Voluntary Voting: Costs and Benefits*

Vijay Krishna[†] and John Morgan[‡]

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Abstract

We study strategic voting in a Condorcet type model in which voters have identical preferences but differential information. Voting is costly and voluntary. We show that under majority rule with voluntary voting, it is an equilibrium to vote sincerely. Thus, in contrast to situations with compulsory voting, there is no conflict between strategic and sincere behavior. In large elections, the equilibrium is shown to be unique. Furthermore, participation rates are such that, in the limit, the correct candidate is elected with probability one. Finally, when voting is costless, a social planner cannot improve over a purely voluntary voting scheme.

1 Introduction

Condorcet's celebrated Jury Theorem states that, when voters have common interests but differential information, sincere voting under majority rule produces the correct outcome in large elections. There are two key components to the theorem. First, it postulates that voting is sincere—that is, voters vote solely according to their private information. Recent theoretical work shows, however, that sincerity is inconsistent with rationality—it is typically not an equilibrium to vote sincerely. The reason is that rational voters will make inferences about others' information and, as a result, will have the incentive to vote against their own private information (Austen-Smith and Banks, 1996).

Equilibrium voting behavior involves the use of mixed strategies—with positive probability, voters vote against their private information. Surprisingly, this does not overturn the conclusion of the Jury Theorem: In large elections, there exist equilibria in which the correct candidate is always chosen despite insincere voting (Feddersen and Pesendorfer, 1998). These convergence results, while powerful, rest on equilibrium behavior that may be deemed implausible. Voting is not only insincere

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[†]Penn State University. E-mail: vkrishna@psu.edu

[‡]University of California, Berkeley. E-mail: morgan@haas.berkeley.edu

but random. Moreover, some voters have *negative* returns to voting—they would rather not vote at all—this is a manifestation of the "swing voter's curse" (Feddersen and Pesendorfer, 1996).

Second, these generalizations of the Jury Theorem rely on the assumption that voter turnout is high. Indeed it is implicitly assumed that voting is compulsory, so all eligible voters show up to vote. When voting is voluntary and costly, however, there is reason to doubt that voters will turn out in large enough numbers to guarantee correct choices. Indeed, even if there were no swing voter's curse, rational voters would correctly realize that a single vote is unlikely to affect the outcome so there is little benefit to voting. This is the "paradox of not voting" (Downs, 1957).

In this paper, we revisit the classic Condorcet Jury Theorem but with two amendments to the environment. First, we relax the assumption that the size of the electorate is fixed and commonly known in favor of one where the size is random as in the Poisson model introduced by Myerson (1998 & 2000). This, by itself, affects none of the findings discussed above but, as Myerson (1998) has demonstrated, leads to a simpler analysis. Second, and more important, we relax the (implicit) assumption that voting is compulsory (i.e., it is not possible to abstain). Specifically, voters incur private costs of voting and may avoid these by abstaining. Voters in our model are fully rational, so the twin problems of strategic voting and the paradox of not voting are present.

We show that under costly and voluntary voting,

1. There exists an equilibrium with endogenously determined participation rates and *sincere voting*. Thus, there is no conflict between rationality and sincerity.

In *large* elections:

2. The equilibrium we study is unique.

3. While the turnout percentage goes to zero, the expected number of voters is unbounded—regardless of the distribution of voting costs.

4. Participation rates are such that the correct candidate is always elected.

Motivated by the observation that many countries have enacted mandatory voting policies, we compare welfare under the voluntary scheme with several alternatives. We find that:

5. Schemes designed to encourage full participation are never optimal; and

6. When voting is costless, voluntary voting maximizes welfare.

To summarize, adding the realistic feature of voluntary and costly voting to the classic Condorcet model restores many of the desirable properties of the original Jury Theorem. Sincere voting obtains as an equilibrium, and, in large elections, the correct candidate is always chosen. Equilibrium payoffs are non-negative and so with voluntary voting, the swing voter's curse is lifted.

To see why voluntary (and costly) voting may lead to sincere voting behavior, consider a two-candidate election in which voters have 50-50 prior beliefs as to the "correct" candidate. Each voter receives a private signal about the suitability of the candidates. Suppose that signals in favor of A are more accurate than those in favor of B. In other words, a signal in favor of A is more likely in situations in which A is the correct candidate (say the chances of this are 75%) than a signal in favor of B is in situations in which B is the correct candidate (say the chances of this are 60%).

First, suppose voting is compulsory and there is a large population. Suppose further that all voters save one, vote sincerely and consider a voter with a signal in favor of A. This voter is pivotal—his vote affects the outcome—if the vote counts are roughly equal. But since signals for A are more accurate than signals for B, a roughly 50-50 vote split is more likely when B is the right candidate. Thus a voter with signal A should rationally vote for B. It is not an equilibrium for everyone to vote sincerely.

Now suppose that voting is voluntary and participation behavior is such that those with information favorable to B are more likely to turn out than those with information favorable to A. The fact that the votes are roughly the same does not automatically imply that the signals are biased towards B: voters in favor of B are more likely to vote, and this mitigates the biased inference from the split vote itself. Our main result exploits this reasoning and shows that, in fact, the endogenously determined participation rates lead to sincere voting behavior.

Related literature Early work on the Condorcet Jury Theorem viewed it as a purely statistical phenomenon—an expression of the law of large numbers. Perhaps this was the way that Condorcet himself viewed it. Game theoretic analyses of the Jury Theorem originate in the work of Austen-Smith and Banks (1996). They show that sincere voting is generally not consistent with equilibrium behavior.

Feddersen and Pesendorfer (1998) derive the ("insincere") equilibria of the voting games specified above—these involve mixed strategies—and then study their limiting properties. They show that, despite the fact that sincere voting is not an equilibrium, large elections still aggregate information correctly. McLennan (1998) views such voting games, in the abstract, as games of common interest and argues on that basis that there are always Pareto efficient equilibria of such games. Apart from the fact that voting is costly and voluntary, our basic setting is the same as that in these papers—there are two candidates, voters have common interests but differential information (sometimes referred to as "common values").

A separate strand of the literature is concerned with costly voting and endogenous participation but in settings in which voter preferences are diverse (sometimes referred to as "private values"). Palfrey and Rosenthal (1985) consider costly voting with privately known costs but where preferences over outcomes are commonly known (see also Palfrey and Rosenthal, 1983 and Ledyard, 1984 for models in which the costs are also common knowledge). These papers are interested in formalizing Downs' paradox of not voting. Börgers (2004) studies majority rule in a costly voting model with private values—that is, with diverse rather than common preferences. He compares voluntary and compulsory voting and argues that individual decisions to vote or not do not properly take into account a "pivot externality"—the casting of a single vote decreases the value of voting for others. As a result, participation rates are too high relative to the optimum and a law that makes voting compulsory would only worsen matters. Krasa and Polborn (2007) show that the externality identified by Börgers' is sensitive to his assumption that the prior distribution of voter preferences is 50-50. With unequal priors, under some conditions, the externality goes in the opposite direction and there are social benefits to encouraging increased turnout via fines for not voting.

Ghosal and Lockwood (2007) reexamine Börgers' result when voters have more general preferences—including common values—and show that it is sensitive to the private values assumption. Finally, Feddersen and Pesendorfer (1996) examine abstention in a common values model when voting is costless. The number of voters is random, some are informed of the state, while others have no information whatsoever. Abstention arises in their model as a result of the aforementioned swing voter's curse—in equilibrium, a fraction of the uninformed voters do not participate.

All of this work postulates a fixed and commonly known population of voters. Myerson (1998 & 2000) argues that precise knowledge of the number of eligible voters is an idealization at best, and suggests an alternative model in which the size of the electorate is a Poisson random variable. He shows that this specification leads to a simpler analysis and derives the mixed equilibrium for the majority rule in *large* elections (in a setting where signal precisions are asymmetric). He then studies its limiting properties as the number of expected voters increases, exhibiting information aggregation results parallel to those derived in the known population models. Feddersen and Pesendorfer (1999) use the Poisson framework to study abstention when voting is costless but preferences are diverse. In large elections, the fraction of informative (as opposed to ideological) voters goes to zero; however information still aggregates. We also find it convenient to adopt Myerson's Poisson game technology but are able to show that there is a *sincere* voting equilibrium for any (expected) size electorate.

The paper is organized as follows. In Section 2 we introduce the basic environment and Myerson's Poisson model. As a benchmark, in Section 3 we first consider the model with compulsory voting and establish that sincere voting is not an equilibrium. In Section 4, we introduce the model with voluntary and costly voting. We first show that under the assumption of sincere voting, there exist positive equilibrium participation levels. We then show that given those participation levels, sincere voting is incentive compatible. Section 5 studies the limiting properties of the equilibria considered in the previous section—it is shown that in the limit, information fully aggregates and the correct candidate is elected with probability one. In Section 6 we show that all equilibria must be sincere and then use the information aggregation properties of large elections to show that there is, in fact, a *unique* equilibrium. Finally, Section 7 compares social welfare under voluntary versus compulsory voting. We show that in large elections, even if it is costless to vote, voluntary voting is welfare superior to compulsory voting.

All proofs are collected in the appendices.

2 The Model

There are two candidates, named A and B, who are competing in an election decided by majority voting.[1] There are two equally likely states of nature, α and β.[2] Candidate A is the better choice in state α while candidate B is the better choice in state β. Specifically, in state α the payoff of any citizen is 1 if A is elected and 0 if B is elected. In state β, the roles of A and B are reversed.

The size of the electorate is a random variable which is distributed according to a *Poisson* distribution with mean n. Thus the probability that there are exactly m eligible voters (or *citizens*) is $e^{-n}n^m/m!$.

Prior to voting, every citizen receives a private signal S_i regarding the true state of nature. The signal can take on one of two values, a or b. The probability of receiving a particular signal depends on the true state of nature. Specifically, each voter receives a conditionally independent signal where

$$\Pr[a \mid \alpha] = r \text{ and } \Pr[b \mid \beta] = s$$

We suppose that both r and s are greater than $\frac{1}{2}$, so that the signals are informative and less than 1, so that they are noisy. Thus, signal a is associated with state α while the signal b is associated with β. The posterior probabilities of the states after receiving signals are

$$q(\alpha \mid a) = \frac{r}{r + (1-s)} \text{ and } q(\beta \mid b) = \frac{s}{s + (1-r)}$$

We assume, without loss of generality, that $r > s$. It may be verified that

$$q(\alpha \mid a) < q(\beta \mid b)$$

Thus the posterior probability of state α given signal a is smaller than the posterior probability of state β given signal b even though the "correct" signal is more likely in state α.

Pivotal Events An *event* is a pair of vote totals (j, k) such that there are j votes for A and k votes for B. An event is *pivotal* for A if a single additional vote for A will affect the outcome of the election. We denote the set of such events by Piv_A. One additional vote for A makes a difference only if either (i) there is a tie; or (ii) A has one vote less than B. Let $T = \{(k, k) : k \geq 0\}$ denote the set of ties and let $T_{-1} = \{(k-1, k) : k \geq 1\}$ denote the set of events in which A is one vote short of a tie. Similarly, Piv_B is defined to be the set of events which are pivotal for B. This set consists of the set T of ties together with events in which A has one vote more

[1] In the event of a tied vote, the winning candidate is chosen by a fair coin toss.

[2] The analysis is unchanged if the states are not equally likely. We study the simple case only for notational ease.

than B. Let $T_{+1} = \{(k, k-1) : k \geq 1\}$ denote the set of events in which A is ahead by one vote.

Let σ_A be the *expected* number of votes for A in state α and let σ_B be the expected number of votes for B in state α. Analogously, let τ_A and τ_B be the expected number of votes for A and B, respectively, in state β. Since it may be possible for voters to abstain, it is only required that $\sigma_A + \sigma_B \leq n$ and $\tau_A + \tau_B \leq n$.

