relations to decide which of the two is preferred. The agent may define a procedural preference $\succsim_{ki}$on $\Pi_{ki} \times F_{ki}$ by means of the pair of correlated similarities and know his aspiration set U at each t (which we identify with the upper contour set of the vector (π_{ki}, f_{ki}) at t , $U = U_\alpha \cup U_\beta \cup U_\delta$; see figure 2). That is, given a pair of vectors $(\overline{\pi}_{ki}, \overline{f}_{ki})$ and (π_{ki}, f_{ki}) in $\Pi_{ki} \times F_{ki}$, the vector $(\overline{\pi}_{ki}, \overline{f}_{ki})$ will be declared to be preferred to (π_{ki}, f_{ki}), i.e. $(\overline{\pi}_{ki}, \overline{f}_{ki}) \succ_{ki} (\pi_{ki}, f_{ki})$, whenever the agent ki perceives that one of the following three conditions is met. (Note that since $(\overline{\pi}_{ki}, \overline{f}_{ki})$ is to be preferred, the conditional similarity relation $S\Pi$ on Π_{ki} given $\overline{f}_{ki}$ and the conditional similarity relation SF on F_{ki} given $\overline{\pi}_{ki}$ and $\overline{f}_{ki}$ are to be used):

Condition α : $\overline{\pi}_{ki} > \pi_{ki}$, and *no* $\overline{\pi}_{ki} S\Pi[\overline{f}_{ki}]\pi_{ki}$; while $\overline{f}_{ki} SF[\overline{\pi}_{ki}, \overline{f}_{ki}] f_{ki}$.

In words, $\overline{\pi}_{ki}$ is bigger than π_{ki} and, given $\overline{f}_{ki}$, $\overline{\pi}_{ki}$ is perceived to be *not similar* to π_{ki} ; while , $\overline{f}_{ki}$ is perceived to be *similar* to f_{ki}. U_α in figure 2 is the area implied by this condition.

Condition β : $\overline{f}_{ki} > f_{ki}$ and *no* $\overline{f}_{ki} SF[\overline{\pi}_{ki}, \overline{f}_{ki}] f_{ki}$;while $\overline{\pi}_{ki} S\Pi[\overline{f}_{ki}]\pi_{ki}$.

In words, $\overline{f}_{ki}$ is bigger than f_{ki} and, given $\overline{\pi}_{ki}$ and $\overline{f}_{ki}$, $\overline{f}_{ki}$ is perceived to be *not similar* to f_{ki}; while, given $\overline{f}_{ki}$, $\overline{\pi}_{ki}$ is perceived to be *similar* to π_{ki}.U_β in Figure 2 is the area implied by this condition.

Condition δ : $\overline{\pi}_{ki} > \pi_{ki}$ and *no* $\overline{\pi}_{ki} S\Pi_{ki}[\overline{f}_{ki}]\pi_{ki}$; $\overline{f}_{ki} > f_{ki}$ and *no* $\overline{f}_{ki} SF[\overline{\pi}_{ki}, \overline{f}_{ki}] f_{ki}$.

That is, vector $(\overline{\pi}_{ki}, \overline{f}_{ki})$ is strictly bigger than (π_{ki}, f_{ki}) and no similarity is perceived in both instances. U_δ in figure 2 is the area implied by this condition.

Indifference:

Whenever both expected payoffs and strategy proportions are perceived to be *similar*, then the two vectors will be declared *indifferent*; i.e. when $\overline{\pi}_{ki} S\Pi[\overline{f}_{ki}]\pi_{ki}$, $\pi_{ki} S\Pi[f_{ki}]\overline{\pi}_{ki}$, $\overline{f}_{ki} SF[\overline{\pi}_{ki}, \overline{f}_{ki}] f_{ki}$ and $f_{ki} SF[\pi_{ki}, f_{ki}]\overline{f}_{ki}$, then $(\overline{\pi}_{ki}, \overline{f}_{ki}) \sim_{ki} (\pi_{ki}, f_{ki})$. When none of these four situations takes place, then the two vectors would be non-comparable (see figure 2).

The distance to the aspiration set U depends on how thick the indifference set of (π_{ki}, f_{ki}) is. We assume here that agents are preference-satisficers; that is, they choose a strategy to reduce the distance from (π_{ki}, f_{ki}) to U. The *Herding agent* can achieve this by reducing doubts by means of playing popular strategies and/or increasing expected payoffs. The smaller (greater) that distance the more satisfied (dissatisfied) the ki-agent will be with his current strategy. It can be seen that the properties (i) and (ii) of λ_{ki} (see below, in the next section) capture the changes in the thickness of the indifference sets. Hence, the λ_{ki} function can be thought of as an index of how dissatisfied the ki-agent is with his current strategy. Notice that a *doubt-less agent*'s indifference classes will consists of almost singletons: $\sim [(\pi_{ki}, f_{ki})] \cong (\pi_{ki}, f_{ki})$ conveying the idea that with almost no doubts about the goodness of the current strategy, the *doubt-less agent* feels very satisfied and, very likely, will not switch to a different strategy.

