Strategy-Proof Coalition Formation

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Abstract

We analyze coalition formation problems in which a group of agents is partitioned into coalitions and agents' preferences only depend on the identity of the members of the coalition they are members of. We study (coalition formation) rules that associate to each profile of agents' preferences a partition of the society. Our main interest is to devise rules that never provide incentives for the agents to misrepresent their preferences. Hence, we analyze strategy-proof rules in restricted domains of preferences as the domain of additively representable or separable preferences. In such restricted domains, we show that a family of rules –single-lapping rules– are the only rules that fulfill the requirements of strategy-proofness, individual rationality, non-bossiness, and minimal flexibility. Single-lapping rules are characterized by severe restrictions on the set of feasible coalitions. However, these rules always select core-stable partitions. Hence, our results highlight the relation between the non-cooperative concept of strategy-proofness and the cooperative concept of core-stability. We also analyze the implications of our results to matching problems as marriage, roommate, or college-admission problems.

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Keywords: Coalition Formation; Strategy-Proofness; Single-Lapping Rules, Core-Stability.

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1 Introduction

We analyze simple coalition formation problems in which a group of agents is partitioned into coalitions and agents have preferences over the coalitions they are members of. Following the terminology proposed by Drèze and Greensberg [8], we focus on problems characterized by the "hedonic" aspect of coalition formation. Agents' preferences only depend on the identity of the members of the coalition to which they belong. Hence, we exclude the existence of externalities among different coalitions. The most relevant examples of such problems are matching problems as the roommate problem, or the formation of social clubs, organizations, teams or societies.

The literature of Coalitional Game Theory has extensively analyzed the existence of stable partitions in hedonic coalition formation problems.[1] Instead, we propose a social choice approach. We study coalition formation rules that associate to each profile of agents' preferences a partition of the group of agents. Our main concern is that our rules satisfy *strategy-proofness*. Strategy-proofness is the strongest decentrability property. Each agent needs to know only her own preferences to compute her best choice.

It is well known that the requirements of strategy-proofness are hard to meet. In the abstract model of social choice, Gibbard [10] and Satterthwaite [15] show that –provided there are more than two alternatives at stake– every strategy-proof social choice rule is dictatorial. However, reasonable strategy-proof rules do exist if appropriate restrictions are imposed on agents' preferences. We focus on restricted domains of preferences over coalitions as the domain of additively representable preferences and the domain of separable preferences, that exclude complementarities among the members of a coalition. For these domains of preferences, possibility results have been obtained in the literature. For instance, in the context of a group of agents choosing a subset from a set of objects (that represent, for instance, candidates who opt to some number of available positions), when agents' preferences over sets of objects are additively representable, then strategy-proof rules can be decomposed into independent rules, one for each object.[2]

Besides strategy-proofness, we would like our rule to satisfy three additional properties that are natural in the context of coalition formation problems. Our rules should be *individually rational, non-bossy* and *flexible*. Individual rationality is a minimal partici-

[1]For further references, see the recent works by Banerjee, Konishi, and Sönmez [3], Barberà and Gerber [4], Bogomolnaia and Jackson [6], and Pàpai [12].

[2]See Barberà, Sonnenschein, and Zhou [5] and Le Breton and Sen [11] for further details.

pation constraint. It means that no agent should be ever worse off than she would be in her own. Non-bossiness is a collusion-proof requirement. It says that if a change in an agent's preferences does not affect the coalition to which this agent is assigned, then the remaining agents are also unaffected by this change of preferences. Flexibility says that every partition formed by a collection of feasible coalitions belongs to the range of the rule. For some agents to form a coalition is not necessary that other coalitions are also formed.

Our main result characterizes a family of rules that fulfills the previous axioms, the family of single-lapping rules. Single lapping rules are characterized by severe restrictions over the set of feasible coalitions. On the other hand, single-lapping rules always select core-stable partitions of the society, in the sense that no feasible coalition of agents may unanimously prefer to join each other rather than to stay at the coalition they are assigned. Hence, our main result provides further evidence on the relation between the Non-Cooperative Game Theory concept of strategy-proofness and the Cooperative Game Theory concept of the core.

Before proceeding with the formal analysis, we review the most related literature. Pàpai [12] is closely related to this work. This author analyzes restrictions over the set of feasible coalitions that ensure the existence of unique core-stable partitions. She introduces the single-lapping property and shows that the single-lapping property is a sufficient condition for unique core-stability. Moreover, it is also shown that single-lapping rules are the only rules that satisfy strategy-proofness, individual rationality, and a weak version of efficiency when agents' preferences over coalitions are not restricted. Our theorems reinforce their results, since we show that, indeed, similar results also hold in much tighter domains of preferences.

Finally, we refer to the works by Alcalde and Revilla [2], Cechlárová and Romero-Medina [7], and Sönmez [17]. All these works study strategy-proof coalition formation rules. However, they focus on different domains of preferences. More specifically, Alcalde and Revilla [2], Cechlárová and Romero-Medina [7] assume that agents' preferences over coalitions are based on the best or the worst group of agents in each coalition. In these environments, they prove the existence of strategy-proof rules that always select core-stable partitions. Sönmez [17] proposes a general model of allocation of indivisible goods that can be applied to coalition formation problems. He focuses on preference domains for which there always exist core-stable partitions. His main result states that there exist strategy-proof, individually rational, and Pareto efficient rules only if the set of core-stable partitions is always essentially single-valued.[3]

The remainder of the paper is organized as follows. In Section 2, we present the model and basic notation. In Section 3, we present different agents' domains of preferences over coalitions. In Section 4, we introduce the main axioms while in Section 5, we provide the characterization results. In Section 6, we prove our main result (Theorem 1). In Section 6, we conclude by analyzing some applications of our result to different classes of coalition formation problems.

2 Basic Notation

Let $N = \{1, \ldots, n\}$ be a society consisting of a finite set of at least 3 agents. We call a non-empty subset $C \subseteq N$ a ***coalition***. We denote by $\mathcal{N}$ the set of all non-empty subsets of N. For each $C \in \mathcal{N}$, let $[C] \equiv \{\{i\} : \ i \in C\}$. A ***collection of coalitions*** is a set of coalitions $\Pi \subseteq \mathcal{N}$ that contains all the singleton sets, $[N] \subseteq \Pi$. We denote by σ a partition of N and by Σ the set of all partitions of N. For each $i \in N$ and each $\sigma \in \Sigma$, we denote by $\sigma_i \in \sigma$ the coalition in σ to which i belongs.