Consider an event where (other than voter 1) the realized electorate is of size m and there are k votes in favor of A and l votes in favor of B. The number of abstentions is thus $m - k - l$. The probability of this event in state α is

$$\Pr[(k, l; m) \mid \alpha] = \frac{e^{-n}}{m!} \binom{m}{k+l} \binom{k+l}{k} (n - \sigma_A - \sigma_B)^{m-k-l} \sigma_A^k \sigma_B^l$$

It is useful to rearrange the expression as follows:

$$\begin{aligned} \Pr[(k, l; m) \mid \alpha] &= e^{-(n-\sigma_A-\sigma_B)} \frac{(n - \sigma_A - \sigma_B)^{m-k-l}}{(m-k-l)!} \\ &\quad \times e^{-\sigma_A} \frac{\sigma_A^k}{k!} e^{-\sigma_B} \frac{\sigma_B^l}{l!} \end{aligned}$$

Of course, the size of the electorate is unknown to voter 1. The probability of the event (k, l), irrespective of the size of the electorate, is

$$\begin{aligned} \Pr[(k, l) \mid \alpha] &= \sum_{m=k+l}^{\infty} \Pr[(k, l; m) \mid \alpha] \\ &= e^{-\sigma_A} \frac{\sigma_A^k}{k!} e^{-\sigma_B} \frac{\sigma_B^l}{l!} \end{aligned}$$

The probability of the event (k, l) in state β may similarly be obtained by replacing σ with τ.

The probability of a tie in state α is

$$\Pr[T \mid \alpha] = e^{-\sigma_A - \sigma_B} \sum_{k=0}^{\infty} \frac{\sigma_A^k}{k!} \frac{\sigma_B^k}{k!} \tag{1}$$

while the probability that A falls one vote short in state α is

$$\Pr[T_{-1} \mid \alpha] = e^{-\sigma_A - \sigma_B} \sum_{k=1}^{\infty} \frac{\sigma_A^{k-1}}{(k-1)!} \frac{\sigma_B^k}{k!} \tag{2}$$

The probability $\Pr[T_{+1} \mid \alpha]$ that A is ahead by one vote may be written by exchanging σ_A and σ_B in (2). The corresponding probabilities in state β are obtained by substituting τ for σ.

In what follows, it will be useful to rewrite the pivot probabilities in terms of *modified Bessel functions* (see Abramowitz and Stegun, 1965), defined by

$$\begin{aligned} I_0(z) &= \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k!} \frac{\left(\frac{z}{2}\right)^k}{k!} \\ I_1(z) &= \sum_{k=1}^{\infty} \frac{\left(\frac{z}{2}\right)^{k-1}}{(k-1)!} \frac{\left(\frac{z}{2}\right)^k}{k!} \end{aligned}$$

In terms of modified Bessel functions, we can rewrite the probabilities associated with close elections as

$$\begin{aligned} \Pr[T \mid \alpha] &= e^{-\sigma_A - \sigma_B} I_0\left(2\sqrt{\sigma_A \sigma_B}\right) \\ \Pr[T_{\pm 1} \mid \alpha] &= e^{-\sigma_A - \sigma_B} \left(\frac{\sigma_A}{\sigma_B}\right)^{\pm \frac{1}{2}} I_1\left(2\sqrt{\sigma_A \sigma_B}\right) \end{aligned} \tag{3}$$

Again, the corresponding probabilities in state β are found by substituting τ for σ.

For our asymptotic results it is useful to note that when z is large, the modified Bessel functions can be approximated as follows[3] (see Abramowitz and Stegun, 1965, p. 377)

$$I_0(z) \approx \frac{e^z}{\sqrt{2\pi z}} \approx I_1(z) \tag{4}$$

3 Compulsory Voting

While our main concern is with situations in which voting is voluntary, it is useful to first study the benchmark case of compulsory voting. Austen-Smith and Banks (1996) showed that sincere voting does not constitute an equilibrium in a model with a fixed number of voters. Here, we show that this conclusion extends to the Poisson framework as well.[4]

Suppose that voting is *sincere*; that is, all those with a signal of a vote for A and all those with a signal of b vote for B. Under compulsory and sincere voting, the expected number of votes for A in state α is simply n times the chance that a voter gets an a signal; that is, $\sigma_A = nr$. The expected number of votes for B in state α is simply n times the probability of a b signal; that is, $\sigma_B = n(1-r)$. Similarly, in state β the expected vote totals are $\tau_A = n(1-s)$ and $\tau_B = ns$.

Since both $\sigma \to \infty$ and $\tau \to \infty$, the formulae in (4) imply that for large n,

$$\frac{\Pr[Piv_A \mid \alpha] + \Pr[Piv_B \mid \alpha]}{\Pr[Piv_A \mid \beta] + \Pr[Piv_B \mid \beta]} \approx \frac{e^{2n\sqrt{r(1-r)}}}{e^{2n\sqrt{s(1-s)}}} \times K(r,s) \tag{5}$$

where $K(r,s)$ is positive and, with compulsory voting, independent of n. If $r > s > \frac{1}{2}$, $s(1-s) > r(1-r)$ and so the expression in (5) goes to zero as n increases. This implies that, when n is large and a voter is pivotal, state β is infinitely more likely

[3] $X(n) \approx Y(n)$ means that $\lim_{n\to\infty}(X(n)/Y(n)) = 1$.

[4] See also Myerson (1998).

than state α. Thus, a type voters will not wish to vote sincerely.[5] It then follows that:

Proposition 1 *If voting is compulsory, sincere voting is not an equilibrium in large elections.*

In Section 7 below, we reexamine compulsory voting in more detail with a view to comparing it to the case of voluntary voting.

4 Voluntary Voting

In this section, we simultaneously introduce two features to the model. First, we allow for the possibility of abstention—every citizen need not vote. Second, we suppose that citizens have heterogeneous costs of going to the polls, which can be avoided by staying at home. Specifically, a citizen's cost of voting is private information and determined by an independent realization from a continuous probability distribution F with support $[0,1]$. We suppose that F admits a density f that is strictly positive on $(0,1)$. Finally, we assume that voting costs are independent of the signal as to who is the better candidate.

Thus prior to the voting decision, each citizen has two pieces of private information—his cost of voting and a signal regarding the state. We will show that there exists an equilibrium of the voting game with the following features.

1. There exists a pair of positive *threshold costs*, c_a and c_b, such that a citizen with a cost realization c and who receives a signal $i = a, b$ votes if and only if $c \leq c_i$. The threshold costs determine differential *participation rates* $F(c_a) = p_a$ and $F(c_b) = p_b$.

2. All those who vote do so sincerely—that is, all those with a signal of a vote for A and those with a signal of b vote for B.

In the model with voluntary and costly voting, our main result is

Theorem 1 *With voluntary voting under majority rule, there exists an equilibrium with positive participation in which all voters vote sincerely. In large elections, the equilibrium is unique, and the right candidate is elected with probability one.*

The result is established in four steps. First, we consider only the participation decision. Under the assumption of sincere voting, we establish the existence of positive threshold costs and the corresponding participation rates. Second, we show that given the participation rates determined in the first step, it is indeed an equilibrium to vote sincerely. Third, we show that in large elections the participation rates are such that, in the limit, information fully aggregates—the right candidate is chosen with probability one. Fourth, we show that in large elections, the equilibrium is unique.

[5] If $r = s$, then the ratio of the pivot probabilities is always 1 and incentive compatibility holds. This corresponds to one of the non-generic cases identified by Austen-Smith and Banks (1996) in a fixed n model. See also Myerson (1998).

4.1 Equilibrium Participation Rates

We now show that when all those who vote do so sincerely, there is an equilibrium in cutoff strategies. That is, there exists a threshold cost $c_a > 0$ such that all voters receiving a signal of a and having a cost $c \leq c_a$ go to the polls and vote for A. Analogously, there exists a threshold cost $c_b > 0$ for voters with a signal of b. Equivalently, one can think of a participation probability, $p_a = F(c_a)$ that a voter with an a signal goes to the polls and a probability $p_b = F(c_b)$ that a voter with a b signal goes to the polls.

Under these conditions, a given voter will vote for A in state α only if he receives the signal a (which happens with probability r) and has a voting cost lower than c_a (which happens with probability p_a). Thus the expected number of votes for A in state α is $\sigma_A = nrp_a$. Similarly, the expected number of votes for B in state α is $\sigma_B = n(1-r)p_b$. The expected number of votes for A and B in state β are $\tau_A = n(1-s)p_a$ and $\tau_B = nsp_b$, respectively.

We look for participation rates p_a and p_b such that a voter with signal a and cost $c_a = F^{-1}(p_a)$ is indifferent between going to the polls and staying home. Formally, this amounts to the condition that

$$U_a(p_a, p_b) \equiv q(\alpha \mid a)\Pr[Piv_A \mid \alpha] - q(\beta \mid a)\Pr[Piv_A \mid \beta] = F^{-1}(p_a) \qquad \text{(IRa)}$$

where the pivot probabilities are determined using the expected vote totals σ and τ as above. Likewise, a voter with signal b and cost $c_b = F^{-1}(p_b)$ must also be indifferent.

$$U_b(p_a, p_b) \equiv q(\beta \mid b)\Pr[Piv_B \mid \beta] - q(\alpha \mid b)\Pr[Piv_B \mid \alpha] = F^{-1}(p_b) \qquad \text{(IRb)}$$

Proposition 2 *There exist participation rates $p_a^* \in (0,1)$ and $p_b^* \in (0,1)$ that simultaneously satisfy IRa and IRb.*

To see why there are positive participation rates, suppose to the contrary that type a voters, say do not participate at all. Consider a citizen with signal a. Since no other a types vote, the only circumstance in which he will be pivotal is either if no b types show up or if only one b type shows up. Conditional on being pivotal, the likelihood ratio of the states is simply the ratio of the pivot probabilities, that is,

$$\frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_A \mid \beta]} = \frac{e^{-n(1-r)p_b}}{e^{-nsp_b}} \times \frac{1 + n(1-r)p_b}{1 + nsp_b}$$

Notice that the ratio of the exponential terms favors state α while the ratio of the linear terms favors state β. It turns out that the exponential terms always dominate. (Formally, this follows from the fact that the function $e^{-x}(1+x)$ is strictly decreasing for $x > 0$ and that $s > 1 - r$.) Since state α is more likely than β for a pivotal a type voter, the payoff from voting is positive.

The next result shows that b type voters are more likely to show up at the polls than a type voters.

Lemma 1 *If $r > s$, then any solution to IRa and IRb satisfies $p_a^* < p_b^*$.*

To see why the result holds, consider the case where the participation rates are the same for both types. In that case, no inference may be drawn from the overall level of turnout, only from the vote totals. Consider a particular voter. When the votes of the others are equal in number, it is clear that a tie among the other voters is more likely in state β than in state α (since b signals are noisier than a signals and everyone is voting sincerely) and this is true whether the voter has an a signal or a b signal. Now consider a voter with an a signal. When the votes of the others are such that A is one behind, then once the voter includes his own a signal (and votes sincerely), the overall vote is tied and by the same reasoning as above, an overall tie is more likely in state β than in α. Finally, consider a voter with a b signal. When the votes of the others are such that B is one behind, then once the voter includes his own b signal (and again votes sincerely), the overall vote is tied once more. Again, this is more likely in β than in α.

Thus if participation rates are equal, chances of being pivotal are greater in state β than in state α. This implies that voting is more valuable for someone with a b signal than for someone with an a signal. But then the participation rates cannot be equal.

The formal proof (in Appendix A) runs along the same lines but applies to all situations in which $p_a \geq p_b$.

The workings of the proposition may be seen in the following example.

Example 1 *Consider an expected electorate $n = 100$. Suppose the signal precisions $r = \frac{3}{4}$ and $s = \frac{2}{3}$ and that the voting costs are distributed according to $F(c) = c^{\frac{1}{3}}$. Then $p_a^* = 0.152$ and $p_b^* = 0.181$.*

Figure 1 depicts the IRa and IRb curves for this example. Notice that neither curve defines a function. In particular, for some values of p_b, there are multiple solutions to IRa. To see why this is the case, notice that for a fixed p_b, when p_a is small there is little chance of a close election outcome and hence little benefit to a types of voting. As the proportion of a types who vote increases, the chances of a close election also increase and hence the benefits from voting rise. However, once p_a becomes relatively large, the chances of a close election start falling and, consequently, so do the benefits from voting.

4.2 Sincere Voting

In this subsection we establish that given the participation rates as determined above, it is a best-response for every voter to vote sincerely.