The S*keptical agent* will have indifference sets that will cover the entire choice space because the similarity intervals on both the Π_{ki} and F_{ki} spaces are degenerate. Thus, we will say that the *Skeptical agent*'s preference relation is *degenerate*. Thus, for tis type of agent any pair of expected payoffs will be similar, as well as any pair of strategy frequencies. Hence, in terms of preferences, the agent will not perceive real differences between any two different vectors in $\Pi_{ki} \times F_{ki}$ and he will declare to be indifferent among them. Thus, having the thickest indifference sets that are possible, the upper contour sets (i.e agents's aspiration set) will appear to be unreachable and the *Skeptical* agent will feel highly dissatisfied.

B The Index of Dissatisfied Agents

Given the expected payoffs and the frequencies attached to each of the pure strategies of population k, we propose the index of dissatisfied agents with pure strategy i to be represented by the agent ki's dissatisfaction level relative to the total dissatisfaction level of population k:

$$\alpha_{ki} = \alpha_{ki}(\pi_{ki}, f_{ki}, \pi_{kj}, f_{kj}) = \frac{\lambda_{ki}}{\Sigma_{i=1}^{m_k} \lambda_{ki}} = \frac{\lambda_{ki}}{\lambda_k},\ i \neq j$$

To avoid the use of different doubt parameters, we will only assume that either they are all *herding agents* or *skeptical* ones; no mixed populations of doubtful agents are allowed.

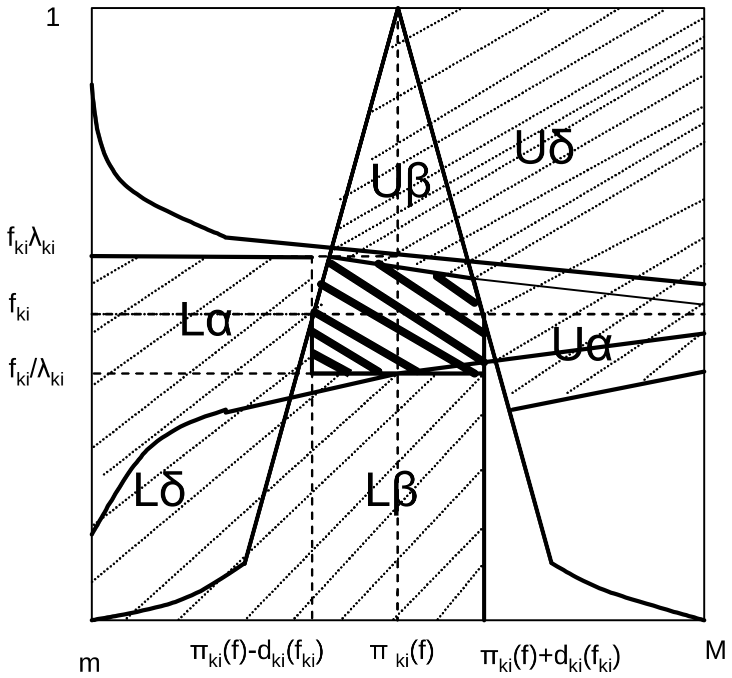

Figure 2: Preference-indifference relation compatible with correlated similarities. Relative to (π_{ki}, f_{ki}), $U = U_\alpha \cup U_\beta \cup U_\delta$ denotes the upper-contour (or aspiration) set, $L = L_\alpha \cup L_\beta \cup L_\delta$ the lower contour set and the darker area is the indifference set.

Further, we assume that the the *herding doubts agents* do perceive the changes in payoffs and frequencies. But we cannot assume the same for the *skeptical agents*. This is so because, as said above, the *skeptical agent* has similarity intervals that are degenerate and, as a consequence, in terms of preferences, he does not distinguish between any two different vectors in $\Pi_{ki} \times F_{ki}$. Therefore, the stimulus intensity received by this agent from any vector (π_{ki}, f_{ki}) would be the same and hence the response probability is the same for each strategy. Furthermore, since a *Cartesian skeptical* agent is endowed with *universal doubts*, he will always be dissatisfied and continuously experimenting with every available strategy, no matter the level of payoffs and popularity attached to each strategy. For this reason, we may say that this type of agents react in a "non-standard" way to the changes in the expected payoffs and strategy frequencies.

Notice that the properties of α_{ki} follow naturally from the properties of λ_{ki} which are as follows:

(i) given f_{ki} and d_{ki}, if π_{ki} increases, $\lambda_{ki}(\pi_{ki})$ decreases and thus, the similarity interval shrinks. This means that when the expected payoffs at stake increase, the discrimination efforts on the frequency space, F_{ki}, increases (generating a kind of horizontal wedge type form, as it is shown in figure 2)

(ii) keeping the function d_{ki}, and π_{ki} constant, if the frequency f_{ki} increases, then $\lambda_{ki}(\pi_{ki})$ decreases and so the similarity intervals of the higher frequency shrink.