For each $i \in N$, let $\mathcal{C}_i = \{C \subseteq N, \ i \in C\}$. That is, $\mathcal{C}_i$ is the set of all coalitions to which i belongs. A ***preference*** for i, $\succsim_i$, is a complete order on $\mathcal{C}_i$.[4] For each $i \in N$, we denote by $\mathcal{D}_i$ the set of all agent i´s admissible preferences. As we assume that agents only care about the coalition they belong to, agents' preferences over partitions are completely defined by their preferences over coalitions. Then, abusing notation, we say that for each $i \in N$, each $\succsim \in \mathcal{D}_i$, and each σ, $\sigma' \in \Sigma$, σ is at least as good as σ', $\sigma \succsim_i \sigma'$, if and only if $\sigma_i \succsim_i \sigma'_i$.

For each $i \in N$, each set of coalitions $\mathcal{X} \subseteq \mathcal{N}$, and each $\succsim_i \in \mathcal{D}_i$, let $\text{top}(\mathcal{X}, \succsim_i)$ be the coalition in $\mathcal{X} \cap \mathcal{C}_i$ that is ranked first according to $\succsim_i$.

Let $\mathcal{D} = \times_{i \in N} \mathcal{D}_i$. We call $\succsim \in \mathcal{D}$ a preference profile. For each $C \subset N$, $\mathcal{D}_C = \times_{i \in C} \mathcal{D}_i$, while for each $\succsim \in \mathcal{D}$, $\succsim_C \in \mathcal{D}_C$ denotes the restriction of profile $\succsim$ to the preferences of the agents in C. Let $\bar{\mathcal{D}} \subset \mathcal{D}$, we say that $\bar{\mathcal{D}}$ is a ***cartesian domain*** if for each $i \in N$ there is $\bar{\mathcal{D}}_i \in \mathcal{D}_i$ such that $\bar{\mathcal{D}} = \times_{i \in N} \bar{\mathcal{D}}_i$.

[3] Under some preference assumptions as strict preferences and no-consumption externalities that are fulfilled in coalition formation problems, Takayima [18] proves that the converse results also holds.

[4] An order is a reflexive, transitive, and antisymmetric binary relation.

We are interested in rules that associate a partition of the society to each profile of agents' preferences.

Let $\bar{\mathcal{D}} \subset \mathcal{D}$ be a cartesian domain. A ***(coalition formation) rule*** defined on the domain $\bar{\mathcal{D}}$ is a mapping $\varphi : \bar{\mathcal{D}} \to \Sigma$.

For each $i \in N$ and each $\succsim \in \bar{\mathcal{D}}$, $\varphi_i(\succsim)$ denotes the coalition in $\varphi(\succsim)$ to which i belongs. We denote by R^φ the range of φ, that is, the set of feasible partitions,

$$R^\varphi \equiv \{\sigma \in \Sigma, \text{ such that there is } \succsim \in \bar{\mathcal{D}},\ \varphi(\succsim) = \sigma\},$$

while, F^φ denotes the set of feasible coalitions,

$$F^\varphi \equiv \{C \in \mathcal{N}, \text{ such that for some } \sigma \in R^\varphi,\ C \in \sigma\}.$$

3 Preferences over Coalitions: Rich Domains

We start by presenting two classes of preferences over sets that play a crucial role in our analysis. We call them *top preferences* and *bottom preferences*. These preferences are obtained by extending orders over single agents to preferences over coalitions. The basic idea behind our preferences over sets is that according to some order of the set of agents, each agent i divides the set of possible mates into two groups. Those agents that she likes, and those agents she dislikes. An agent equipped with top preferences prioritizes joining the agents she likes the most with respect to avoiding the agents she dislikes. On the other hand, an agent equipped with bottom preferences prioritizes avoiding the agents she dislikes the most with respect to joining the agents she likes. In order to present both domains of preferences, we introduce first additional notation.

Let $\mathcal{P}$ be the set of all complete orders over N. For each $P \in \mathcal{P}$, R denotes the weak order associated to P and it is defined in the usual way. For each $C \subseteq N$ and each $P \in \mathcal{P}$, we denote by $\max(C, P)$ and $\min(C, P)$, respectively, the agents of C who are the first-ranked and last-ranked agents by P. Next, for each $i \in N$, each $P \in \mathcal{P}$, and each $C \in \mathcal{C}_i$, let $C_i^+(P) \equiv \{j \in C, \text{ s.t. } j\ R\ i\}$, and $C_i^-(P) \equiv \{j \in C \text{ s.t. } i\ R\ j\}$. Now, define $C_i^+(1, P) \equiv \max(C_i^+, P)$ and $C_i^-(1, P) \equiv \min(C_i^-, P)$. Once $C_i^+(t, P)$ and $C_i^-(t, P) \equiv \min(C_i^-, P)$ are defined for some $t \geq 1$, iteratively, let

$$C_i^+(t+1, P) \equiv \max\left(\left[C_i^+(P) \setminus \cup_{k=1}^t C_i^+(k, P)\right], P\right), \text{ and,}$$
$$C_i^-(t+1, P) \equiv \min\left(\left[C_i^-(P) \setminus \cup_{k=1}^t C_i^-(k, P)\right], P\right).$$

Let $i \in N$ and $P \in \mathcal{P}$.

The preference $\succsim_i \in \mathcal{D}_i$ is the ***top preference associated to*** P ***by*** i, $\succsim_i = \succsim_i^+ (P)$ if for each two distinct coalitions $C, C' \in \mathcal{C}_i$, $C \succ C'$ if and only if

- $C_i^+(P) \neq C_i'^+(P)$ and $C_i^+(t,P)\ P\ C_i'^+(t,P)$, where t is the first integer such that $C_i^+(t,P) \neq C_i'^+(t,P)$
- $C_i^+(P) = C_i'^+(P)$ and $C_i^-(t',P)\ P\ C_i'^-(t',P)$, where t' is the first integer such that $C_i^-(t',P) \neq C_i'^-(t',P)$.

The preference $\succsim_i \in \mathcal{D}_i$ is the ***bottom preference associated to*** P ***by*** i, $\succsim_i = \succsim_i^- (P)$ if for each two distinct coalitions $C, C' \in \mathcal{C}_i$, $C \succ C'$ if and only if

- $C_i^-(P) \neq C_i'^-(P)$, and $C_i^-(t,P)\ P\ C_i'^-(t,P)$, where t is the first integer such that $C_i^-(t,P) \neq C_i'^-(t,P)$
- $C_i^-(P) = C_i'^-(P)$ and $C_i^+(t',P)\ P\ C_i'^+(t',P)$, where t' is the first integer such that $C_i^+(t',P) \neq C_i'^+(t',P)$.

For each $i \in N$, let

$$\mathcal{D}_i^+ \equiv \{\succsim_i \in \mathcal{D}_i \text{ such that for some } P \in \mathcal{P},\ \ \succsim_i = \succsim_i^+ (P)\}\ ,$$
$$\mathcal{D}_i^- \equiv \{\succsim_i \in \mathcal{D}_i \text{ such that for some } P \in \mathcal{P},\ \ \succsim_i = \succsim_i^- (P)\}\ ,$$
$$\mathcal{D}_i^* \equiv \mathcal{D}_i^+ \cup \mathcal{D}_i^- \ \text{ and,}$$
$$\mathcal{D}^* \equiv \times_{i \in N} \mathcal{D}_i^*.$$

Let $\bar{\mathcal{D}} \subseteq \mathcal{D}$. We say that $\bar{\mathcal{D}}$ is a ***rich domain*** if $\bar{\mathcal{D}}$ is cartesian and $\mathcal{D}^* \subseteq \bar{\mathcal{D}}$.