Likelihood Ratios The following result is key in establishing this—it compares the likelihood ratio of α to β conditional on the event Piv_B to that conditional on the event Piv_A. It requires only that the voting behavior is such that expected number of votes for A is greater in state α than in state β and the reverse is true for B. While

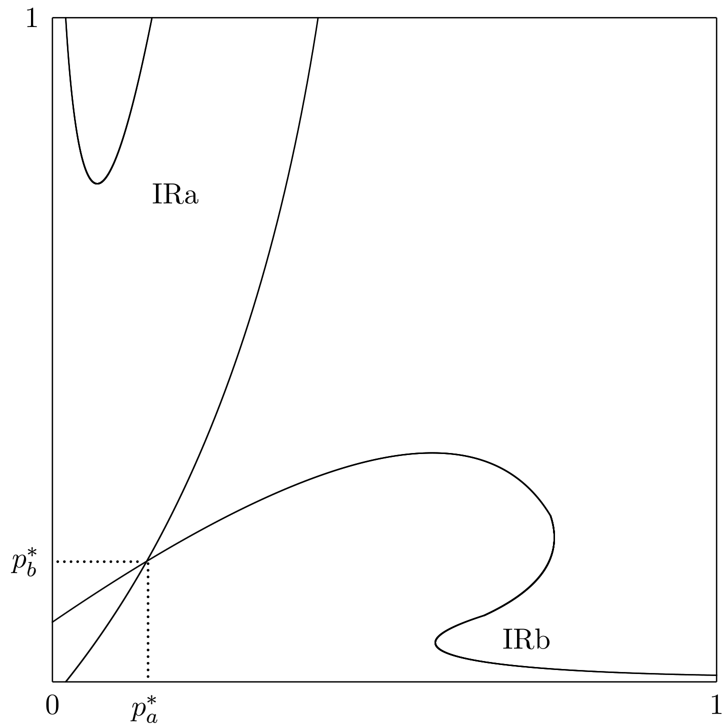

Figure 1: Equilibrium Participation Rates

the lemma is more general, it is easy to see that sincere voting behavior satisfies the assumptions of the lemma.

Lemma 2 (Likelihood Ratio) *If voting behavior is such that* $\sigma_A > \tau_A$ *and* $\sigma_B < \tau_B$*, then*

$$\frac{\Pr[Piv_B \mid \alpha]}{\Pr[Piv_B \mid \beta]} > \frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_A \mid \beta]} \tag{6}$$

Since $\sigma_A > \tau_A$ and $\sigma_B < \tau_B$, then, on "average," the ratio of A to B votes is higher in state α than in state β. Of course, voters' decisions do not depend on the average outcome, but rather on pivotal outcomes. The lemma shows that even when one considers the set of "marginal" events where the vote totals are close (and a voter is pivotal) it is still the case that A is more likely to be leading in state α and more likely to be trailing in state β (details are provided in Appendix A).

Incentive Compatibility With the Likelihood Ratio Lemma in hand, we now examine the incentives to vote sincerely. Let (p_a^*, p_b^*) be equilibrium participation rates. A voter with signal a and cost $c_a^* = F^{-1}(p_a^*)$ is just indifferent between voting and staying home, that is,

$$q(\alpha \mid a) \Pr[Piv_A \mid \alpha] - q(\beta \mid a) \Pr[Piv_A \mid \beta] = F^{-1}(p_a^*) \tag{IRa}$$

We want to show that sincere voting is optimal for a "type a" voter if others are voting sincerely. That is,

$$\begin{aligned} & q(\alpha \mid a)(\Pr[Piv_A \mid \alpha] - q(\beta \mid a) \Pr[Piv_A \mid \beta]) \\ & \geq q(\beta \mid a) \Pr[Piv_B \mid \beta] - q(\alpha \mid a) \Pr[Piv_B \mid \alpha] \end{aligned} \tag{ICa}$$

The left-hand side is the payoff to a type a voter from voting for A whereas the right-hand side is the payoff to voting for B.

Now notice that since $p_a^* > 0$, IRa implies

$$\frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_A \mid \beta]} > \frac{q(\beta \mid a)}{q(\alpha \mid a)}$$

and so applying Lemma 2 it follows that,

$$\frac{\Pr[Piv_B \mid \alpha]}{\Pr[Piv_B \mid \beta]} > \frac{q(\beta \mid a)}{q(\alpha \mid a)}$$

which is equivalent to

$$q(\beta \mid a) \Pr[Piv_B \mid \beta] - q(\alpha \mid a) \Pr[Piv_B \mid \alpha] < 0$$

and so the payoff from voting for B with a signal of a is negative. Thus ICa holds. We have argued that if (p_a^*, p_b^*) are such that a voter with signal a and cost $F^{-1}(p_a^*)$ is just indifferent between participating or not, then all voters with a signals who have lower costs, have the incentive to vote sincerely. Recall that this was not the case under compulsory voting.

What about voters with b signals? Again, since (p_a^*, p_b^*) are equilibrium participation rates, then a voter with signal b and cost $c_b^* = F^{-1}(p_b^*)$ is just indifferent between voting and staying home, that is,

$$q(\beta \mid b) \Pr[Piv_B \mid \beta] - q(\alpha \mid b) \Pr[Piv_B \mid \alpha] = F^{-1}(p_b^*) \tag{IRb}$$

We want to show that a voter with signal b is better off voting for B over A, that is

$$\begin{aligned} & q(\beta \mid b) \Pr[Piv_B \mid \beta] - q(\alpha \mid b) \Pr[Piv_B \mid \alpha] \\ & \geq q(\alpha \mid b) \Pr[Piv_A \mid \alpha] - q(\beta \mid b) \Pr[Piv_A \mid \beta] \end{aligned} \tag{ICb}$$

As above, since $p_b^* > 0$, the left-hand side of ICb is strictly positive and Lemma 2 implies that the right-hand side is negative.

We have thus established,

Proposition 3 *Under voluntary participation, sincere voting is incentive compatible.*

Proposition 3 shows that it is optimal for each participating voter to vote according to his or her own private signal alone, provided that others are doing so. One may speculate that equilibrium participation rates are such that, conditional on being pivotal, the posterior assessment of α and β is 50-50. Thus, a voter's own signal "breaks the tie" and sincere voting is optimal. This simple intuition turns out to be incorrect, however. In Example 1, for instance, this posterior assessment favors state β slightly; that is, $\Pr[\alpha \mid Piv_A \cup Piv_B] < \frac{1}{2}$. But once an a type voter takes his own signal also into account, the posterior assessment favors α, that is, $\Pr[\alpha \mid a, Piv_A \cup Piv_B] > \frac{1}{2}$.

5 Large Elections

Together, Propositions 2 and 3 show that there exist a pair of positive equilibrium participation rates which induce sincere voting. In this section, we study the limiting behavior of these rates. We will show that although the participation rates go to zero as n increases, they do so sufficiently slowly so that the expected number of voters goes to infinity.

The approximation in (4) implies that if $\sqrt{\sigma_A \sigma_B} \to \infty$, as $n \to \infty$ then, for large n

$$\Pr[T \mid \alpha] \approx \frac{e^{-\left(\sigma_A + \sigma_B - 2\sqrt{\sigma_A \sigma_B}\right)}}{\sqrt{4\pi\sqrt{\sigma_A \sigma_B}}} = \frac{e^{-\left(\sqrt{\sigma_A} - \sqrt{\sigma_B}\right)^2}}{\sqrt{4\pi\sqrt{\sigma_A \sigma_B}}} \tag{7}$$

Also, the probability of "offset" events of the form T_{+1} or T_{-1} can be approximated as follows

$$\Pr[T_{\pm 1} \mid \alpha] \approx \Pr[T \mid \alpha] \times \left(\frac{\sigma_A}{\sigma_B}\right)^{\pm\frac{1}{2}} \tag{8}$$

And of course, the corresponding probabilities in state β can again be approximated by substituting τ for σ.

The probabilities of the pivotal events defined in Section 2 can then be approximated by using (7) and (8).[6] In state α,

$$\Pr[Piv_A \mid \alpha] \approx \tfrac{1}{2}\Pr[T \mid \alpha] \times \left(1 + \sqrt{\tfrac{\sigma_B}{\sigma_A}}\right) \tag{9}$$

$$\Pr[Piv_B \mid \alpha] \approx \tfrac{1}{2}\Pr[T \mid \alpha] \times \left(1 + \sqrt{\tfrac{\sigma_A}{\sigma_B}}\right) \tag{10}$$

Again, the pivot probabilities in state β can similarly be obtained by substituting τ for σ.

As a first step we have[7]

Lemma 3 *In any sequence of sincere voting equilibria, the participation rates tend to zero; that is,* $\limsup p_a(n) = \limsup p_b(n) = 0$.

To see why this is the case, suppose to the contrary that one or both types of voters participated at positive rates even in the limit. Then an infinite number of

[6] The approximation formulae for the pivot probabilities also follow from Myerson (2000).

[7] Unless otherwise specified, all limits are taken as $n \to \infty$.

voters would turn out and the gross benefit to voting would go to zero since there is no chance that an individual's vote would be pivotal. Since voting is costly, a voter would be better off staying at home than voting under these circumstances. Of course, this contradicts the notion that participation rates are positive in the limit.

On its face, Lemma 3 seems inconsistent with observed turnout rates in large elections. Indeed, a general criticism of costly voting models is that they predict implausibly low rates of voter participation. However, when voting costs are heterogeneous, this is no longer the case. For a fixed (expected) electorate n, there exist voting cost distributions F, and signal precisions, r and s, that are capable of rationalizing observed turnout rates. Consider a typical Congressional election in the US.[8] The average number of the eligible voters in a Congressional district is about 400,000 and typical turnout rates in non-presidential election years are about 45%.

Example 2 *Consider an expected electorate* $n = 400,000$. *Suppose the signal precisions* $r = 0.65$ *and* $s = 0.55$ *and that the voting costs are distributed according to* $F(c) = c^{\frac{1}{4000}}$. *Then* $p_a^* \approx 0.4$ *and* $p_b^* \approx 0.5$.

While Lemma 3 shows that, for a fixed cost distribution F, participation rates go to zero as the number of potential voters goes to infinity, there is, in fact, a race between the shrinking participation rates and the growing size of the electorate. A common intuition is that the outcome of this race depends on the shape of the cost distribution—particularly in the neighborhood of 0. As we show below, however, sincere voting equilibria have the property that the number of voters (of either type) becomes unbounded regardless of the shape of the cost distribution. In other words, the problem of too little participation does not arise in the limit—even though voting is voluntary and costly. Formally,

Proposition 4 *In any sequence of sincere voting equilibria, the expected number of voters with either signal tends to infinity; that is,*

$$\liminf np_a(n) = \infty = \liminf np_b(n)$$

Proof. The proof is a direct consequence of Lemmas 6 and 7 in Appendix B. ■

On its face, the result seems intuitive. If there is only a finite turnout in expectation, then there is a positive probability that a voter is pivotal and, one might guess, this would mean that there is a positive benefit from voting; thus contradicting the idea that the cost thresholds go to zero in the limit. However, the mere fact of being pivotal with positive probability is no guarantee of a positive benefit from voting. It may well be that, conditional on being pivotal, the likelihood ratio is exactly 50-50 under sincere voting. In that case, there would be no benefit from voting whatsoever and hence the cost threshold would, appropriately, go to zero.

To gain some intuition for why this is never the case, it is helpful to consider what happens when a and b signals are equally precise, that is, when $r = s$. It is

[8] The Electoral College system in US Presidential races complicates what it means to be pivotal. Hence, we illustrate the model using House races.

easy to see that in that case, the participation rates for a and b voters will be the same, and hence the likelihood of a given state will depend only on the relative vote totals. Consider a voter with an a signal when aggregate turnout is finite. This voter is pivotal under two circumstances—when A is behind by a vote and when the vote total is tied. When A is behind by a vote, the inclusion of the voter's own a signal leads to a 50-50 likelihood of α versus β. In other words, when the voter includes her own signal, these events are not decisive as to the likelihood of α versus β. When the vote total is tied, the likelihood ratio favors α. Thus, the overall likelihood ratio favors α.