Then the properties of α_{ki} for the *herding agent ki* are:

1. The proportion of dissatisfied agents with their current pure strategy $i \in S_k$ will decrease if expected payoffs to strategy $i \in S_k$, π_{ki}, increase.

$$\frac{\partial \alpha_{ki}}{\partial \pi_{ki}} = \frac{\frac{\partial \lambda_{ki}}{\partial \pi_{ki}} \lambda_k - \frac{\partial \lambda_{ki}}{\partial \pi_{ki}} \lambda_{ki}}{\lambda_k^2} = \frac{\frac{-d_{ki}(f_{ki})}{(\pi_{ki} - d_{ki}(f_{ki}))^2} (\lambda_k - \lambda_{ki})}{\lambda_k^2} < 0$$

2. The proportion of dissatisfied agents with their current pure strategy $i \in S_k$ should increase if expected payoffs to strategy $j \in S_k$, π_{kj}, increase.

$$\frac{\partial \alpha_{ki}}{\partial \pi_{kj}} = \frac{-\frac{\partial \lambda_{kj}}{\partial \pi_{kj}} \lambda_{ki}}{\lambda_k^2} = \frac{\frac{-d_{kj}(f_{kj})}{\left(\pi_{kj} - d_{kj}(f_{kj})\right)^2} (-\lambda_{ki})}{\lambda_k^2} > 0$$

3. If agents ki's doubts decrease, because the popularity of strategy i, f_{ki}, has increased,

the proportion of dissatisfied should decrease too.

$$\frac{\partial \alpha_{ki}}{\partial f_{ki}} = \frac{\frac{\partial \lambda_{ki}}{\partial f_{ki}}\lambda_k - \frac{\partial \lambda_{ki}}{\partial f_{ki}}\lambda_{ki}}{\lambda_k^2} = \frac{\frac{\pi_{ki}\frac{\partial d_{ki}(f_{ki})}{\partial f_{ki}}}{(\pi_{ki}-d_{ki}(f_{ki}))^2}\left(\lambda_k - \lambda_{ki}\right)}{\lambda_k^2} < 0$$

4. If the popularity of strategy $j \in S_k$, f_{kj}, increases, the proportion of dissatisfied agents with their current pure strategy $i \in S_k$ should increase.

$$\frac{\partial \alpha_{ki}}{\partial f_{kj}} = \frac{-\frac{\partial \lambda_{kj}}{\partial f_{kj}}\lambda_{ki}}{\lambda_k^2} = \frac{\frac{\pi_{kj}\frac{\partial d_{kj}(f_{kj})}{\partial f_{kj}}}{\left(\pi_{kj}-d_{kj}(f_{kj})\right)^2}\left(-\lambda_{ki}\right)}{\lambda_k^2} > 0$$

C Proofs of propositions

Let

	(y) L	R
(x)U	a_{11}, b_{11}	a_{12} , b_{12}
D	a_{21}, b_{21}	a_{22}, b_{22}

denote the 2×2 constant-sum game G, and $I^* \equiv [(x^*, 1-x^*), (y^*, 1-y^*)]$,with $x^* > 0$ and $y^* > 0$, the Mixed strategy Nash Equilibrium of G. To get this equilibrium, we may assume, without loss of generality, that $a_{11} > a_{21}$, $b_{11} < b_{12}$, $a_{12} < a_{22}$, and $b_{22} < b_{21}$. Recall that payoffs are normalized so that they take values on $[m, M]$. To avoid the use of four different doubt parameters, we shall assume that the four doubt functions are the same: $d_D = d_U = d_R = d_L = d$. The doubt-based selection dynamics (for definition (a) of λ_{ki}) are represented by the following system:

$$\begin{aligned} \dot{x} &= \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)}(\pi_U d_D - \pi_D d_U) \qquad \text{(C.1)} \\ &= \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)}((a_{11}y + a_{12}(1-y))d_D(1-x) - (a_{21}y + a_{22}(1-y))d_U(x)) \\ &\equiv G_1(x,y)F_1(x,y) \end{aligned}$$

$$\begin{aligned}
\dot{y} &= \frac{y(1-y)}{\pi_L(\pi_R - d_R) + \pi_R(\pi_L - d_L)}(\pi_L d_R - \pi_R d_L) \qquad \text{(C.2)} \\
&= \frac{y(1-y)}{\pi_L(\pi_R - d_R) + \pi_R(\pi_L - d_L)}((b_{11}x + b_{21}(1-x))d_R(1-y) - (b_{12}x + b_{22}(1-x))d_L(y)) \\
&\equiv G_2(x,y)F_2(x,y)
\end{aligned}$$

Proof of Proposition 2:

1. We must first show that a Mixed Strategy Nash Equilibrium (MSNE) converges to a Mixed Strategy Doubt-Full Equilibrium (MSDFE) as δ converges to zero in the class of doubt functions $D^{1-\delta} \subset D_M$.