Next, we present two domains of preferences that have been extensively analyzed in the social choice literature, the domains of additive preferences and the domain of separable preferences. Both domains exclude the possibility of (negative or positive) complementarities among the members of a coalition.

Let $i \in N$. A ***utility function*** for agent i is a mapping $u_i : N \to \mathbb{R}$ such that $u_i(i) = 0$. A preference for agent i, $\succsim_i \in \mathcal{D}_i$ is ***additively representable*** if there is a utility function u_i such that for each $A, B \in \mathcal{C}_i$, $A \succsim_i B$ if and only if $\sum_{a \in A} u_i(a) \geq \sum_{b \in B} u_i(b)$. For each $i \in$N, $\mathcal{A}_i$ denotes the set of all i's additively representable preferences for agent i and let $\mathcal{A} \equiv \times_{i \in N} \mathcal{A}_i$.

A preference for i, $\succsim_i \in \mathcal{D}_i$, is ***separable*** if for each $j \in N$ and each $C \in \mathcal{C}_i$ such that $j \notin C$, $\{i,j\} \succ_i \{i\}$ if and only if $(C \cup \{j\}) \succ_i C$. Let $\mathcal{S}_i$ be the set of all agent i's separable preferences and let $\mathcal{S} \equiv \times_{i \in N} \mathcal{S}_i$.

The following remark shows that the additive and the separable domains are rich domains.

Remark 1. *Let $i \in N$.*

(a) *If $n \geq 4$, then $\mathcal{D}_i^* \subset \mathcal{A}_i \subset \mathcal{S}_i$.*

(b) *If $n = 3$, then $\mathcal{D}_i^* = \mathcal{A}_i = \mathcal{S}_i$.*

Proof. It is well-known that additive preferences are separable. Hence, we only prove the inclusion $\mathcal{D}_i^* \subset \mathcal{A}_i$. Let $\succsim_i \in \mathcal{D}_i$ be such that for some $P \in \mathcal{P}$, $\succsim_i = \succsim_i^+ (P)$. Let $t^* \equiv \{t \in \mathbb{N} : \ i = N_i^+(t, P)\}$, and $\bar{t} \equiv n - t^*$. For each $j \in N_i^+(P) \setminus \{i\}$, if $j = N_i^+(k, P)$, then let $u_i(j) = n^{n-k}$, whereas for each $j' \in N_i^-(P) \setminus i$, if $j' = N_i^-(k', P)$, then let $u_i(j') = -(n^{\bar{t}-k-1})$. Now, let $\succsim_i' \in \mathcal{D}_i$ be such $\succsim_i' = \succsim_i^- (P)$. For each $j \in N_i^+(P) \setminus \{i\}$, if $j = N_i^+(k, P)$, then let $u_i'(j) = n^{t^*-k-1}$, whereas for each $j' \in N_i^-(P) \setminus i$, if $j' = N_i^-(k', P)$, then let $u_i'(j') = -(n^{n-k})$.

The proof of (b) is just a matter of checking. Let $N = \{i, j, k\}$. Note that $\mathcal{D}_i^*$, $\mathcal{A}_i$ and $\mathcal{S}_i$ consist of the following eight preferences:

$\succsim_i^1$:	$\succsim_i^2$:	$\succsim_1^3$:	$\succsim_i^4$:	$\succsim_i^5$:	$\succsim_i^6$:	$\succsim_i^7$:	$\succsim_i^8$:
$\{i,j,k\}$	$\{i,j,k\}$	$\{i,j\}$	$\{i,j\}$	$\{i,k\}$	$\{i,k\}$	$\{i\}$	$\{i\}$
$\{i,j\}$	$\{i,k\}$	$\{i,j,k\}$	$\{i\}$	$\{i,j,k\}$	$\{i\}$	$\{i,j\}$	$\{i,k\}$
$\{i,k\}$	$\{i,j\}$	$\{i\}$	$\{i,j,k\}$	$\{i\}$	$\{i,j,k\}$	$\{i,k\}$	$\{i,j\}$
$\{i\}$	$\{i\}$	$\{i,k\}$	$\{i,k\}$	$\{i,j\}$	$\{i,j\}$	$\{i,j,k\}$	$\{i,j,k\}$

Note that $\succsim_i^1, \succsim_i^2, \succsim_i^7, \succsim_i^8 \in \mathcal{D}_i^+ \cap \mathcal{D}_i^-$, while $\succsim_i^3, \succsim_i^5 \in \mathcal{D}_i^+ \setminus \mathcal{D}_i^-$, and $\succsim_i^4, \succsim_i^6 \in \mathcal{D}_i^- \setminus \mathcal{D}_i^+$. ■

We close this section with a final remark on the size of the domain $\mathcal{D}_i^*$. It is clear that by focusing on the domain $\mathcal{D}_i^*$ we are restricting considerably the set of admissible preferences.

Remark 2. *For each $i \in N$, $\#\mathcal{D}_i^* = 2(n-1)(n-1)!)$ while $\#\mathcal{D}_i = (2^{(n-1)})!$.*

4 Axioms

This section introduces four properties that rules may satisfy. Let $\bar{\mathcal{D}} \subseteq \mathcal{D}$ be a cartesian domain. Let $\varphi : \bar{\mathcal{D}} \to \Sigma$ be a rule defined on $\bar{\mathcal{D}}$.

Our main axiom is an incentive constraint. A rule should provide incentives for the agents to report their true preferences. Only if a rule elicits the true preferences from the agents the social choice will be based upon the correct information. Of course, this property refers to the specific domain in which the rule is defined.

Strategy-Proofness. *For each $i \in N$, each $\succsim \in \bar{\mathcal{D}}$, and each $\succsim_i' \in \bar{\mathcal{D}}_i$, $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim_{-i}, \succsim_i')$. Conversely, φ is* **manipulable** *if φ is not strategy-proof in $\bar{\mathcal{D}}$.*

The Gibbard-Satterthwaite Theorem states that every *strategy-proof* rule on an unrestricted domain either is dictatorial or its range contains only two elements.[5] As we assume that agents' preferences over social outcomes are restricted to depend only on the coalitions they are members of and we focus on rich domains, the negative consequences of the Gibbard-Satterthwaite Theorem do not apply to our framework.

We also consider a minimal participation constraint. Agents should not prefer to stay on their own rather than to belong to the coalition that the rule assigns them.