Of course, when signal precisions are not the same, turnout rates are no longer equal and the inference from the vote totals is more complicated. However, when voting is efficient (that is, A is more likely to win in state α), then the same basic intuition obtains. Voters endogenously participate in such a way that the likelihood ratios turn on the tie events rather than on the events in which A is either ahead or behind by one vote. As a consequence, the likelihood ratio for a voter with an a signal favors α and hence there is a strictly positive benefit to voting. This, in turn, implies that the expected number of voters becomes unbounded. For the inefficient case, the argument is more delicate. The formal proof, which is somewhat involved, shows, however, that the likelihood ratio cannot be 50-50 for both sides.

We now turn to the question of whether the equilibrium is efficient under costly voting. In other words, is it the case that in large elections, the "right" candidate is elected? One may have thought that we have, in effect, already answered this question (in the affirmative) by showing that voting is sincere and expected participation is unbounded in large elections. However, this ignores that the fact that voters with different signals turn out at different rates. If turnout is too lop-sided in favor of B versus A, then even with sincere voting, the election could still fail to choose the "right" candidate.

5.1 Information Aggregation

In large elections, candidate A is chosen in state α if and only if $rp_a > (1-r)\, p_b$ and candidate B is chosen in state β if and only if $(1-s)\, p_a < sp_b$. Information aggregation thus requires that for large n, the equilibrium participation rates satisfy

$$\frac{1-r}{r} < \frac{p_a}{p_b} < \frac{s}{1-s} \tag{11}$$

First, recall from Lemma 1 that any solution to the threshold equations satisfies $p_a < p_b$. Thus in large elections, in equilibrium, b types turn out to vote at higher rates than do a types. Since $s > \frac{1}{2}$, this implies that the second inequality holds and so in large elections, B wins in state β with probability 1.

In state α, however, the larger turnout for B is detrimental. We now argue that in large elections, the first inequality also holds.

First, note that since with sincere voting, it follows from Lemma 7 (in Appendix

B) that

$$\limsup \left(\frac{\sigma_A}{\sigma_B}\right)^{\pm\frac{1}{2}} < \infty \text{ and } \limsup \left(\frac{\tau_A}{\tau_B}\right)^{\pm\frac{1}{2}} < \infty$$

Hence in the expressions for the pivot probabilities (specifically, (9) and the corresponding formula in state β), the exponential terms dominate in the limit. Thus we have

$$\frac{\Pr\left[Piv_A \mid \alpha\right]}{\Pr\left[Piv_A \mid \beta\right]} = \frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}} \times K\left(\sigma_A, \sigma_B, \tau_A, \tau_B\right)$$

where K is a function that stays finite in the limit.

Thus it must be the case that in the limit

$$\left(\sqrt{\sigma_A} - \sqrt{\sigma_B}\right)^2 = \left(\sqrt{\tau_A} - \sqrt{\tau_B}\right)^2 \tag{12}$$

In particular, suppose that the left-hand side of (12) was greater than the right-hand side. In that case,

$$\lim \frac{\Pr\left[Piv_A \mid \alpha\right]}{\Pr\left[Piv_A \mid \beta\right]} = 0$$

and it would then follow that state β is infinitely more likely in the event Piv_A than is state α. This, however, would imply that the gross benefit to a voter with signal a from voting is negative, which contradicts Lemma 3. Similarly, if the left-hand side was smaller then it would then follow that state α is infinitely more likely in the event Piv_B than is state β. This, however, would then imply that the gross benefit to a voter with signal b from voting is negative, which also contradicts Lemma 3. Thus (12) must hold in the limit.

Under sincere voting $\sigma_A = nrp_a$; $\sigma_B = n(1-r)p_b$; $\tau_A = n(1-s)p_a$ and $\tau_B = nsp_b$, and so (12) can be rewritten as

$$\sqrt{s} - \sqrt{1-s}\sqrt{\tfrac{p_a}{p_b}} \approx \pm\left(\sqrt{r}\sqrt{\tfrac{p_a}{p_b}} - \sqrt{1-r}\right)$$

and the left-hand side is positive since $p_b > p_a$. Now observe that if $(1-r)p_b \geq rp_a$, then we have

$$\sqrt{s} - \sqrt{1-s}\sqrt{\tfrac{p_a}{p_b}} \approx \sqrt{1-r} - \sqrt{r}\sqrt{\tfrac{p_a}{p_b}}$$

and this is impossible since both r and s are greater than $\frac{1}{2}$ (Lemma 7 in Appendix B ensures that $\frac{p_a}{p_b}$ is bounded). Thus we must have, that for large n, $rp_a > (1-r)p_b$.

We have thus shown that information fully aggregates in large elections.

Proposition 5 *In any sequence of sincere voting equilibria, the probability that the right candidate is elected in each state (A in state α and B in β) goes to one.*

Note that as a result of the reasoning above, we know that

$$\sqrt{s} - \sqrt{1-s}\sqrt{\tfrac{p_a}{p_b}} \approx \sqrt{r}\sqrt{\tfrac{p_a}{p_b}} - \sqrt{1-r}$$

and so we obtain that ratio of the participation probabilities satisfies

$$\lim \sqrt{\frac{p_a}{p_b}} = \frac{\sqrt{1-r}+\sqrt{s}}{\sqrt{r}+\sqrt{1-s}} \tag{13}$$

6 Uniqueness

In this section, we show that with voluntary and costly voting, there is a unique equilibrium when n is large. Recall that the equilibrium derived in the previous sections has the following features: (i) voting is sincere; and (ii) the cost thresholds are determined by IRa and IRb.

The uniqueness of the equilibrium is established in two steps. In the first step, we show that *all* equilibria must involve sincere voting (this result does not require n to be large). It is easy to see that at least one type must vote sincerely. Let $U(A,a)$ denote type a's payoff from voting for A. Similarly, define $U(B,a)$, $U(A,b)$ and $U(B,b)$. If neither type votes sincerely, then we have

$$U(A,a) > U(A,b) \geq U(B,b)$$

where the first inequality follows from the fact that all else being equal, voting for A must be better having received a signal in favor of A than a signal in favor of B. The second inequality follows from the fact that b types find it profitable to vote insincerely. At the same time, we also have

$$U(B,b) > U(B,a) \geq U(A,a)$$

and the two inequalities contradict each other. To show that, in fact, both types vote sincerely we show that the Likelihood Ratio Lemma holds even if voting is insincere. The Likelihood Ratio condition then shows that it cannot be a best response for either type to vote insincerely (Lemma 10 in Appendix C). Thus all equilibria involve sincere voting.

The second and final step is to show that when n is large, there is a unique solution to the cost thresholds. We know that in the limit, all sincere voting equilibria are efficient: A wins in state α and B wins in state β. Thus, for large n, the equilibrium participation probabilities satisfy (11). It can be shown that for any pair of participation probabilities satisfying (11), the IRa curve is steeper than the IRb curve (Lemma 11 in Appendix C). Thus they can intersect only once and so we obtain,

Proposition 6 *In large elections, there is a unique equilibrium.*

Proof. See Appendix C. ■

7 Welfare

We now turn to the welfare properties of the voluntary voting model. As a benchmark, consider a planner who can choose both turnout rates and voting behavior. Suppose further that voting costs are of no concern; the planner only wishes to maximize the probability that the right candidate is chosen. The optimal policy seems obvious: Surely the planner can do no better than to require that all voters come to the polls and vote sincerely. After all, this uses all of society's available information in making a choice and, presumably more information leads to better choices.

While this intuition seems compelling, it is, in fact, incorrect. The flaw is that, while the average informational contribution of a voter is positive, the marginal contribution need not be. To see this, consider the supposedly ideal situation where everyone is participating and voting sincerely. What happens if the participation by voters with a signals decreases by a small amount? The welfare impact of decreased participation comes only from tied or near-tied outcomes. Since voters have noisier signals in state β, ties and near-ties are more likely in this state.[9] Thus, a decrease in participation by a types increases the error rate in state α, but reduces it in state β. Since the latter is more likely, the net effect of reduced participation by a types is to *increase* welfare. Proposition 7 formalizes this argument.

Proposition 7 *Suppose that* $n \geq \frac{\ln\left(\frac{r}{1-s}\right)}{2\left(\sqrt{s(1-s)}-\sqrt{r(1-r)}\right)}$. *Then full participation and sincere voting are not welfare optimal.*

Proof. The probability that A wins in state α is

$$W(\alpha) = \tfrac{1}{2}\Pr[T \mid \alpha] + \sum_{m=1}^{\infty} \Pr[T_m \mid \alpha]$$

where T_m denotes the set of events in which A beats B by m votes. Similarly, the probability that B wins in state β is

$$W(\beta) = \tfrac{1}{2}\Pr[T \mid \beta] + \sum_{m=1}^{\infty} \Pr[T_{-m} \mid \beta]$$

where T_{-m} denotes the set of events in which B beats A by m votes. Let $W = \frac{1}{2}W(\alpha) + \frac{1}{2}W(\beta)$ denote the overall probability that the right candidate wins.

We will argue that when $p_a = 1$ and $p_b = 1$; that is, there is full participation and voting is sincere,

$$\frac{\partial W}{\partial p_a} < 0$$

To see this note that for all m,

$$\begin{aligned} \frac{\partial \Pr[T_m \mid \alpha]}{\partial p_a} &= nr\left(\Pr[T_{m-1} \mid \alpha] - \Pr[T_m \mid \alpha]\right) \\ \frac{\partial \Pr[T_{-m} \mid \beta]}{\partial p_a} &= n(1-s)\left(\Pr[T_{-m-1} \mid \beta] - \Pr[T_{-m} \mid \beta]\right) \end{aligned}$$

[9] This requires a modestly large number of voters. Obviously, if there is only a single voter, ties are equally likely in both states. A precise definition of "modestly large" is offered in the proposition below.

and some routine calculations using the formulae for $W(\alpha)$ and $W(\beta)$ shows that

$$\frac{\partial W}{\partial p_a} = \tfrac{1}{2} nr \Pr[Piv_A \mid \alpha] - \tfrac{1}{2} n (1-s) \Pr[Piv_A \mid \beta]$$

Next observe that when $p_a = 1$ and $p_b = 1$,

$$nr \Pr[Piv_A \mid \alpha] = \tfrac{1}{2} e^{-n} n \left(r I_0 \left(2n\sqrt{r(1-r)} \right) + \sqrt{r(1-r)} I_1 \left(2n\sqrt{r(1-r)} \right) \right) \tag{14}$$

whereas

$$n(1-s) \Pr[Piv_A \mid \beta] = \tfrac{1}{2} e^{-n} n \left((1-s) I_0 \left(2n\sqrt{s(1-s)} \right) + \sqrt{s(1-s)} I_1 \left(2n\sqrt{s(1-s)} \right) \right) \tag{15}$$

Notice that since $r > s$, $s(1-s) > r(1-r)$ and I_1 is an increasing function, the second term in (15) is greater than the second term in (15). A sufficient condition for the first term in (15) to be greater than the first term in (15) is that

$$\frac{\ln\left(\frac{r}{1-s}\right)}{2\left(\sqrt{s(1-s)} - \sqrt{r(1-r)}\right)} < n$$

This last inequality comes from the fact that $I_0(x)/I_0(y) > e^{x-y}$ (see Joshi and Bissu, 1991). ■

Remark 1 *The lower bound on n in the proposition is not too stringent. For instance, if $r = \frac{3}{4}$ and $s = \frac{2}{3}$ then $n \geq 11$ is sufficient.*

The proposition carries with it a surprising implication: policies designed to raise turnout can be harmful purely on informational grounds. The problem of low turnout in elections has led over 40 countries, including Australia, Belgium, Italy as well as most of South America, to adopt mandatory voting laws. Some countries, such as Australia, impose fines to ensure compliance. Others, such as Greece, make it difficult for non-voting citizens to obtain (and renew) driver's licenses and passports. In Bolivia, the sanctions are even more extreme—abstaining can result in the freezing of a citizen's bank account. Proposition 7 suggests that, if successful in achieving full turnout, such schemes are welfare reducing—even if one ignores voting costs entirely.

But what is the optimal scheme? In general, the planner will trade off voting costs of increased participation against informational benefits. The exact resolution of this trade-off will, of course, depend on the cost distribution, and offers little new insight. Obviously, the planner will simply equate the marginal benefit of additional participation with the marginal voting cost.