An interior rest point of (C.1)-(C.2),(i.e. a MSDE), satisfies:

$$\begin{aligned}
(a_{11}y + a_{12}(1-y))\, d_D(1-x) - (a_{21}y + a_{22}(1-y))\, d_U(x) &= 0 \\
(b_{11}x + b_{21}(1-x))\, d_R(1-y) - (b_{12}x + b_{22}(1-x))\, d_L(y) &= 0
\end{aligned}$$

Then, if $d_i \in D^{1-\delta}$ for $i \in \{U, D, L, R\}$,

$$\lim_{\delta \to 0} \frac{d_U(x)}{d_D(1-x)} = \lim_{\delta \to 0} \frac{d_L(y)}{d_R(1-y)} = 1, \text{ for all } (x,y) \in (0,1) \times (0,1)$$

Now suppose that we are in the MSNE, $(x^*, y^*) \in (0,1) \times (0,1)$, of G and that $d_i \in D^{1-\delta}$. Then, the strategies available to each player get the same expected payoff; that is $a_{11}y^* + a_{12}(1-y^*) = a_{21}y^* + a_{22}(1-y^*)$ and $b_{11}x^* + b_{21}(1-x^*) = b_{12}x^* + b_{22}(1-x^*)$. Thus,

$$\lim_{\delta \to 0} \frac{(a_{11}y^* + a_{12}(1-y^*))\, d_D(1-x^*)}{(a_{21}y^* + a_{22}(1-y^*))\, d_U(x^*)} = \lim_{\delta \to 0} \frac{(b_{11}x^* + b_{21}(1-x^*))\, d_R(1-y^*)}{(b_{12}x^* + b_{22}(1-x^*))\, d_L(y^*)} = 1$$

This, plus continuity, establishes the result.

2. We show that for all $(x', y') \in (0,1) \times (0,1)$, there exists a sequence of functions $d^\delta \in D^\delta$ and a δ' low enough that the rest point of (C.1)-(C.2) cannot be any $C \neq [(x', 1-x'), (y', 1-y')]$ for any $\delta \leq \delta'$ and then the result follows.

An interior rest point of (C.1)-(C.2) must satisfy:

$$\begin{aligned}
(a_{11}y + a_{12}(1-y))\, d_D(1-x) - (a_{21}y + a_{22}(1-y))\, d_U(x) &= 0 \\
(b_{11}x + b_{21}(1-x))\, d_R(1-y) - (b_{12}x + b_{22}(1-x))\, d_L(y) &= 0
\end{aligned}$$

which implies that

$$\begin{aligned}\left(a_{11}y+a_{12}\left(1-y\right)\right)\frac{d_D\left(1-x\right)}{d_U\left(x\right)}-\left(a_{21}y+a_{22}\left(1-y\right)\right) &= 0\\ \left(b_{11}x+b_{21}\left(1-x\right)\right)\frac{d_R\left(1-y\right)}{d_L\left(y\right)}-\left(b_{12}x+b_{22}\left(1-x\right)\right) &= 0\end{aligned}$$

Let first $x' \leq 1/2$. We construct the doubt functions $d_U(x)$ and $d_D(1-x)$ in D as follows:

$$d_{ki}^{x'}(f_{ki}) = \begin{cases} m\delta\left(1-f_{ki}\right) & \text{if } f_{ki} \leq x' \\ m\delta\left(1-f_{ki}\right)\frac{(1-f_{ki})^{1/\delta}}{(1-x')^{1/\delta}} & \text{if } f_{ki} > x' \end{cases}$$

where $ki \in \{U, D\}$ and $\delta > 0$. Note that as δ approaches 0, the graph of $d_{ki}^{x'}$ function approaches the horizontal axis and the agent is said to be in a *doubt-less* mode.

Now, for $x > x'$

$$\frac{d_D\left(1-x\right)}{d_U\left(x\right)} = \frac{mx}{m\left(1-x\right)\frac{(1-x)^{1/\delta}}{(1-x')^{1/\delta}}} = \frac{x}{1-x}\left(\frac{1-x'}{1-x}\right)^{1/\delta}$$

Since $1-x' > 1-x$ we can make $\left(\frac{1-x'}{1-x}\right)^{1/\delta}$ as big as we want by choosing a sufficiently small δ. Then

$$\frac{x}{1-x}\left(\frac{1-x'}{1-x}\right)^{1/\delta} > \frac{\left(a_{21}y+a_{22}\left(1-y\right)\right)}{\left(a_{11}y+a_{12}\left(1-y\right)\right)}$$