Individual Rationality. For each $i \in N$ and each $\succsim \in \bar{\mathcal{D}}$, $\varphi_i(\succsim) \succsim_i \{i\}$.

Note that, for every *individually rational* rule, its set of feasible allocations is a collection of coalitions.

We consider rules such that whenever a change in an agent's preference does not change the coalition she is assigned to, then the assignment for the remaining agents does not change.

Non-Bossiness. For each $i \in N$, each $\succsim \in \bar{\mathcal{D}}$, and each $\succsim_i \in \bar{\mathcal{D}}_i$, $\varphi_i(\succsim) = \varphi_i(\succsim_{-i}, \succsim_i')$ implies $\varphi(\succsim) = \varphi(\succsim_{-i}, \succsim_i')$.

Although in our model there does not exist any transferable private good, we can interpret *non-bossiness* as a collusion-proof or bribe-proof condition. A violation of its

[5] A rule defined in the domain $\bar{\mathcal{D}}$ is ***dictatorial*** if there is $i \in N$ (a dictator) such that for each $\succsim \in \bar{\mathcal{D}}$, $\varphi_i(\succsim) = \mathrm{top}(F^{\varphi}, \succsim_i)$.

requirements implies a possibility of collusion. An agent might misrepresent her preferences in exchange for a transfer (of a private good) from those who benefit from her lie.

We also introduce a minimal flexibility condition on the range of the rule. We assume that the range of a rule is determined by the set of feasible coalitions.

Flexibility. For each $\sigma = \{C_1, \ldots, C_m\} \in \Sigma$, $C_t \in F^{\varphi}$ for each $t = 1, \ldots, m$, implies $\sigma \in R^{\varphi}$.

Flexibility is a mild condition. It does not assume the feasibility of any coalition and it does not restrict significatively the range of applications of our model. It avoids interdependence among coalitions.

Although in principle, we are not interested in efficiency, we include now a minimal efficiency requirement.

Pareto Efficiency (on the Range). There is no $\sigma \in R^{\varphi}$ such that for each $i \in N$ $\sigma_i \succsim_i \varphi_i(\succsim)$, and for some $j \in N$, $\sigma_j \succ_j \varphi_j(\succsim)$.

5 Characterization Results

In this section we analyze the implications of the axioms listed above over rules defined on rich domains. First, we introduce additional notation due to Pàpai [12]. This author proposes a property over sets of coalitions – the single-lapping property– that ensures the existence of a unique *core-stable* partition for every preference profile. We make use of this property to define a class of rules.

A collection of coalitions Π satisfies the ***single-lapping property*** if

Condition (a): For each $C, C' \in \Pi$, $C \neq C'$ implies $\#(C \cap C') \leq 1$.

Condition (b): For each $\{C_1, \ldots, C_m\} \subseteq \Pi$ with $m \geq 3$ and for each $t = 1, \ldots, m$, $\#(C_t \cap C_{t+1}) \geq 1$ (where $m + 1 = 1$), there is $i \in N$ such that for each $t = 1, \ldots, m$, $C_t \cap C_{t+1} = \{i\}$.

Condition (a) states that if there is an overlap between any two coalitions in the collection, there cannot be more than one agent who is member of these two coalitions. *Condition (b)* is a non-cycle condition. It requires that if a set of coalitions in the collection form a cycle in which every two neighbor coalitions have a common member, then all these coalitions have the same common member. A prominent property of single-lapping collection of coalitions is that for every preference profile, there is a coalition in the collection such that all its members think that this coalition is the best coalition in the collection.[6] This fact implies that for every single-lapping collection of coalitions and every preference profile there is a unique *core-stable* partition of the society. Moreover, Pàpai [12] also shows that the single-lapping property is a necessary condition for the existence of a unique *core-stable* partition when agents' preferences over coalitions are unrestricted. Furthermore, she also presents the following algorithm that allows us to find such partition.

For each $\succsim \in \mathcal{D}^*$ and each single-lapping collection of coalitions $\Pi \subset \mathcal{N}$, the ***core-stable partition associated to*** Π ***at profile*** $\succsim$, $\bar{\sigma}^{\Pi}(\succsim)$, can be identified by the following algorithm:

Algorithm: **Pàpai [12]** Find $C \in \Pi$ such that for each $i \in C$, $\text{top}(\Pi, \succsim_i) = C$. As Π is single-lapping, such coalition exists. Note that there may be several such coalitions, and all these coalitions are disjoint. Let $M^{\Pi}(1, \succsim)$ denote the set of all coalitions that are obtained in this first stage. Let $\Pi(1, \succsim) \equiv \Pi$. Let $T^{\Pi}(1, \succsim)$ denote the set of agents that are matched in the first stage. Then,

$$M^{\Pi}(1, \succsim) \equiv \{C \in \Pi \text{ such that for each } i \in C, \text{top}(\Pi, \succsim_i) = C\}$$
$$T^{\Pi}(1, \succsim) \equiv \cup_{C \in M^{\Pi}(1, \succsim)} C$$

Once $\Pi(t, \succsim)$, $M^{\Pi}(t, \succsim)$, and $T^{\Pi}(t, \succsim)$ are defined for some $t \geq 1$, let,

$$\Pi(t+1, \succsim) \equiv \{C \in \Pi \text{ such that } C \cap T^{\Pi}_t = \{\varnothing\}\},$$
$$M^{\Pi}(t+1, \succsim) \equiv \{C \in \Pi(t+1, \succsim) \text{ such that for each } i \in C, \text{top}(\Pi(t+1, \succsim), \succsim_i) = C\} \text{ and,}$$
$$T^{\Pi}(t+1, \succsim) \equiv \cup_{C \in M^{\Pi}(1, \succsim) \cup \ldots \cup M^{\Pi}(t+1, \succsim)} C.$$

Note that, for each $t = 1, \ldots, m$, $\Pi(t, \succsim) \subset \Pi$, $\Pi(t, \succsim)$ is a collection of coalitions for the reduced society $N \setminus T^{\Pi}(t, \succsim)$. Moreover, $\Pi(t, \succsim)$ satisfies the single-lapping property. Then, the algorithm identifies a unique partition, $\bar{\sigma}^{\Pi}(\succsim) \equiv \{C \in$

[6] See Lemma 1, Pàpai [12].

Π such that for some $t \leq m,\ C \in M_t^{\Pi}\}$, where $m \leq n$ is the smallest integer such that $T^{\Pi}(m, \succsim) = N$..

As for each single-lapping collection of coalitions and each preference profile there is a unique core-stable partition, each single-lapping collection of coalitions defines a unique rule.

Let $\bar{\mathcal{D}} \subseteq \mathcal{D}$ be a cartesian domain of preferences and let φ be a rule defined on $\bar{\mathcal{D}}$. The rule φ is a ***single-lapping rule*** if there is a single-lapping collection of coalition Π such that for each $\succsim \in \bar{\mathcal{D}}$, $\varphi(\succsim) = \bar{\sigma}^{\Pi}(\succsim)$.