Instead, we consider a frictionless environment where voting costs are entirely absent, and derive the optimal scheme. First, consider a purely voluntary scheme. When n is large, this scheme has a *unique* equilibrium as the following proposition shows:

Proposition 8 *In large elections under voluntary voting with zero costs, there is a unique equilibrium: (i) all b types vote; (ii) a types vote with probability p_a; and (iii) all those who vote, vote sincerely. The sequence $p_a(n)$ satisfies*

$$\lim p_a(n) = \left(\frac{\sqrt{1-r} + \sqrt{s}}{\sqrt{r} + \sqrt{1-s}} \right)^2 \tag{16}$$

Proof. It is routine to verify that there is an equilibrium in which all b types vote and a types mix between voting and staying at home and so U_a the payoff of an a type must be zero. The limit of the mixing probability can be found by the approximation formulae for the pivotal probabilities.

We now argue that there is only one equilibrium when n is large. First, note that Lemma 10 continues to hold even with zero costs so that in *any* equilibrium, voting must be sincere. Thus it remains to show that equilibrium participation rates are uniquely determined. The equilibrium participation rates p_a and p_b cannot both be strictly less than one because then the payoffs of both types, U_a and U_b, must be zero. But Lemmas 4 and 5 together imply that this is impossible. Hence at least one of the two types must participate fully, that is, either $p_a = 1$ or $p_b = 1$. Since, in equilibrium we must have $p_b \geq p_a$ (see Lemma 1), it must be that $p_b = 1$. Finally, in large elections, we know from the proof of Lemma 11 that

$$\frac{\partial U_a}{\partial p_a} < 0$$

and so there is a unique p_a such that $U_a(p_a, 1) = 0$. ■

While it is nice to know that equilibrium multiplicity is not a problem under the voluntary scheme, Proposition 8 has deeper welfare implications. Absent voting costs, majority-rule voting is a common interest game and so there exists an efficient equilibrium (McLennan, 1998). Since there is a *unique* equilibrium under the voluntary scheme, it then follows that it must be efficient. That is:

Proposition 9 *In large elections with zero voting costs, voluntary voting is optimal.*

Proposition 9 implies, among other things, that schemes designed to raise turnout offer no informational benefit. At best, such schemes are equivalent to voluntary voting. Is voluntary voting is welfare equivalent to compulsory voting when costs are zero?[10] After all, both schemes lead to full information aggregation in the limit. And one might conjecture that the equilibrium under voluntary voting with private costs is merely a purification of the mixed equilibrium under compulsory voting.[11]

This is not the case, however, since pivotality considerations in the two schemes differ. Under voluntary voting, the behavior of a types is determined by a comparison

[10] In practice, a voter may show up to the polls while still abstaining by handing in a blank ballot. This, however, is not what proponents of compulsory voting would view as an ideal situation. In our setting, we thus equate compulsory voting with voting for one of the two candidates.

[11] Equilibrium of the voluntary voting model with small private costs is, of course, a purification of the mixed equilibrium of the voluntary voting model with zero costs.

of the payoffs from voting for A versus abstaining—that is, by the Piv_A events alone. Under compulsory voting, the behavior of a types is determined by a comparison of the payoffs from voting for A versus voting for B—that is, by both the Piv_A events and the Piv_B events. We now turn to a more detailed comparison.

In Section 3 we showed that when voting is compulsory, it is not an equilibrium for all voters to vote sincerely. While b type voters vote sincerely in equilibrium, a type voters mix between voting for A and voting for B. The mixing probability is determined by the condition that for a types, the likelihood ratio of α to β conditional on being pivotal is one. This is equivalent to requiring that

$$\begin{aligned} U(A,a) &\equiv q(\alpha \mid a)\Pr[Piv_A \mid \alpha] - q(\beta \mid a)\Pr[Piv_A \mid \beta] \\ &= q(\beta \mid a)\Pr[Piv_B \mid \beta] - q(\alpha \mid a)\Pr[Piv_B \mid \alpha] \equiv U(B,a) \end{aligned} \tag{17}$$

That is, the returns to voting for A are equal to the returns from voting for B. As we show in the next proposition, for large elections, there is a unique mixing probability satisfying equation (17).

Proposition 10 *In large elections under compulsory voting, there is a unique equilibrium: (i) all b types vote for B; (ii) all a types vote for A with probability $\mu < 1$. The sequence $\mu(n)$ satisfies*

$$\lim \mu(n) = \frac{1}{1+r-s} \tag{18}$$

Again, we omit a detailed proof. The limit of the mixing probability μ for a type voters is the condition that a types are indifferent between voting for A and voting for B. Since the exponential terms dominate in the limit, the condition that $U(A,a) = U(B,a)$ requires that

$$e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2} = e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}$$

and this is easily verified to be equivalent to

$$\mu = \frac{1}{1+r-s}$$

It may be readily verified that the expected vote shares in each state under the compulsory scheme are not equal to those under the voluntary scheme. However, the uniqueness result contained in Proposition 10 implies that it is efficient in the class of all schemes involving full turnout. It remains to compare welfare under the two schemes directly. The social welfare $W(\alpha)$ in state α can be written as:

$$W(\alpha) = 1 - e^{-(\sigma_A+\sigma_B)}\left(\tfrac{1}{2}I_0\left(2\sqrt{\sigma_A\sigma_B}\right) + \sum_{m=1}^{\infty}\left(\sqrt{\tfrac{\sigma_B}{\sigma_A}}\right)^m I_m\left(2\sqrt{\sigma_A\sigma_B}\right)\right)$$

where,

$$I_m(z) = \sum_{k=m}^{\infty} \frac{\left(\frac{z}{2}\right)^{k-m}}{(k-m)!} \frac{\left(\frac{z}{2}\right)^{k}}{k!}$$

is associated with the event that A wins by m votes. Since for all m, the leading term of $I_m(z)$ is $\frac{e^z}{\sqrt{2\pi z}}$ (Abramowitz and Stegun, 1965, p. 377), then, under the assumption that $\sigma_B < \sigma_A$, we obtain the approximation

$$W(\alpha) \approx 1 - e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2} \left(\frac{1}{2} + \frac{\sqrt{\frac{\sigma_B}{\sigma_A}}}{1-\sqrt{\frac{\sigma_B}{\sigma_A}}} \right) \frac{1}{\sqrt{4\pi\sqrt{\sigma_A \sigma_B}}} \tag{19}$$

The welfare in state β can be written similarly by substituting τ for σ and exchanging A and B.We are now in a position to show:

Proposition 11 *In large elections with zero costs, voluntary voting is welfare superior to compulsory voting.*

Proof. We prove that welfare in state α is higher under voluntary voting than under compulsory voting. The proof for state β is analogous.

In the limit, under voluntary voting, $\sigma_A = nrp_a$ and $\sigma_B = n(1-r)$ whereas, in the limit, under compulsory voting, $\sigma_A = nr\mu$ and $\sigma_B = n(1-r\mu)$.

From (19), it follows that in large elections, a welfare comparison rests only on the exponential term. Specifically, we will show that the term $\sqrt{\sigma_A} - \sqrt{\sigma_B}$ is higher under voluntary voting than under compulsory voting; that is,

$$\sqrt{rp_a} - \sqrt{1-r} > \sqrt{r\mu} - \sqrt{1-r\mu} \tag{20}$$

Substituting from (18) and (16) the inequality in (20) can be rewritten as

$$\sqrt{r}\frac{\sqrt{s}+\sqrt{1-r}}{\sqrt{r}+\sqrt{1-s}} - \sqrt{1-r} > \frac{\sqrt{r}-\sqrt{1-s}}{\sqrt{1+r-s}}$$

and we will establish the inequality

$$\sqrt{r}\frac{\sqrt{s}+\sqrt{1-r}}{\sqrt{r}+\sqrt{1-s}} - \sqrt{1-r} > \sqrt{r}-\sqrt{1-s} \tag{21}$$

which is is stronger because $r > s$, and so $1 + r - s > 1$.

The inequality in (21) may be rearranged as

$$\sqrt{r}\sqrt{s} > r + s + \sqrt{1-r}\sqrt{1-s} - 1 \tag{22}$$

Now when $r = s$, the two sides are equal and it may be verified that for fixed s, the derivative of the left-hand side of (22) with respect to r is greater than the derivative of the right-hand side. This completes the proof. ■

To gain some intuition for the result, notice that under the compulsory scheme, players with a signals earn *negative* expected payoffs in equilibrium. They would rather not vote at all. These players earn zero payoffs in equilibrium under the voluntary scheme, so this clearly represents an improvement. Of course, it is possible that the sacrifice of the a types under the compulsory scheme is more than compensated by welfare gains to the b types. This turns out not to be the case, but requires the formal machinery of Proposition 11 to show.

8 Conclusions

Rational choice models of voting behavior have long been criticized on behavioral grounds. They require voters to employ mixed strategies, they imply that swing voters would prefer not to come to the polls, and when voting is costly, they beg the question as to why anyone should bother to vote at all.

Many of these problems disappear if one amends the standard model to allow for realistic features such as the possibility of abstention and heterogeneous costs of going to the polls. With these additions, there is no longer a conflict between sincere and strategic voting and swing voters willingly participate. Moreover, voting in large elections nearly always produces the "right" outcome.

The model allows for comparisons of various policies designed to increase turnout with a purely voluntary scheme. The common intuition is that more turnout produces better outcomes. We show that this intuition is incorrect—full participation is never optimal even when voting is costless. Moreover, when voting costs are not an important welfare consideration, an even sharper result emerges: laissez-faire is best. A social planner cannot improve over a purely voluntary voting scheme.

A Appendix: Equilibrium

Proof of Proposition 2. It is useful to rewrite IRa and IRb in terms of threshold costs rather than participation probabilities. Let $V_a(c_a, c_b)$ denote the payoff to a voter with signal a from voting for A when the two threshold costs are $c_a = F^{-1}(p_a)$ and $c_b = F^{-1}(p_b)$; that is, $V_a(c_a, c_b) \equiv U_a(F(c_a), F(c_b))$. Similarly, let $V_b(c_a, c_b) \equiv U_b(F(c_a), F(c_b))$. We will show that there exist $(c_a, c_b) \in (0,1)^2$ such that $V_a(c_a, c_b) = c_a$ and $V_b(c_a, c_b) = c_b$.

The function $V = (V_a, V_b) : [0,1]^2 \to [-1,1]^2$ maps a pair of threshold costs to a pair of payoffs from voting sincerely. Note that payoffs may be negative.

Consider the function $V^+ : [0,1]^2 \to [0,1]^2$ defined by

$$\begin{aligned} V_a^+(c_a, c_b) &= \max\{0, V_a(c_a, c_b)\} \\ V_b^+(c_a, c_b) &= \max\{0, V_b(c_a, c_b)\} \end{aligned}$$

Since V is a continuous function, V^+ is also continuous and so by Brouwer's Theorem V^+ has a fixed point, say $(c_a^*, c_b^*) \in [0,1]^2$.

We argue that c_a^* and c_b^* are strictly positive. Suppose that $c_a^* = 0$. Then $p_a^* = F(c_a^*)$ is also zero and so there are no a types who vote. Consider an individual who receives a signal of a. The only events in which a vote for A is pivotal is if either (i) no b types show up to vote; or (ii) a single b type shows up. Thus

$$\begin{aligned}\Pr[Piv_A \mid \alpha] &= \tfrac{1}{2}e^{-n(1-r)p_b^*}\left(1+n(1-r)p_b^*\right)\\ \Pr[Piv_A \mid \beta] &= \tfrac{1}{2}e^{-nsp_b^*}\left(1+nsp_b^*\right)\end{aligned}$$

where $p_b^* = F(c_b^*)$. We claim that $\Pr[Piv_A \mid \alpha] > \Pr[Piv_A \mid \beta]$. This follows from the fact that the function $e^{-x}(1+x)$ is strictly decreasing for $x > 0$ and $s > 1-r$. Hence, if $p_a^* = 0$

$$q(\alpha \mid a)\Pr[Piv_A \mid \alpha] - q(\beta \mid a)\Pr[Piv_A \mid \beta] > 0$$

since $q(\alpha \mid a) > \frac{1}{2}$. Since $c_a^* = 0$, this is equivalent to

$$V_a^+(c_a^*, c_b^*) > c_a^*$$

contradicting the assumption that (c_a^*, c_b^*) was a fixed point. Thus $c_a^* > 0$.