Hence,

$$\left(a_{11}y+a_{12}\left(1-y\right)\right)\frac{x}{1-x}\left(\frac{1-x'}{1-x}\right)^{1/\delta}-\left(a_{21}y+a_{22}\left(1-y\right)\right) > 0$$

Now, for $x < x'$

$$\frac{d_D\left(1-x\right)}{d_U\left(x\right)} = \frac{mx\left(\frac{x}{1-x'}\right)^{1/\delta}}{m\left(1-x\right)} = \frac{x}{1-x}\left(\frac{x}{1-x'}\right)^{1/\delta}$$

since $x < x' \leq 1/2$, we have that $1-x' > x$ so we can make $\left(\frac{x}{1-x'}\right)^{1/\delta}$ as small as we want by choosing a sufficiently small δ. Then

$$\frac{x}{1-x}\left(\frac{x}{1-x'}\right)^{1/\delta} < \frac{\left(a_{21}y+a_{22}\left(1-y\right)\right)}{\left(a_{11}y+a_{12}\left(1-y\right)\right)}$$

Hence

$$\left(a_{11}y+a_{12}\left(1-y\right)\right)\frac{x}{1-x}\left(\frac{x}{1-x'}\right)^{1/\delta}-\left(a_{21}y+a_{22}\left(1-y\right)\right) < 0$$

When $x' > 1/2$ let $d_U(x)$ and $d_D(1-x)$ in D as follows:

$$d_{ki}^{x'}(f_{ki}) = \begin{cases} m\delta\left(1-f_{ki}\right) & \text{if } f_{ki} \leq x' \\ m\delta\left(1-f_{ki}\right)\frac{(1-f_{ki})^{1/\delta}}{x'^{1/\delta}} & \text{if } f_{ki} > x' \end{cases}$$

where $ki \in \{U, D\}$ and $\delta > 0$. Now, for $x > x'$

$$\frac{d_D\left(1-x\right)}{d_U\left(x\right)} = \frac{mx}{m\left(1-x\right)\frac{(1-x)^{1/\delta}}{x'^{1/\delta}}} = \frac{x}{1-x}\left(\frac{x'}{1-x}\right)^{1/\delta}$$

Since $x > x' > 1/2$, $1-x < 1/2$ we can make$\left(\frac{x'}{1-x}\right)^{1/\delta}$ as big as we want by choosing a sufficiently small δ. Then

$$\frac{x}{1-x}\left(\frac{x'}{1-x}\right)^{1/\delta} > \frac{\left(a_{21}y + a_{22}\left(1-y\right)\right)}{\left(a_{11}y + a_{12}\left(1-y\right)\right)}$$

Hence,

$$\left(a_{11}y + a_{12}\left(1-y\right)\right)\frac{x}{1-x}\left(\frac{x'}{1-x}\right)^{1/\delta} - \left(a_{21}y + a_{22}\left(1-y\right)\right) > 0$$

For $x < x'$

$$\frac{d_D\left(1-x\right)}{d_U\left(x\right)} = \frac{mx\left(\frac{x}{x'}\right)^{1/\delta}}{m\left(1-x\right)} = \frac{x}{1-x}\left(\frac{x}{x'}\right)^{1/\delta}$$

Since $x < x'$, we can make$\left(\frac{x}{x'}\right)^{1/\delta}$ as small as we want by choosing a sufficiently small δ. Then

$$\frac{x}{1-x}\left(\frac{x}{x'}\right)^{1/\delta} < \frac{\left(a_{21}y + a_{22}\left(1-y\right)\right)}{\left(a_{11}y + a_{12}\left(1-y\right)\right)}$$

Hence

$$\left(a_{11}y + a_{12}\left(1-y\right)\right)\frac{x}{1-x}\left(\frac{x}{x'}\right)^{1/\delta} - \left(a_{21}y + a_{22}\left(1-y\right)\right) < 0$$

The argument for y is analogous. ■

Proof of Proposition 3

Let $I^* \equiv \left[\left(x^*, 1-x^*\right), \left(y^*, 1-y^*\right)\right] \in (0,1) \times (0,1)$ be an interior Mixed Strategy Nash Equilibrium (MSNE) of G. In this equilibrium, expected payoffs are equalized across strategies; that is, $\pi_U = \pi_D$ and $\pi_L = \pi_R$. From Proposition 2, we also know that an MSNE is a Mixed Strategy Doubt-Full equilibrium (MSDFE); that is, $\pi_U d_D(1-x^*) = \pi_D d_U(x^*)$ and $\pi_L d_R(1-y^*) = \pi_R d_L(y^*)$. Hence, an interior MSNE is a stationary state of the system (C.1)-(C.2) if all agents are *Cartesian Skeptical.*