Pàpai [12] has proved that single-lapping rules there are not other rules defined on $\mathcal{D}$ satisfying *strategy-proofness*, *individual rationality*, and *Pareto efficiency*. Of course, single-lapping rules also satisfy those axioms when defined on smaller domains. Moreover, single-lapping rules also satisfy *non-bossiness* and *flexibility*. The fact that for each single-lapping rule and each preference profile there is a feasible coalition such that all its members think that this is their best preferred coalition is the key point on the non manipulability of single-lapping rules.

Theorem 1. *Let $\bar{\mathcal{D}} \subseteq \mathcal{D}$ be a rich domain of preferences. If the rule φ defined on $\bar{\mathcal{D}}$ is a single-lapping rule, then φ satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility.*

Proof. Let $F^{\varphi} = \Pi$. As φ is a single-lapping rule, Π is a single-lapping collection of coalitions. Let us check that φ satisfies *strategy-proofness*. Let $\succsim \in \bar{\mathcal{D}}$. For each $i \in T^{\Pi}(1, \succsim)$, $\varphi_i(\succsim) = \text{top}(\Pi, \succsim_i)$. Then, agents in $T^{\Pi}(1, \succsim)$ cannot manipulate. Moreover, by the definition of single-lapping rule for each $\succsim' \in \bar{\mathcal{D}}$ such that for each $i \in T^{\Pi}(1, \succsim)$ $\succsim_i = \succsim'_i$, $\varphi_i(\succsim) = \varphi_i(\succsim')$. Now, let $j \in T^{\Pi}(2, \succsim)$. If there exists $C \in \Pi$ such that $C \succ_j \varphi_j(\succsim)$, then there is $i \in T^{\Pi}(1, \succsim))$ such that $i \in C$. As for each $\succsim'_j \in \bar{\mathcal{D}}_j$, for each $i \in T^{\Pi}(1, \succsim)$, $\varphi_i(\succsim_{N\setminus\{j\}}, \succsim'_j) = T^{\Pi}(1, \succsim)$, $\varphi_j(\succsim) \succsim_j \varphi_j(\succsim_{N\setminus\{j\}}, \succsim'_j)$, and j cannot manipulate. Repeating iteratively the argument with the remaining steps of the algorithm, we obtain that no agent can manipulate. Let us check that φ satisfies *individual rationality*. By the definition of single-lapping rule, for each $i \in N$ and each $\succsim \in \bar{\mathcal{D}}$, there is $t \leq n$ such that $\varphi_i(\succsim) \in M^{\Pi}(t, \succsim)$. Note that $\{i\} \in \Pi(t, \succsim)$. By the definition of single-lapping rule, $\varphi_i(\succsim) \equiv \text{top}(\Pi_(t, \succsim), \succsim_i)$. Then, $\varphi_i(\succsim) \succsim_i \{i\}$. Let us check that φ satisfies *non-bossiness*. Let $i \in N$, $\succsim \in \bar{\mathcal{D}}$, and $\succsim'_i \in \bar{\mathcal{D}}_i$ be such that $\varphi_i(\succsim) = \varphi_i(\succsim_{N\setminus\{i\}}, \succsim'_i)$. Let $i \in T^{\Pi}(t, \succsim)$. As φ

is a single-lapping rule, for each $j \in \cup_{t' \leq t} T^{\Pi}(t', \succsim)$, $\varphi_j(\succsim) = \varphi_j(\succsim_{N\setminus\{i\}}, \succsim'_i)$. Moreover, as $\varphi_i(\succsim) = \varphi_i(\succsim_{N\setminus\{i\}}, \succsim'_i)$, for each $k \in \cup_{t' \geq t} T^{\Pi}(t', \succsim)$, also $\varphi_k(\succsim) = \varphi_k(\succsim_{N\setminus\{i\}}, \succsim'_i)$. Then, $\varphi(\succsim) = \varphi(\succsim')$. Finally, let us check that φ satisfies *flexibility*. Let $\sigma = \{C_1, \ldots, C_m\} \in \Sigma$ be such that for each $t = 1, \ldots, k$, $C_t \in \Pi$. Let $\succsim \in \bar{\mathcal{D}}$ be such that for each $t = 1, \ldots, m$ and each $i \in C_t$, $\text{top}(\mathcal{N}, \succsim_i) = C_t$. By the definition of single-lapping rule, $\varphi(\succsim) = \sigma$ and $\sigma \in R^{\varphi}$. ■

Next, we present our main result. We prove that even if we restrict dramatically the domain of admissible preferences and we focus on the smallest rich domain, single-lapping rules are the only rules that satisfy our list of axioms.

Theorem 2. *Let $\varphi : \mathcal{D}^* \to \Sigma$. If φ satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility then φ is a single-lapping rule.*

Proof. See Section 5. ■

The intuition behind the proof of Theorem 2 runs as follows. First, for every rule that satisfies our axioms, whenever a feasible coalition of individuals agrees that they are the best coalition available, they should become together. The result follows immediately once we check that the set of feasible coalitions satisfies the single-lapping property. This step is far from being immediate, although the analysis is relatively simple for three agents societies. An induction argument extends the results to arbitrary societies.

From Proposition 1 and Theorem 2, we obtain the following characterization theorem.

Theorem 3. *A rule φ defined on a rich domain of preferences satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility if and only if φ is a single-lapping rule.*

At this point, we have to relate our results those by Sönmez [17]. They are logically independent, but all highlight the close relation between the concepts of *strategy-proofness* and *core-stability*. Sönmez [17] proves that for coalition formation problems for which there is always a core-stable partition, there is a rule that satisfies *strategy-proofness*, *individual rationality*, and *Pareto efficient* if there is always a unique core-stable partition.[7] Besides the different set of axioms that we analyze, the main difference between our framework and Sönmez's one relies on the domain of preferences over coalitions. Sönmez

[7]Takayima [18] shows that in fact the converse result is also true.

[17] assumes the existence of certain preferences that need not to exist on a rich domain. Basically, Sönmez [17] assumes that for each $i \in N$, and each $A \in (F^\varphi \cap \mathcal{C}_i)$, if there is an admissible preference $\succsim_i$ such that $A \succ_i \{i\}$ then there is another admissible preference $\succsim'_i$ such that for each $B \in (F^\varphi \cap \mathcal{C}_i) \setminus \{i\}$, $B \succsim'_i A$ if and only if $B \succsim_i A$, while $A \succsim_i B$ if and only if $A \succsim'_i B$ and $A \succsim'_i \{i\} \succsim'_i B$. However, there are rich domains, as the domain of additively representable preferences, for which such preferences are not admissible. Let $i, j, k \in N$, and assume $\{i,j\}, \{i,k\}, \{i,j,k\} \in F^\varphi$. Let $\succsim_i \in \mathcal{A}_i$ be such that $\{i,j,k\} \succ_i \{i,j\} \succ \{i,k\} \succ \{i\}$, but there is no $\succsim'_i \in \mathcal{A}_i$ such that $\{i,j,k\} \succ'_i \{i\}$, $\{i\} \succ'_i \{i,j\}$, and $\{i\} \succ'_i \{i,k\}$.