A similar argument shows that $c_b^* > 0$.

Since both c_a^* and c_b^* are strictly positive, we have that

$$V^+(c_a^*, c_b^*) = V(c_a^*, c_b^*) = (c_a^*, c_b^*)$$

Thus (c_a^*, c_b^*) is also a fixed point of V and so solves IRa and IRb.

Next, notice that at any point $(1, p_b)$

$$q(\alpha \mid a)\Pr[Piv_A \mid \alpha] - q(\beta \mid a)\Pr[Piv_A \mid \beta] < 1$$

Thus if (c_a^*, c_b^*) is a fixed point of V then we also have that both c_a^* and c_b^* are also less than one. ■

Proof of Lemma 1. We claim that if $p_a \geq p_b$, then $U_a(p_a, p_b) < U_b(p_a, p_b)$. A rearrangement of the relevant expressions shows that $U_a(p_a, p_b) < U_b(p_a, p_b)$ is equivalent to

$$(q(\alpha \mid a) + q(\alpha \mid b))\Pr[T \mid \alpha] + q(\alpha \mid a)\Pr[T_{-1} \mid \alpha] + q(\alpha \mid b)\Pr[T_{+1} \mid \alpha] \qquad (23)$$

being less than

$$(q(\beta \mid b) + q(\beta \mid a))\Pr[T \mid \beta] + q(\beta \mid a)\Pr[T_{-1} \mid \beta] + q(\beta \mid b)\Pr[T_{+1} \mid \beta] \qquad (24)$$

We will show that each term in (23) is less than the corresponding term in (24).

With sincere voting, $\sigma_A = nrp_a$, $\sigma_B = n(1-r)p_b$, $\tau_A = n(1-s)p_a$ and $\tau_B = nsp_b$.

First, since $r > s > \frac{1}{2}$, we have $\sigma_A\sigma_B < \tau_A\tau_B$ and since $p_a \geq p_b$, $\sigma_A + \sigma_B \geq$

$\tau_A + \tau_B$. Thus,

$$\begin{aligned}\Pr\left[T \mid \alpha\right] &= e^{-\sigma_A-\sigma_B}\sum_{k=0}^{\infty}\frac{\sigma_A^k}{k!}\frac{\sigma_B^k}{k!}\\ &< e^{-\tau_A-\tau_B}\sum_{k=0}^{\infty}\frac{\tau_A^k}{k!}\frac{\tau_B^k}{k!}\\ &= \Pr\left[T \mid \beta\right]\end{aligned}$$

It is also easily verified that $q(\alpha \mid a) + q(\alpha \mid b) < q(\beta \mid b) + q(\beta \mid a)$.

Second, since $r > s > \frac{1}{2}$, we have for all $k \geq 1$, $r\sigma_A^{k-1}\sigma_B^k < (1-s)\,\tau_A^{k-1}\tau_B^k$. Thus,

$$\begin{aligned}q(\alpha \mid a)\Pr\left[T_{-1} \mid \alpha\right] &= e^{-\sigma_A-\sigma_B}\frac{r}{r+1-s}\sum_{k=1}^{\infty}\frac{\sigma_A^{k-1}}{(k-1)!}\frac{\sigma_B^k}{k!}\\ &< e^{-\tau_A-\tau_B}\frac{1-s}{r+1-s}\sum_{k=1}^{\infty}\frac{\tau_A^{k-1}}{(k-1)!}\frac{\tau_B^k}{k!}\\ &= q(\beta \mid a)\Pr\left[T_{-1} \mid \beta\right]\end{aligned}$$

Third, a similar argument establishes that

$$q(\alpha \mid b)\Pr\left[T_{+1} \mid \alpha\right] < q(\beta \mid b)\Pr\left[T_{+1} \mid \beta\right]$$

Combining these three facts establishes that (23) is less than (24)

This means that if $p_a^* \geq p_b^*$, then (p_a^*, p_b^*) cannot satisfy IRa and IRb. Thus $p_a^* < p_b^*$. ■

Proof of Lemma 2. Consider the functions

$$\begin{aligned}G(x,y) &= I_0(z) + \sqrt{\tfrac{y}{x}}I_1(z)\\ H(x,y) &= I_0(z) + \sqrt{\tfrac{x}{y}}I_1(z)\end{aligned}$$

where $z = 2\sqrt{xy}$. Note that inequality (6) is equivalent to

$$\frac{G(\tau_A,\tau_B)}{H(\tau_A,\tau_B)} > \frac{G(\sigma_A,\sigma_B)}{H(\sigma_A,\sigma_B)} \tag{25}$$

We will argue that G/H is decreasing in x and increasing in y. Since $\sigma_A > \tau_A$ and $\sigma_B < \tau_B$, this will establish the inequality (25).

It may be verified that

$$\begin{aligned}
HG_x - GH_x &= \left(I_0(z) + \sqrt{\tfrac{x}{y}} I_1(z)\right)\left(\tfrac{y}{x} I_0(z) + \left(1 - \tfrac{1}{x}\right)\sqrt{\tfrac{y}{x}} I_1(z)\right) \\
&\quad - \left(I_0(z) + \sqrt{\tfrac{y}{x}} I_1(z)\right)^2 \\
&= -\tfrac{1}{x}\left((y-x)\left(I_1(z)^2 - I_0(z)^2\right) + \sqrt{\tfrac{y}{x}} I_0(z) I_1(z) + I_1(z)^2\right) \\
&= -\tfrac{1}{x} g(x,y)
\end{aligned}$$

where

$$g(x,y) = (y-x)\left(I_1(z)^2 - I_0(z)^2\right) + \sqrt{\tfrac{y}{x}} I_0(z) I_1(z) + I_1(z)^2$$

We claim that $g(x,y) > 0$, whenever x and y are positive. Note that for any $y > 0$,

$$\lim_{x \to 0} g(x,y) = 0$$

Some routine calculations show that

$$g_x(x,y) = \left(I_0(z) + \sqrt{\tfrac{y}{x}} I_1(z)\right)^2 + \left(I_0(z)^2 - I_1(z)^2\right) - \frac{1}{x} g(x,y)$$

Thus, if $g(x,y) \leq 0$, then $g_x(x,y) > 0$ (recall that $I_0(z) > I_1(z)$). This implies that for all $x > 0$, $g(x,y) > 0$ and so $HG_x - GH_x < 0$.

It may also be verified that

$$\begin{aligned}
HG_y - GH_y &= \left(I_0(z) + \sqrt{\tfrac{x}{y}} I_1(z)\right)^2 \\
&\quad - \left(I_0(z) + \sqrt{\tfrac{y}{x}} I_1(z)\right)\left(\tfrac{x}{y} I_0(z) + \left(1 - \tfrac{1}{y}\right)\sqrt{\tfrac{x}{y}} I_1(z)\right) \\
&= \tfrac{1}{y}\left((x-y)\left(I_1(z)^2 - I_0(z)^2\right) + \sqrt{\tfrac{x}{y}} I_0(z) I_1(z) + I_1(z)^2\right) \\
&= \tfrac{1}{y} h(x,y)
\end{aligned}$$

where $h(x,y) = g(y,x)$. The same reasoning now shows that so for all $y > 0$, $HG_y - GH_y > 0$.

This completes the proof. ■

B Appendix: Large Elections

Proof of Lemma 3. Suppose to the contrary, that for some sequence, $\lim c_a(n) > 0$. In that case, the gross benefits (excluding the costs of voting) to voters with a signals from voting must be positive; that is

$$\lim \left(q(\alpha \mid a) \Pr[Piv_A \mid \alpha] - q(\beta \mid a) \Pr[Piv_A \mid \beta]\right) > 0$$

where it is understood that the probabilities depend on n.

We know that along the given sequence, $\lim p_a(n) > 0$. This implies that $\lim \sigma_A(n) = \lim nrp_a(n) = \infty$.

First, suppose that there is a subsequence along which $\lim \sqrt{\sigma_A \sigma_B} < \infty$. In that case,

$$\Pr[Piv_A \mid \alpha] = \tfrac{1}{2} e^{-\sigma_A - \sigma_B} \left(I_0 \left(2\sqrt{\sigma_A \sigma_B}\right) + \sqrt{\tfrac{\sigma_B}{\sigma_A}} I_1 \left(2\sqrt{\sigma_A \sigma_B}\right) \right)$$

and since $\lim \left(e^{-\sigma_A} / \sqrt{\sigma_A}\right) = 0$ and $\limsup e^{-\sigma_B} \sqrt{\sigma_B} < \infty$, along any such subsequence,

$$\lim \Pr[Piv_A \mid \alpha] = 0$$

Second, suppose that there is a subsequence along which $\lim \sqrt{\sigma_A \sigma_B} = \infty$. In that case,

$$\Pr[Piv_A \mid \alpha] \approx \tfrac{1}{2} \frac{e^{-\left(\sigma_A + \sigma_B - 2\sqrt{\sigma_A \sigma_B}\right)}}{\sqrt{4\pi \sqrt{\sigma_A \sigma_B}}} \left(1 + \sqrt{\tfrac{\sigma_B}{\sigma_A}}\right)$$

Notice that the denominator is unbounded while the numerator is always bounded. Hence, along any such subsequence,

$$\lim \Pr[Piv_A \mid \alpha] = 0$$

An identical argument applies for $\tau_A(n)$ and $\tau_B(n)$. Therefore,

$$\lim \Pr[Piv_A \mid \beta] = 0$$

But this means that the gross benefit of voting for A when the signal is a tends to zero. This contradicts the assumption that $\lim c_a(n) > 0$. ■

Proof of Proposition 4. The result is a consequence of a series of lemmas.

Lemma 4 *Suppose that there is a sequence of sincere voting equilibria such that* $\lim np_a(n) = n_a < \infty$ *and* $\lim np_b(n) = n_b < \infty$. *If* $rn_a \geq (1-r) n_b$, *then* $U_b = 0$ *implies* $U_a > 0$.

Proof. The condition that $U_b = 0$ is equivalent to

$$s \Pr[Piv_B \mid \beta] = (1-r) \Pr[Piv_B \mid \alpha]$$

whereas $U_a > 0$ is equivalent to

$$r \Pr[Piv_A \mid \alpha] > (1-s) \Pr[Piv_A \mid \beta]$$

We will argue that

$$\frac{r \Pr[Piv_A \mid \alpha]}{(1-s) \Pr[Piv_A \mid \beta]} > \frac{(1-r) \Pr[Piv_B \mid \alpha]}{s \Pr[Piv_B \mid \beta]}$$

or equivalently,

$$\frac{rn_a\left(\Pr\left[T\mid\alpha\right]+\Pr\left[T_{-1}\mid\alpha\right]\right)}{\left(1-s\right)n_a\left(\Pr\left[T\mid\beta\right]+\Pr\left[T_{-1}\mid\beta\right]\right)}>\frac{\left(1-r\right)n_b\left(\Pr\left[T\mid\alpha\right]+\Pr\left[T_{+1}\mid\alpha\right]\right)}{sn_b\left(\Pr\left[T\mid\beta\right]+\Pr\left[T_{+1}\mid\beta\right]\right)}$$

Now note that

$$rn_a\Pr\left[T_{-1}\mid\alpha\right]=\left(1-r\right)n_b\Pr\left[T_{+1}\mid\alpha\right]$$

and

$$\left(1-s\right)n_a\Pr\left[T_{-1}\mid\beta\right]=sn_b\Pr\left[T_{+1}\mid\beta\right]$$

and the required inequality follows from the fact that $rn_a\geq\left(1-r\right)n_b$ and $\left(1-s\right)n_a<sn_b$. ■

Lemma 5 *Suppose that there is a sequence of sincere voting equilibria such that* $\lim np_a\left(n\right)=n_a<\infty$ *and* $\lim np_b\left(n\right)=n_b<\infty$. *If* $rn_a<\left(1-r\right)n_b$, *then* $U_a>0$.

Proof. Consider the function

$$K\left(x,y\right)=e^{-x-y}\left(xI_0\left(z\right)+\tfrac{1}{2}zI_1\left(z\right)\right)$$

where $z=2\sqrt{xy}$.