Thus, $F_1(x^*, y^*) = 0$ and $F_2(x^*, y^*) = 0$, where

$$\begin{aligned} F_1(x,y) &= \pi_U d_D(1-x) - \pi_D d_U(x) \\ &= (a_{11}y + a_{12}(1-y))d_D\,(1-x) - (a_{21}y + a_{22}(1-y))d_U\,(x) \\ F_2(x,y) &= \pi_L d_R(1-y) - \pi_R d_L(y) \\ &= (b_{11}x + b_{21}(1-x))d_R\,(1-y) - (b_{12}x + b_{22}(1-x))d_L\,(y) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial F_1(x,y)}{\partial x} &= \pi_U \frac{\partial d_D(1-x)}{\partial x} - \pi_D \frac{\partial d_U(x)}{\partial x} \\ \frac{\partial F_1(x,y)}{\partial y} &= (a_{11} - a_{12})\, d_D(1-x) + (a_{22} - a_{21})\, d_U(x) \\ \frac{\partial F_2(x,y)}{\partial x} &= (b_{11} - b_{21})\, d_R(1-y) + (b_{22} - b_{12})\, d_L(y) \\ \frac{\partial F_2(x,y)}{\partial y} &= \pi_L \frac{\partial d_R(1-y)}{\partial y} - \pi_R \frac{\partial d_L(y)}{\partial y} \end{aligned}$$

On the other hand, the Jacobian $J(x,y)$ of the dynamic system (C.1)-(C.2) evaluated at the steady state (x^*, y^*) is:

$$J(x^*,y^*) = \begin{bmatrix} G_1(x^*,y^*) \left.\frac{\partial F_1(x,y)}{\partial x}\right|_{I^*} & G_1(x^*,y^*) \left.\frac{\partial F_1(x,y)}{\partial y}\right|_{I^*} \\ G_2(x^*,y^*) \left.\frac{\partial F_2(x,y)}{\partial x}\right|_{I^*} & G_2(x^*,y^*) \left.\frac{\partial F_2(x,y)}{\partial y}\right|_{I^*} \end{bmatrix}$$

In an MSNE, $\pi_U = \pi_D$, $\pi_L = \pi_R$. If, on the other hand, agents are playing in a *doubt-full mode*, (that is, $d_i \in D^{1-\delta}$ for $i \in \{U, D, L, R\}$ with $\lim_{\delta \to \delta^*} d_U\,(x) = \lim_{\delta \to \delta^*} d_D\,(1-x) = \lim_{\delta \to \delta^*} d_L\,(y) = \lim_{\delta \to \delta^*} d_R\,(1-y)$ and being nearly 1, for all $(x,y) \in (0,1) \times (0,1)$; $\delta^* > 0$ but nearly zero, as in Proposition 2). Then, writing $d_i\,(.) = 1$, we would also have $\pi_U d_D = \pi_D d_U$ and $\pi_L d_R = \pi_R d_L$.

Hence, in an MSNE as an MSDFE :

$$
\begin{aligned}
G_1(x^*, y^*) &= \frac{x^*(1-x^*)}{2\pi_U\pi_D - \pi_U d_D(1-x^*) - \pi_D d_U(x^*)} \\
&= \frac{x^*(1-x^*)}{\pi_U(2\pi_U - d_D(1-x^*) - d_U(x^*))} \\
&= \frac{x^*(1-x^*)}{2\pi_U(\pi_U - 1)} \\
G_2(x^*, y^*) &= \frac{y^*(1-y^*)}{2\pi_L(\pi_L - 1)}
\end{aligned}
$$

Thus, the elements of the Jacobian matrix are the following:

$$
\begin{aligned}
j_{11} &= G_1(x^*, y^*) \left. \frac{\partial F_1(x,y)}{\partial x} \right|_{I^*} \\
&= \frac{x^*(1-x^*)}{2\pi_U(\pi_U - 1)} \left(\pi_U \frac{\partial d_D(1-x)}{\partial x} - \pi_U \frac{\partial d_U(x)}{\partial x} \right)_{I^*} \\
&= \frac{x^*(1-x^*)}{2(\pi_U - 1)} \left(\frac{\partial d_D(1-x)}{\partial x} - \frac{\partial d_U(x)}{\partial x} \right)_{I^*}
\end{aligned}
$$

$$
\begin{aligned}
j_{12} &= G_1(x^*, y^*) \left. \frac{\partial F_1(x,y)}{\partial y} \right|_{I^*} \\
&= \frac{x^*(1-x^*)}{2\pi_U(\pi_U - 1)} ((a_{11} - a_{12})\, d_D(1-x^*) + (a_{22} - a_{21})\, d_U(x^*))
\end{aligned}
$$

$$
\begin{aligned}
j_{21} &= G_2(x^*, y^*) \left. \frac{\partial F_2(x,y)}{\partial y} \right|_{I^*} \\
&= \frac{y^*(1-y^*)}{2\pi_L(\pi_L - 1)} ((b_{11} - b_{21})\, d_R(1-y^*) + (b_{22} - b_{12})\, d_L(y^*))
\end{aligned}
$$

$$
\begin{aligned}
j_{22} &= G_2(x^*, y^*) \left. \frac{\partial F_2(x,y)}{\partial y} \right|_{I^*} \\
&= \frac{y^*(1-y^*)}{2(\pi_L - 1)} \left(\frac{\partial d_R(1-y)}{\partial y} - \frac{\partial d_L(y)}{\partial y} \right)_{I^*}
\end{aligned}
$$