Richness is a severe restriction on the domain of admissible preferences. This fact makes our negative result stronger. Similar results hold for other plausible domains of preferences as the domains of additively representable or separable preferences. Then, we can state the following result.

Corollary 1. *Let $\tilde{\varphi} : \mathcal{A} \to \Sigma$. Then, $\tilde{\varphi}$ satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility if and only if $\tilde{\varphi}$ is a single-lapping rule.*

Corollary 2. *Let $\tilde{\varphi} : \mathcal{S} \to \Sigma$. Then, $\tilde{\varphi}$ satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility if and only if $\tilde{\varphi}$ is a single-lapping rule.*

Theorems 2 and 3 are tight if there are at least four agents. When there are only three agents, *flexibility* is directly implied by *individual rationality*. The following examples show the independence of the axioms.

Example 1 (*Strategy-proofness*)**.** *For each $i \in N$ and each $\succsim \in \mathcal{D}^*$, let*

$$IR_i(\succsim) \equiv \{C \in \mathcal{C}_i, \text{ such that for each } j \in C, C \succsim_j \{j\}\}.$$

Let $i \in N$. Let φ^{-SP} be such that for each $\succsim \in \mathcal{D}^*$, $\varphi_i^{-SP}(\succsim) \equiv \mathrm{top}(IR_i(\succsim), \succsim_i)$ and for each $j \notin \mathrm{top}(IR_i(\succsim), \succsim_i)$, $\varphi_j^{-SP}(\succsim) \equiv \{j\}$. Note that φ^{-SP} satisfies individual rationality, non-bossiness, and flexibility. However, φ^{-SP} violates strategy-proofness.[8]

[8] In order to check that φ^{-SP} is manipulable, let $N = \{i,j,k\}$, $\succsim \in \mathcal{D}^*$, and $\succsim'_j \in \mathcal{D}^*_j$ be such that $\{i,j\} \succ_i \{i,j,k\} \succ_i \{i\}$, $\{i,j,k\} \succ_j \{i,j\} \succ_j \{j,k\} \succ_j \{j\}$, and $\{i,k\} \succ_k \{i,j,k\} \succ_k \{k\}$; while $\{j,k\} \succ'_j \{1,j,k\} \succ'_j \{j\}$. Note that $\varphi^{-SP}(\succsim) = (\{i,j\}, \{k\})$, while $\varphi^{-SP}(\succsim_{N\setminus\{j\}}, \succsim'_j) = \{i,j,k\}$. Then, $\varphi_j^{-SP}(\succsim_{N\setminus\{j\}}, \succsim'_j) \succ_j \varphi_j^{-SP}(\succsim)$.

Example 2 (*Individual rationality*)**.** *Let* $N = \{i, j, k\}$*. Assume that agents* i *and* j *form a club and have to vote on whether they admit agent* k *in the club. In order to join the club, agent* k *only needs the support of one of the other agents. Then, for each* $\succsim \in \mathcal{D}^*$*, let*

$$\varphi^{-IR}(\succsim) \equiv \begin{cases} \{i,j,k\} & \text{if } \{i,j,k\} \succ_k \{k\}, \text{ and either } \{i,k\} \succ_i \{i\}, \text{ or } \{j,k\} \succ_j \{j\}, \\ (\{i,j\},\{k\}) & \text{otherwise.} \end{cases}$$

The rule φ^{-IR} *is an instance of voting by committees as presented by Barberà et al. [5]. It is clear that* φ^{-IR} *satisfies strategy-proofness, non-bossiness, and flexibility. However,* φ^{-IR} *violates individual rationality.*

Example 3 (*Non-Bossiness*)**.** *Let* $N = \{i, j, k\}$*. Let* φ^{-NB} *be such that for each* $\succsim \in \mathcal{D}^*$*,*

$$\varphi^{-NB}(\succsim) = \begin{cases} \{i,j,k\} & \text{if for each } i' \in N, \{i,j,k\} \succsim_{i'} \{i'\}, \\ (\{i,j\},\{k\}) & \text{if } \{i,j\} \succ_i \{i\}, \{i,j\} \succ_j \{j\} \text{ and } \mathrm{top}(\mathcal{N}, \succsim_k) = \{k\}, \\ (\{i\},\{j\},\{k\}) & \text{otherwise.} \end{cases}$$

It is not difficult to check that φ^{-NB} *satisfies individual rationality, strategy-proofness, flexibility. However,* φ^{-NB} *violates non-bossiness.*[9]

Example 4 (*Flexibility*)**.** *Let* $N = \{i, j, k, l\}$*. Let* φ^{-F} *be such that for each* $\succsim \in \mathcal{A}$*,*

$$\varphi^{-F}(\succsim) = \begin{cases} (\{i,j\},\{k,l\}) & \text{if for each } m \in N, \ (\{i,j\},\{k,l\}) \succsim_m \{m\}, \\ (\{i\},\{j\},\{k\},\{l\}) & \text{otherwise.} \end{cases}$$

It is immediate to check that φ^{-F} *satisfies individual rationality, strategy-proofness, and non-bossiness. However,* φ^{-R} *violates flexibility.*

6 Proof of Theorem 2

We begin this section by introducing some properties that are implied by our axioms. These properties incorporate the idea that a rule cannot be against the preferences of the members of the society. When there is a partition that each agent considers at least as good as every other partition, a rule should choose that best-preferred partition. A

[9] In order to check that φ^{-NB} violates *non-bossiness*, let $\succsim \in \mathcal{A}$, $\succsim'_k \in \mathcal{A}^k$ be such that $\{i,j\} \succ_i \{i\}$, $\{i,j\} \succ_j \{j\}$, $\mathrm{top}(\mathcal{N}, \succsim_k) = \{k\}$, while $\{j,k\} \succ'_k \{k\} \succ'_k \{i,j,k\}$. Note that $\varphi(\succsim) = (\{i,j\},\{k\})$ and $\varphi(\succsim_{N\setminus\{k\}}, \succsim'_k) = [\{i\},\{j\},\{k\}]$.

stronger requirement would be that whenever the members of a coalition consider this coalition as the best coalition, a rule should allow them to join, independently of the preferences of the remaining agents in society.

Unanimity (on the Range). Let $\sigma = \{C_1, \ldots, C_m\} \in \Sigma$ be such that for each $t = 1, \ldots, m$, $C_t \in F^{\varphi}$. For each $\succsim \in \mathcal{D}^*$, each $t = 1, \ldots, m$, and each $i \in C_t$, $\text{top}(F^{\varphi}, \succsim_i) = C_t$ implies $\varphi(\succsim) = \sigma$.