Note that if $\sigma_A=rn_a$ and $\sigma_B=\left(1-r\right)n_b$, then

$$\begin{aligned}rn_a\Pr\left[Piv_A\mid\alpha\right]&=\tfrac{1}{2}\sigma_Ae^{-\sigma_A-\sigma_B}\left(I_0\left(2\sqrt{\sigma_A\sigma_B}\right)+\sqrt{\tfrac{\sigma_B}{\sigma_A}}I_1\left(2\sqrt{\sigma_A\sigma_B}\right)\right)\\&=\tfrac{1}{2}K\left(\sigma_A,\sigma_B\right)\end{aligned}$$

Similarly, if $\tau_A=\left(1-s\right)n_a$ and $\tau_B=sn_b$, then

$$\left(1-s\right)n_a\Pr\left[Piv_A\mid\beta\right]=\tfrac{1}{2}G\left(\tau_A,\tau_B\right)$$

We will show that when $x<y$, $K\left(x,y\right)$ is increasing in x and decreasing in y. Observe that

$$\begin{aligned}K_x\left(x,y\right)&=&&e^{-x-y}\left(I_0\left(z\right)+xI_0'\left(z\right)z_x+\tfrac{1}{2}\left(zI_1\left(z\right)\right)'z_x-xI_0\left(z\right)-\tfrac{1}{2}zI_1\left(z\right)\right)\\&=&&e^{-x-y}\left(I_0\left(z\right)+xI_1\left(z\right)z_x+\tfrac{1}{2}zI_0\left(z\right)z_x-xI_0\left(z\right)-\tfrac{1}{2}zI_1\left(z\right)\right)\\&=&&e^{-x-y}\left(1+y-x\right)I_0\left(z\right)\\&>&&0\end{aligned}$$

where we have used the fact that $I_0'\left(z\right)=I_1\left(z\right)$ and $\left(zI_1\left(z\right)\right)'=zI_0\left(z\right)$. Also, $xz_x=\frac{1}{2}z$ and $\frac{1}{2}zz_x=y$.

Also,

$$\begin{aligned} K_y(x,y) &= e^{-x-y}\left(xI_0'(z)\,z_y+\tfrac{1}{2}(zI_1(z))'\,z_y-xI_0(z)-\tfrac{1}{2}zI_1(z)\right)\\ &= e^{-x-y}\left(xI_1(z)\,z_y+\tfrac{1}{2}zI_0(z)\,z_y-xI_0(z)-\tfrac{1}{2}zI_1(z)\right)\\ &= e^{-x-y}\left(xz_y-\tfrac{1}{2}z\right)I_1(z)\\ &< 0 \end{aligned}$$

where we have used the fact that, $z_y z = 2x$ and $z_y = \sqrt{\frac{x}{y}} < 1$ and $x < \frac{1}{2}z$.

Finally, notice that since $rn_a < (1-r)\,n_b$

$$(1-s)\,n_a < rn_a < (1-r)\,n_b < sn_b$$

which is the same as

$$\tau_A < \sigma_A < \sigma_B < \tau_B$$

and since $K_x > 0$ and $K_y < 0$ for $x < y$, we have

$$\frac{r\Pr[Piv_A \mid \alpha]}{(1-s)\Pr[Piv_A \mid \beta]} = \frac{K(\sigma_A,\sigma_B)}{K(\tau_A,\tau_B)} > 1$$

and so

$$U_a = \frac{r}{1+1-s}\Pr[Piv_A \mid \alpha] - \frac{1-s}{r+1-s}\Pr[Piv_A \mid \beta] > 0$$

■

Lemma 6 *In any sequence of sincere voting equilibria, either* $\lim np_a(n) = \infty$ *or* $\lim np_b(n) = \infty$.

Proof. Lemma 3 then implies that both

$$\lim U_a(p_a(n), p_b(n)) = 0 \text{ and } \lim U_b(p_a(n), p_b(n)) = 0$$

Suppose to the contrary that $\lim np_a(n) < \infty$ and $\lim np_b(n) < \infty$. But now Lemmas 4 and 5 lead to a contradiction. ■

Our next lemma shows that in the limit, the participation rates are of the same order of magnitude.

Lemma 7 *In any sequence of sincere voting equilibria, (i)* $\liminf \frac{p_a(n)}{p_b(n)} > 0$*; and (ii)* $\liminf \frac{p_b(n)}{p_a(n)} > 0$.

Proof. To prove part (i), suppose to the contrary that $\liminf \frac{p_a(n)}{p_b(n)} = 0$. Lemma 6 implies that $\liminf np_b(n) = \infty$.

Consider the probability of outcome (k,l) in state α

$$\Pr[(k,l) \mid \alpha] = e^{-nrp_a}\frac{(nrp_a)^k}{k!}e^{-n(1-r)p_b}\frac{(n(1-r)p_b)^l}{l!}$$

and the corresponding probability $\Pr[(k,l) \mid \beta]$, which is obtained by substituting $(1-s)$ for r in the expression above.

The likelihood ratio

$$\frac{\Pr[(k,l) \mid \alpha]}{\Pr[(k,l) \mid \beta]} = e^{np_b(r+s-1)\left(1-\frac{p_a}{p_b}\right)} \times \frac{r}{(1-s)^k} \frac{(1-r)^l}{s}$$

Since along some sequence, $\frac{p_a}{p_b} \to 0$ and $np_b \to \infty$

$$e^{np_b(r+s-1)\left(1-\frac{p_a}{p_b}\right)} \to \infty$$

Moreover, in all events in the set Piv_B, $|k-l| \leq 1$.

Thus, there exists an n_0 such that for all $n \geq n_0$

$$\frac{\Pr[Piv_B \mid \alpha]}{\Pr[Piv_B \mid \beta]} > \frac{q(\beta \mid b)}{q(\alpha \mid b)}$$

But this contradicts the fact that for all n, the participation thresholds are positive, that is

$$q(\beta \mid b)\Pr[Piv_B \mid \beta] - q(\alpha \mid b)\Pr[Piv_B \mid \alpha] = F^{-1}(p_b) > 0$$

Part (ii) is, of course, an immediate consequence of Lemma 1. ■

C Appendix: Uniqueness

The purpose of this appendix is to provide a proof of Proposition 6.

First, we show that in any equilibrium, voting behavior must be sincere. This now means that all equilibria must be of the kind we have studied—and the only way there could be multiple equilibria is that there are multiple solutions to the equilibrium participation rates. We complete the proof by showing that when n is large, there can be only one pair of equilibrium participation rates.

To show that all equilibria involve sincere voting, we first rule out equilibria in which voters with a signals and voters with b signals both vote against their own signals with positive probability.

Lemma 8 *In any equilibrium, at least one type votes sincerely.*

Proof. Suppose to the contrary that neither type votes sincerely.

Let $U(A,a)$ denote the gross payoff (not including costs of voting) of voting for A to a voter with an a signal. Similarly, define $U(B,a)$, $U(A,b)$ and $U(B,b)$.

Then we have that

$$U(A,a) > U(A,b) \geq U(B,b)$$

where the first inequality follows from the fact that all else being equal, a vote of A is more valuable with signal a than with with signal b. The second inequality follows from the fact that, by assumption, b types vote for A with positive probability.

On the other hand, similar reasoning leads to

$$U(B,b) > U(B,a) \geq U(A,a)$$

and the two inequalities contradict each other. Hence it cannot be that neither type votes sincerely. ■

Lemma 9 *There cannot be an equilibrium in which both types always vote for the same candidate.*

Proof. Suppose that all voters vote for A (say). Then we have that

$$U(A,a) > U(A,b) \geq U(B,b) > U(B,a)$$

Moreover, since b types participate,

$$\begin{aligned} U(A,b) &= q(\alpha \mid b) \Pr[Piv_A \mid \alpha] - q(\beta \mid b) \Pr[Piv_A \mid \beta] \\ &= q(\alpha \mid b) \tfrac{1}{2} e^{-n(rp_a+(1-r)p_b)} - q(\beta \mid b) \tfrac{1}{2} e^{-n((1-s)p_a+sp_b)} \\ &\geq 0 \end{aligned}$$

since the only circumstances in which a vote for A is pivotal is if no one else shows up. Since $r > 1-s$, a necessary condition for this to hold is that $rp_a + (1-r)p_b < (1-s)p_a + sp_b$.

We claim that

$$U(B,b) - U(A,b) > 0$$

which is equivalent to

$$q(\beta \mid b)(\Pr[Piv_A \mid \beta] + \Pr[Piv_B \mid \beta]) > q(\alpha \mid b)(\Pr[Piv_A \mid \alpha] + \Pr[Piv_B \mid \alpha])$$

Notice that

$$\begin{aligned} \Pr[Piv_A \mid \beta] + \Pr[Piv_B \mid \beta] &= e^{-n((1-s)p_a+sp_b)} \left(1 + \tfrac{1}{2} n((1-s)p_a + sp_b)\right) \\ \Pr[Piv_A \mid \alpha] + \Pr[Piv_B \mid \alpha] &= e^{-n(rp_a+(1-r)p_b)} \left(1 + \tfrac{1}{2} n(rp_a + (1-r)p_b)\right) \end{aligned}$$

and the first term is greater since the function $e^{-x}\left(1 + \frac{1}{2}x\right)$ is decreasing for $x > 0$ and $rp_a + (1-r)p_b < (1-s)p_a + sp_b$.

Thus,

$$U(B,b) - U(A,b) > 0$$

which contradicts the assumption that b types vote for A. ■

Lemma 10 *In any equilibrium, voting is sincere.*

Proof. Lemmas 8 and 9 imply that any equilibrium must have the following form: one type votes sincerely and the other type votes sincerely with positive probability.

First, suppose that a types vote sincerely and b types vote sincerely with probability $\mu < 1$. In this case,

$$\begin{aligned} \sigma_A &= n\left(rp_a + (1-r)(1-\mu)p_b\right); \quad \sigma_B = n(1-r)\mu p_b \\ \tau_A &= n\left((1-s)p_a + s(1-\mu)p_b\right); \quad \tau_B = ns\mu p_b \end{aligned} \tag{26}$$

Since b types are indifferent between voting for A and voting for B, we have

$$0 \le U(B,b) = U(A,b) < U(A,a)$$

where the inequality follows from the fact that, all else being equal, the payoff from voting for A when the signal is a is higher than when the signal is b. Thus the gross payoff of b types is lower than the gross payoff of a types and so $p_b < p_a$. If $p_b < p_a$, then using (26), it is easy to verify that $\sigma_A > \tau_A$ and $\sigma_B < \tau_B$. Hence voting behavior in any such equilibrium satisfies the conditions of the Likelihood Ratio Lemma (Lemma 2). The gross payoff to a b type from voting is

$$U(B,b) = q(\beta \mid b)\Pr[Piv_B \mid \beta] - q(\alpha \mid b)\Pr[Piv_B \mid \alpha] \ge 0$$

where the pivot probabilities are computed using the expected vote totals in (26). The inequality $U(B,b) \ge 0$ may be rewritten as

$$\frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_B \mid \alpha]} \ge \frac{q(\alpha \mid b)}{q(\beta \mid b)}$$

Lemmas 2 then implies that,

$$\frac{\Pr[Piv_A \mid \beta]}{\Pr[Piv_A \mid \alpha]} > \frac{q(\alpha \mid b)}{q(\beta \mid b)}$$

which is equivalent to

$$U(A,b) = q(\alpha \mid b)\Pr[Piv_A \mid \alpha] - q(\beta \mid b)\Pr[Piv_A \mid \beta] < 0$$

which is a contradiction.

Second, suppose that b types vote sincerely and a types vote sincerely with probability $\mu < 1$. In this case,

$$\begin{aligned} \sigma_A &= nr\mu p_a; \quad & \sigma_B &= n\left(r(1-\mu)p_a + (1-r)p_b\right) \\ \tau_A &= n(1-s)\mu p_a; \quad & \tau_B &= n\left((1-s)(1-\mu)p_a + sp_b\right) \end{aligned} \tag{27}$$

An analogous argument shows that now $p_b > p_a$ and again the conditions of Lemma 2 are satisfied. As above, this implies that a types cannot be indifferent. ∎

We have thus shown that all equilibria must involve sincere voting. Note that this does not require n to be large.