Recall that the real part of the eigenvalues of $J(x^*, y^*)$ only depends on the sum of the diagonal terms (the trace of the matrix):

$$\begin{aligned}\text{Trace of } J(x^*, y^*) &= G_1(x^*, y^*) \left.\frac{\partial F_1(x,y)}{\partial x}\right|_{I^*} + G_2(x^*, y^*) \left.\frac{\partial F_2(x,y)}{\partial y}\right|_{I^*} \\ &= \frac{x^*(1-x^*)}{2(\pi_U - 1)} \left(\frac{\partial d_D(1-x)}{\partial x} - \frac{\partial d_U(x)}{\partial x}\right)_{I^*} \\ &\quad + \frac{y^*(1-y^*)}{2(\pi_L - 1)} \left(\frac{\partial d_R(1-y)}{\partial y} - \frac{\partial d_L(y)}{\partial y}\right)_{I^*}\end{aligned}$$

Since the expected values $\pi_U = a_{11}y^* + a_{12}(1-y^*)$ and $\pi_L = b_{11}x^* + b_{21}(1-x^*)$ are smaller than 1, both $\frac{x^*(1-x^*)}{2(\pi_U-1)}$ and $\frac{y^*(1-y^*)}{2(\pi_L-1)}$ are negative. The sign of $\left(\frac{\partial d_D(1-x)}{\partial x} - \frac{\partial d_U(x)}{\partial x}\right)_{I^*}$ and $\left(\frac{\partial d_R(1-y)}{\partial y} - \frac{\partial d_L(y)}{\partial y}\right)_{I^*}$ is clearly positive (that is, the signs of the derivatives of $d_D(1-x)$ and $d_R(1-y)$ with respect to x and y, respectively, are positive and those of $d_U(x)$ and $d_L(y)$ are negative). Thus, $j_{11} < 0$ and $j_{22} < 0$ and so the sign of the trace is negative

$$sign \left[G_1(x^*, y^*) \left.\frac{\partial F_1(x,y)}{\partial x}\right|_{I^*} + G_2(x^*, y^*) \left.\frac{\partial F_2(x,y)}{\partial y}\right|_{I^*}\right] < 0$$

Without loss of generality, we may assume, for an interior equilibrium, that $a_{11} > a_{21}$, $b_{11} < b_{12}$, $a_{12} < a_{22}$, and $b_{22} < b_{21}$. Then it can be seen that the sign of $j_{21} \times j_{12}$ is negative, when the agents are playing in the *absent* or *doubt-full mode:*

$$j_{21} \times j_{12} = \left(\frac{y^*(1-y^*)}{2\pi_L(\pi_L - 1)}((b_{11} - b_{12}) + (b_{22} - b_{21})\right) \times \left(\frac{x^*(1-x^*)}{2\pi_U(\pi_U - 1)}((a_{11} - a_{21}) + (a_{22} - a_{12}))\right) < 0$$

Thus, the determinant associated to $J(x^*, y^*)$, Det $J(x^*, y^*) = j_{11} \times j_{22} - j_{21} \times j_{12}$, has a positive sign. Therefore, when every agent is *Cartesian skeptical*, the MSNE, $I^* \equiv [(x^*, 1-x^*), (y^*, 1-y^*)]$, is a *sink* and therefore is an asymptotically stable equilibrium.■

Proof of Proposition 4:

Using the same procedure as in Proposition 3, we can easily prove that, under the *doubtless* mode, the MSDLE $[(1/2, 1/2), (1/2, 1/2)]$ is a *source.* Now, to see the trajectory of initial points different from $[(1/2, 1/2), (1/2, 1/2)]$, we might use the doubt function constructed for the proof of part 2 of Proposition 2. .

$$\dot{x} = \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)}(\pi_U d_D - \pi_D d_U)$$

$$\dot{y} = \frac{y(1-y)}{\pi_L(\pi_R - d_R) + \pi_R(\pi_L - d_L)}(\pi_L d_R - \pi_R d_L)$$

Note that the denominators of (C.1)-(C.2) are positive in the *doubt-less mode* of play. Hence the sign of $\dot{x}$ and $\dot{y}$ depend on the sign of $(\pi_U d_D - \pi_D d_U)$ and $(\pi_L d_R - \pi_R d_L)$, respectively. Now we can proceed as in the proof of part 2 of Proposition 2.