Top-Coalition (on the Range). Let $C \in F^{\varphi}$ and $\succsim \in \mathcal{D}^*$. If for each $i \in C$, $\text{top}(F^{\varphi}, \succsim_i) = C$, then for each $i \in C$, $\varphi_i(\succsim) = C$.

It is clear that *top-coalition* implies *unanimity.* Note that *top-coalition* is a property of rules. Banerjee *et al.* [3] use the term top-coalition to name a a property of preference profiles. These authors say that a preference profile satisfies the top-coalition property if for every group of agents $V \subseteq N$ there is a coalition $C \subseteq V$ that is mutually the best for all the members of C. Basically, our *top-coalition* implies that if a preference profile satisfies the Banerjee *et al.*'s top-coalition property, then the rule selects a partition in which the coalition that all its members consider that coalition as the best is formed.

Lemma 1. *Let $\varphi : \mathcal{D}^* \to \Sigma$ satisfy strategy-proofness non-bossiness, and flexibility. Then, φ satisfies unanimity.*

Proof. Let $\sigma = \{C_1, \ldots, C_m\} \in \Sigma$ be such that for each $t = 1, \ldots, m$, $C_t \in F^{\varphi}$. Let $\succsim \in \mathcal{D}^*$ be such that for each $t = 1, \ldots, t$ and each $i \in C_t$, $\text{top}(F^{\varphi}, \succsim_i) = C_t$. By *flexibility*, $\sigma \in R^{\varphi}$. Then, there is $\succsim' \in \mathcal{D}^*$, such that $\varphi(\succsim') = \sigma$. Let $i \in N$. Let $\succsim'' \in \mathcal{A}$ be such that $\succsim''_i = \succsim_i$ while for each $j \in N \setminus \{i\}$, $\succsim''_j = \succsim'_j$. By *strategy-proofness*, $\varphi_i(\succsim'_{N\setminus\{i\}}, \succsim_i) \succsim_i \varphi_i(\succsim') = \text{top}(F^{\varphi}, \succsim_i)$. Then, $\varphi_i(\succsim'_{N\setminus\{i\}}, \succsim_i) = \varphi_i(\succsim') = \text{top}(F^{\varphi}, \succsim_i)$. By *non-bossiness*, $\varphi(\succsim'_{N\setminus\{i\}}, \succsim_i) = \varphi(\succsim')$. Repeating the argument as many times as necessary, we obtain $\varphi(\succsim) = \varphi(\succsim')$. ■

Lemma 2. *Let $\varphi : \mathcal{D}^* \to \Sigma$ satisfy strategy-proofness, individual rationality, non-bossiness, and flexibility. Then, φ satisfies top-coalition.*

Proof. Let $C \in F^{\varphi}$. Let $\succsim \in \mathcal{D}^*$ be such that for each $i \in C$, $\text{top}(F^{\varphi}, \succsim_i) = C$. If $\#C = 1$, the result follows from *individual rationality.* If $C = N$, the result is immediate by *unanimity.* Let $\succsim' \in \mathcal{D}^*$ be such that for each $i \in C$, $\text{top}(F^{\varphi}, \succsim'_i) = C$, and for each

$C' \subseteq N$, such that there is $j \in (C' \setminus C)$, $\{i\} \succ_i C'$, while for each $k \notin C$, $\succsim_k = \succsim'_k$. By *individual rationality*, for each $i \in C$, $\varphi_i(\succsim) \subseteq C$. Let $\succsim'' \in \mathcal{A}$ be such that for each $i \in C$, $\succsim'_i = \succsim''_i$ while for each $k \in (N \setminus C)$, $\varphi_k(\succsim') = \text{top}(F^\varphi, \succsim''_k)$. By *strategy-proofness*, $\varphi_k(\succsim'_{N\setminus\{k\}}, \succsim''_k) = \varphi_k(\succsim')$. By *non-bossiness*, $\varphi(\succsim'_{N\setminus\{k\}}, \succsim''_k) = \varphi(\succsim')$. Repeating the arguments for each $k \in (N \setminus C)$, $\varphi(\succsim') = \varphi(\succsim'')$. By *unanimity*, for each $i \in C$, $\varphi_i(\succsim'') = C$. Then, $\varphi_i(\succsim') = C$. Finally, let $i \in C$. By *strategy-proofness*, $\varphi_i(\succsim'_{N\setminus\{i\}}, \succsim_i) \succsim_i \varphi_i(\succsim')$. Then, $\varphi_i(\succsim'_{N\setminus\{i\}}, \succsim_i) = C$. Repeating the argument as many times as necessary, we obtain that for each $i \in C$, $\varphi_i(\succsim) = C$. ■

In the following lemma we prove that agents' preferences over unfeasible coalitions are irrelevant for the social choice.

Lemma 3. *Let $\varphi : \mathcal{D}^* \to \Sigma$ satisfy strategy-proofness and non-bossiness. Then, for each $\succsim, \succsim' \in \mathcal{D}^*$ such that for each $i \in N$, and each $C, C' \in (F^\varphi \cap \mathcal{C}_i)$, $C \succ_i C'$ if and only if $C \succ'_i C'$, $\varphi(\succsim) = \varphi(\succsim')$.*

Proof. Let $\succsim, \succsim' \in \mathcal{D}^*$ be such that for each each $i \in N$, and each $C, C' \in (F^\varphi \cap \mathcal{C}_i)$, $C \succ_i C'$ if and only if $C \succ'_i C'$. Let $i \in N$. By *strategy-proofness*, $\varphi_i(\succsim_{N\setminus\{i\}}, \succsim'_i) \succsim'_i \varphi_i(\succsim)$ and $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim_{N\setminus\{i\}}, \succsim'_i)$. Then, as for each $C, C' \in (F^\varphi \cap \mathcal{C}_i)$, $C \succ_i C'$ if and only if $C \succ'_i C'$, $\varphi_i(\succsim) = \varphi_i(\succsim_{N\setminus\{i\}}, \succsim'_i)$. By *non-bossiness*, $\varphi(\succsim) = \varphi(\succsim_{N\setminus\{i\}}, \succsim'_i)$. Repeating the argument as many times as necessary, we get $\varphi(\succsim) = \varphi(\succsim')$. ■

The following lemma presents the crucial step in the proof of Theorem 2.

Lemma 4. *Let $\varphi : \mathcal{D}^* \to \Sigma$ satisfy strategy-proofness, individual rationality, non-bossiness, and flexibility. Then, F^φ satisfies the single-lapping property.*

Proof. The proof is by induction on the number of agents. We first focus on three-agent societies. Then, we extend the result to arbitrary societies.

Claim 1. Let $n = 3$, then F^φ satisfies *Condition (a)* of the single-lapping property.