It remains to show that given sincere voting, there is a unique set of participation rates—that is, there is a unique solution (p_a^*, p_b^*) to IRa and IRb. As we show next, this is also true in large elections.[12]

Lemma 11 *In large elections, there is a unique solution to the cost threshold equations IRa and IRb.*

Proof. Equilibrium cost thresholds are determined by the equations

$$U_a(p_a, p_b) \equiv q(\alpha \mid a) \Pr[Piv_A \mid \alpha] - q(\beta \mid a) \Pr[Piv_A \mid \beta] = F^{-1}(p_a) \tag{IRa}$$

$$U_b(p_a, p_b) \equiv q(\beta \mid b) \Pr[Piv_B \mid \beta] - q(\alpha \mid b) \Pr[Piv_B \mid \alpha] = F^{-1}(p_b) \tag{IRb}$$

We will show that when n is large, at any intersection of the two, the curve determined by IRa is steeper than that determined by IRb, that is,

$$-\left(\frac{\partial U_a}{\partial p_a} - \left(F^{-1}(p_a)\right)'\right) \div \frac{\partial U_a}{\partial p_b} > -\frac{\partial U_b}{\partial p_a} \div \left(\frac{\partial U_b}{\partial p_b} - \left(F^{-1}(p_b)\right)'\right) \tag{28}$$

The calculation of the partial derivatives is facilitated by using the following simple fact. If we write,

$$\Pr[(l,k) \mid \alpha] = e^{-nrp_a} \frac{(nrp_a)^l}{l!} e^{-n(1-r)p_b} \frac{(n(1-r)p_b)^k}{k!}$$

as the probability of outcome (l, k) in state α, then

$$\begin{aligned} \frac{\partial \Pr[(l,k) \mid \alpha]}{\partial p_a} &= nr \Pr[(l-1,k) \mid \alpha] - nr \Pr[(l,k) \mid \alpha] \\ \frac{\partial \Pr[(l,k) \mid \alpha]}{\partial p_b} &= n(1-r) \Pr[(l,k-1) \mid \alpha] - n(1-r) \Pr[(l,k) \mid \alpha] \end{aligned}$$

Similar expressions obtain for the partial derivatives of $\Pr[(l,k) \mid \beta]$.

Since the probability of a pivotal term Piv_C where $C = A, B$ is just a sum of terms of the form $\Pr[(l,k) \mid \alpha]$, we obtain

$$\begin{aligned} \frac{\partial \Pr[Piv_C \mid \alpha]}{\partial p_a} &= nr \Pr[Piv_C - (1,0) \mid \alpha] - nr \Pr[Piv_C \mid \alpha] \\ \frac{\partial \Pr[Piv_C \mid \alpha]}{\partial p_b} &= n(1-r) \Pr[Piv_C - (0,1) \mid \alpha] - n(1-r) \Pr[Piv_C \mid \alpha] \end{aligned}$$

Again, similar expressions obtain for the partial derivatives of $\Pr[Piv_C \mid \beta]$ where $C = A, B$.

Myerson (2000) has shown that when the expected number of voters is large, the

[12] This result does not hold in a corresponding model of costly voting with a fixed population. Ghosal and Lockwood (2007) provide an example with the majority rule in which there are multiple cost thresholds and hence, multiple equilibria.

probabilities of the "offset" events in state α are

$$\begin{aligned}
\Pr[Piv_C - (1,0) \mid \alpha] &\approx \Pr[Piv_C \mid \alpha]\, x^{\frac{1}{2}} \\
\Pr[Piv_C - (0,1) \mid \alpha] &\approx \Pr[Piv_C \mid \alpha]\, x^{-\frac{1}{2}}
\end{aligned}$$

where

$$x = \frac{1-r}{r}\frac{p_b}{p_a}$$

Similarly, the probabilities of the offset events in state β are

$$\begin{aligned}
\Pr[Piv_C - (1,0) \mid \alpha] &\approx \Pr[Piv_C \mid \beta]\, y^{\frac{1}{2}} \\
\Pr[Piv_C - (0,1) \mid \alpha] &\approx \Pr[Piv_C \mid \beta]\, y^{-\frac{1}{2}}
\end{aligned}$$

where

$$y = \frac{s}{1-s}\frac{p_b}{p_a}$$

Using Myerson's offset formulae it follows that

$$\begin{aligned}
\frac{\partial U_a}{\partial p_a} &\approx nq(\alpha \mid a)\, r \Pr[Piv_A \mid \alpha]\,(x^{\frac{1}{2}} - 1) - nq(\beta \mid a)(1-s)\Pr[Piv_A \mid \beta]\,(y^{\frac{1}{2}} - 1) \\
\frac{\partial U_a}{\partial p_b} &\approx nq(\alpha \mid a)(1-r)\Pr[Piv_A \mid \alpha]\,(x^{-\frac{1}{2}} - 1) - nq(\beta \mid a)\, s \Pr[Piv_A \mid \beta]\,(y^{-\frac{1}{2}} - 1)
\end{aligned}$$

and similarly,

$$\begin{aligned}
\frac{\partial U_b}{\partial p_a} &\approx nq(\beta \mid b)(1-s)\Pr[Piv_B \mid \beta]\,(y^{\frac{1}{2}} - 1) - nq(\alpha \mid b)\, r \Pr[Piv_B \mid \alpha]\,(x^{\frac{1}{2}} - 1) \\
\frac{\partial U_b}{\partial p_b} &\approx nq(\beta \mid b)\, s \Pr[Piv_B \mid \beta]\,(y^{-\frac{1}{2}} - 1) - nq(\alpha \mid b)(1-r)\Pr[Piv_B \mid \alpha]\,(x^{-\frac{1}{2}} - 1)
\end{aligned}$$

We have argued that when n is large, any point of intersection of IRa and IRb, say (p_a, p_b), results in efficient electoral outcomes—A wins in state α and B wins in state β. This requires that (p_a, p_b) satisfy

$$\frac{1-r}{r}\frac{p_b}{p_a} < 1 \text{ and } \frac{s}{1-s}\frac{p_b}{p_a} > 1 \tag{29}$$

and by definition this is the same as

$$x < 1 \text{ and } y > 1 \tag{30}$$

From this it follows that at any point (p_a, p_b) satisfying (29),

$$\frac{\partial U_a}{\partial p_a} < 0 \text{ and } \frac{\partial U_a}{\partial p_b} > 0$$

and similarly,

$$\frac{\partial U_b}{\partial p_a} > 0 \text{ and } \frac{\partial U_b}{\partial p_b} < 0$$

Thus at any (p_a, p_b) satisfying (29), the curves determined by IRa and IRb are both positively sloped.

Since $\left(F^{-1}(p_a)\right)'$ and $\left(F^{-1}(p_b)\right)'$ are both positive, in order to establish the inequality in (28), it is sufficient to show that

$$\left(-\frac{\partial U_a}{\partial p_a}\right) \div \frac{\partial U_a}{\partial p_b} > \frac{\partial U_b}{\partial p_a} \div \left(-\frac{\partial U_b}{\partial p_b}\right)$$

which is equivalent to

$$\frac{q(\alpha \mid a)\, r \Pr[Piv_A \mid \alpha](1 - x^{\frac{1}{2}}) + q(\beta \mid a)(1-s)\Pr[Piv_A \mid \beta](y^{\frac{1}{2}} - 1)}{q(\alpha \mid a)(1-r)\Pr[Piv_A \mid \alpha](x^{-\frac{1}{2}} - 1) + q(\beta \mid a)\, s \Pr[Piv_A \mid \beta](1 - y^{-\frac{1}{2}})}$$
$$> \frac{q(\alpha \mid b)\, r \Pr[Piv_B \mid \alpha](1 - x^{\frac{1}{2}}) + q(\beta \mid b)(1-s)\Pr[Piv_B \mid \beta](y^{\frac{1}{2}} - 1)}{q(\alpha \mid b)(1-r)\Pr[Piv_B \mid \alpha](x^{-\frac{1}{2}} - 1) + q(\beta \mid b)\, s \Pr[Piv_B \mid \beta](1 - y^{-\frac{1}{2}})}$$

Using

$$q(\alpha \mid a) = \frac{r}{r + (1-s)} \text{ and } q(\beta \mid b) = \frac{s}{s + (1-r)}$$

and writing

$$L_A = \frac{\Pr[Piv_A \mid \beta]}{\Pr[Piv_A \mid \alpha]} \text{ and } L_B = \frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_B \mid \alpha]}$$

as the two likelihood ratios, the inequality above is the same as

$$\frac{(r)^2(1 - x^{\frac{1}{2}}) + (1-s)^2(y^{\frac{1}{2}} - 1)L_A}{r(1-r)(x^{-\frac{1}{2}} - 1) + s(1-s)(1 - y^{-\frac{1}{2}})L_A} > \frac{r(1-r)(1 - x^{\frac{1}{2}}) + s(1-s)(y^{\frac{1}{2}} - 1)L_B}{(1-r)^2(x^{-\frac{1}{2}} - 1) + s(1 - y^{-\frac{1}{2}})L_B}$$

Cross-multiplying and cancelling terms, further reduces the inequality to

$$\left(\frac{(1-r)(1-s)}{rs}(x^{-\frac{1}{2}} - 1)(y^{\frac{1}{2}} - 1) - (1 - x^{\frac{1}{2}})(1 - y^{-\frac{1}{2}})\right) \times L_A$$
$$> \left(\frac{(1-r)(1-s)}{rs}(x^{-\frac{1}{2}} - 1)(y^{\frac{1}{2}} - 1) - (1 - x^{\frac{1}{2}})(1 - y^{-\frac{1}{2}})\right) \times \frac{rs}{(1-r)(1-s)} \times L_B \tag{31}$$

We claim that for all (p_a, p_b) satisfying (29),

$$\frac{(1-r)(1-s)}{rs}(x^{-\frac{1}{2}} - 1)(y^{\frac{1}{2}} - 1) - (1 - x^{\frac{1}{2}})(1 - y^{-\frac{1}{2}}) < 0 \tag{32}$$

To see this, note that by definition,

$$\begin{aligned} y &= \frac{s}{1-s}\frac{p_b}{p_a} \\ &= \frac{rs}{(1-r)(1-s)}\frac{1-r}{r}\frac{p_b}{p_a} \\ &= \frac{rs}{(1-r)(1-s)}x \\ &= Rx \end{aligned}$$

where $R = \frac{rs}{(1-r)(1-s)}$. Substituting $y = Rx$ we obtain

$$R(x^{-\frac{1}{2}}-1)(y^{\frac{1}{2}}-1)-(1-x^{\frac{1}{2}})(1-y^{-\frac{1}{2}}) = R^{-1}(x^{-\frac{1}{2}}-1)(R^{\frac{1}{2}}x^{\frac{1}{2}}-1)-(1-x^{\frac{1}{2}})(1-R^{-\frac{1}{2}}x^{-\frac{1}{2}})$$

Now consider the function

$$\phi(x) = R^{-1}(x^{-\frac{1}{2}}-1)(R^{\frac{1}{2}}x^{\frac{1}{2}}-1)-(1-x^{\frac{1}{2}})(1-R^{-\frac{1}{2}}x^{-\frac{1}{2}})$$

Since $x < 1 < y = Rx$, we have $R^{-1} < x < 1$. Notice that $\phi(1) = 0 = \phi(R^{-1})$. It is routine to verify that ϕ is convex and so $\phi(x) < 0$ for all $x \in (R^{-1}, 1)$. Thus we have established (32).

Now because of (32), the inequality in (31) reduces to

$$\frac{\Pr[Piv_A \mid \beta]}{\Pr[Piv_A \mid \alpha]} < R \times \frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_B \mid \alpha]} \tag{33}$$

Finally, notice that IRa and IRb imply, respectively, that

$$\frac{r}{1-s} = \frac{q(\alpha \mid a)}{q(\beta \mid a)} > \frac{\Pr[Piv_A \mid \beta]}{\Pr[Piv_A \mid \alpha]} \text{ and } \frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_B \mid \alpha]} > \frac{q(\alpha \mid b)}{q(\beta \mid b)} = \frac{1-r}{s}$$

and this immediately implies (33), thereby completing the proof that at any point of intersection of IRa and IRb, the slope of IRa is greater than the slope of IRb. This means that the curves cannot intersect more than once. ■

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