Let first $x' \leq 1/2$. We construct the doubt functions $d_U(x)$ and $d_D(1-x)$ in D as follows:

$$d_{ki}^{x'}(f_{ki}) = \begin{cases} m\delta(1-f_{ki}) & \text{if } f_{ki} \leq x' \\ m\delta(1-f_{ki})\frac{(1-f_{ki})^{1/\delta}}{(1-x')^{1/\delta}} & \text{if } f_{ki} > x' \end{cases}$$

This means that if $x > x'$

$$sign\left[\dot{x}\right] = sign\left[\left(\pi_U - \pi_D\frac{(1-x)^{1/\delta}}{(1-x')^{1/\delta}}\right)\right]$$

Then there is a δ' low enough such that for all $0 < \delta \leq \delta'$, $(1-x)^{1/\delta} / (1-x')^{1/\delta}$ is sufficiently small so that $sign\left[\dot{x}\right] > 0$ and hence if $x(0) > x'$, then $\lim_{t\to\infty} x(t) = 1$.

If on the other hand $x < x'$

$$sign\left[\dot{x}\right] = sign\left[\left(\pi_U\frac{x^{1/\delta}}{(1-x')^{1/\delta}} - \pi_D\right)\right]$$

Since $x < x' \leq 1/2$, we have that $1 - x' > x$ so there is a δ' low enough such that for all $0 < \delta \leq \delta'$, $x^{1/\delta}/(1-x')^{1/\delta}$ is sufficiently small so that $sign\left[\dot{x}\right] < 0$ and hence if $x(0) < x'$, then $\lim_{t\to\infty} x(t) = 0$.

When $x' > 1/2$, we let $d_U(x)$ and $d_D(1-x)$ in D as follows:

$$d_{ki}^{x'}(f_{ki}) = \begin{cases} m\delta(1-f_{ki}) & \text{if } f_{ki} \leq x' \\ m\delta(1-f_{ki})\frac{(1-f_{ki})^{1/\delta}}{x'^{1/\delta}} & \text{if } f_{ki} > x' \end{cases}$$

This means that if $x > x'$

$$sign\left[\dot{x}\right] = sign\left[\left(\pi_U - \pi_D\frac{(1-x)^{1/\delta}}{x'^{1/\delta}}\right)\right]$$

Since $x > x' > 1/2$, $1-x < 1/2$, there is a δ' low enough such that for all $0 < \delta \leq \delta'$, we can make$(x')^{1/\delta} / (1-x)^{1/\delta}$ is sufficiently big so that $sign\left[\dot{x}\right] > 0$ and hence if $x(0) > x'$, then $\lim_{t\to\infty} x(t) = 1$.

If on the other hand $x < x'$

$$sign\left[\dot{x}\right] = sign\left[\left(\pi_U \frac{x^{1/\delta}}{x'^{1/\delta}} - \pi_D\right)\right]$$

Since $x < x'$, there is a $\delta^{'}$ low enough such that for all $0 < \delta \leq \delta^{'}$, $x^{1/\delta}/x'^{1/\delta}$ is sufficiently small so that $sign\left[\dot{x}\right] < 0$ and hence if $x(0) < x'$, then $\lim_{t\to\infty} x(t) = 0$. The argument for y is analogous.

■

Proof Proposition 5:

(a) Let $S_k = \{1, 2\}$ be player population k's strategy set. Without loss of generality, let us refer to the dynamics of strategy 1. Then, by equation (7), we have

$$\begin{aligned} \overset{\bullet}{f}_{k1} &= f_{k1}\overline{\phi_k} - \phi_{k1}\phi_k \\ &= \frac{\epsilon_k}{\pi_{k1}(\pi_{k2} - \epsilon_k) + \pi_{k2}(\pi_{k1} - \epsilon_k)} f_{k1}(\pi_{k1} - \overline{\pi}_k) \\ &= \frac{\epsilon_k}{D(f)} f_{ki}[\pi_{ki} - \overline{\pi}_k] \end{aligned} \tag{C.3}$$

where $D(f) \equiv \pi_{k1}(\pi_{k2} - \epsilon_k) + \pi_{k2}(\pi_{k1} - \epsilon_k) > 0$.

By equation (C.3), the growth rates $\frac{\overset{\bullet}{f}_{ki}}{f_{ki}}$ equal payoff differences $[\pi_{ki} - \overline{\pi}_k]$ multiplied by a (Lipschitz) continuous, positive function $\frac{\epsilon_k}{D(f)}$. This concludes the proof. (Note that, given ϵ_k, a payoff difference $[\pi_{ki} - \overline{\pi}_k]$ will have stronger dynamic effect if $D(f)$ is low than if it is high; if ϵ_k decreases, the dynamic effect of $[\pi_{ki} - \overline{\pi}_k]$ decreases).

(b) Easy.■