Let $N = \{i, j, k\}$. Assume to the contrary that F^φ does not satisfy *Condition (a)*. Then, there are $C, C' \in F^\varphi$ such that $\#(C \cap C') \geq 2$. We have two cases.

Case (*a.1*): $F^\varphi = \{\{i\}, \{j\}, \{k\}, \{i, j\}, \{i, j, k\}\}$.

Let $\bar{\succsim}_k \in \mathcal{D}_k^*$ be such that $\{i,j,k\}\bar{\succ}_k\{i,k\}\bar{\succ}_k\{j,k\}\bar{\succ}_k\{k\}$. Let the rule $\bar{\varphi}^{\{i,j\}} : \mathcal{D}^*_{\{i,j\}} \to \Sigma$ be such that for each $\succsim_{\{i,j\}}\in \mathcal{D}^*_{\{i,j\}}$, $\bar{\varphi}^{\{i,j\}}(\succsim_{\{i,j\}}) \equiv \varphi(\succsim_{\{i,j\}}, \bar{\succsim}_k)$. By φ's *strategy-proofness*, $\bar{\varphi}^{\{i,j\}}$ satisfies *strategy-proofness*. By φ's *top-coalition*,

$$R^{\bar{\varphi}^{\{i,j\}}} = \{(\{i\},\{j\},\{k\}),(\{i,j\},\{k\}),\{i,j,k\}\}.$$

Note that, agent i and agent j's preferences over the partitions in $R^{\bar{\varphi}^{\{i,j\}}}$ are unrestricted. Hence, $\bar{\varphi}^{\{i,j\}}$ satisfies *strategy-proofness*, its range contains three elements, and agents' preferences over the elements of the range are unrestricted. Then, by the Gibbard-Satterthwaite Theorem, $\bar{\varphi}^{\{i,j\}}$ is *dictatorial*. Assume that i is the dictator for $\bar{\varphi}^{\{i,j\}}$. Let $\succsim_{\{i,j\}}\in \mathcal{D}^*_{\{i,j\}}$ be such that $\{i,j,k\} \succ_i \{i,j\} \succ_i \{i\}$ and $\{j\} \succ_j \{i,j\} \succ_j \{i,j,k\}$. Then, $\varphi(\succsim_{\{i,j\}}, \bar{\succsim}_k) = \{i,j,k\}$, but $\{j\} \succ_j \varphi_j(\succsim)$, which violates *individual rationality*.

Case *(a.2)* $\{\{i\},\{j\},\{k\},\{i,j\},\{j,k\},\{i,j,k\}\} \subseteq F^{\varphi}$.

Let $\succsim^1\in \mathcal{D}^*$ be such that,

$\succsim_i^1$:	$\succsim_j^1$:	$\succsim_k^1$:
$\{i,j\}$	$\{i,j\}$	$\{j,k\}$
$\{i\}$	$\{j\}$	$\{i,j,k\}$
$\{i,j,k\}$	$\{i,j,k\}$	$\{k\}$
$\{i,k\}$	$\{j,k\}$	$\{i,k\}$

By *top-coalition*, $\varphi(\succsim^1) = (\{i,j\},\{k\})$.

Let $\succsim^2\in \mathcal{D}^*$ be such that $\succsim^2_{N\setminus\{i\}}=\succsim^1_{N\setminus\{i\}}$ and $\{i,j,k\} \succsim_i^2 \{i,j\} \succsim_i^2 \{i,k\} \succsim_2^i \{i\}$. By *strategy-proofness*, $\varphi_i(\succsim^2) \succsim_i^2 \varphi_i(\succsim^1)$. Then, $\varphi_i(\succsim^2)$ is either $\{i,j,k\}$ or $\{i,j\}$. As $\{j\} \succ_j^2 \{i,j,k\}$, by *individual rationality*, $\varphi_i(\succsim^2) = \{i,j\}$. Then, by *non-bossiness*, $\varphi(\succsim^2) = \varphi(\succsim^1)$.

Let $\succsim^3\in \mathcal{D}^*$ be such that $\succsim^3_{N\setminus\{j\}}=\succsim^2_{N\setminus\{j\}}$ and $\{i,j\} \succsim_j^3 \{i,j,k\} \succsim_j^3 \{j\}$. By *strategy-proofness*, $\varphi_j(\succsim^3) \succsim_j^3 \varphi_j(\succsim^2)$. Then, $\varphi_j(\succsim^3) = \{i,j\}$. By *non-bossiness*, $\varphi(\succsim^3) = \varphi(\succsim^2)$.

Now, let $\succsim^4\in \mathcal{D}^*$ be such that $\succsim^4_{N\setminus\{i\}}=\succsim^3_{N\setminus\{i\}}$ and $\{i,k\} \succsim_i^4 \{i,j,k\} \succsim_i^4 \{i\}$. Then,

$\succsim_i^4$:	$\succsim_j^4$:	$\succsim_k^4$:
$\{i,k\}$	$\{i,j\}$	$\{j,k\}$
$\{i,j,k\}$	$\{i,j,k\}$	$\{i,j,k\}$
$\{i\}$	$\{j\}$	$\{k\}$
$\{i,j\}$	$\{j,k\}$	$\{i,k\}$

By *individual rationality,* $\varphi_k(\succsim^4) \neq \{i,k\}$ and $\varphi_j(\succsim^4) \neq \{j,k\}$. By *strategy-proofness,* $\varphi_i(succsim^3) \succsim_i^3 \varphi(\succsim^4)$. Note that, $\{i,j,k\} \succ_i^3 \varphi(\succsim^3)$. Then, $\varphi(\succsim^4) = (\{i\},\{j\},\{k\})$.

Let $\succsim^5 \in \mathcal{D}^*$ be such that $\succsim_i^5 = \succsim_i^4$, $\{j,k\} \succ_j^5 \{j\} \succ_j^5 \{i,j,k\} \succsim_j^5 \{i,j\}$, and $\{i,j,k\} \succsim_k^5 \{j,k\} \succsim_k^5 \{i,k\} \succsim_k^5 \{k\}$. By *top-coalition,* $\varphi_k(\succsim_{N\setminus\{k\}}^5, \succsim_k^4) = \{j,k\}$. By *strategy-proofness,* $\varphi_k(\succsim^5) \succsim_k^5 \{j,k\}$. As $\{j\} \succ_j^5 \{i,j,k\}$, by *individual rationality,* $\varphi(\succsim^5) = (\{i\},\{j,k\})$.

Let $\succsim^6 \in \mathcal{D}^*$ be such that $\succsim_{N\setminus\{j\}}^6 = \succsim_{N\setminus\{j\}}^5$ and $\{i,j,k\} \succsim_j^6 \{j,k\} \succsim_j^6 \{i,j\} \succsim_j^6 \{j\}$. Note that, by *unanimity,* $\varphi(\succsim_{N\setminus\{i\}}^6, \succsim_i^3) = \{i,j,k\}$. Hence, by *strategy-proofness,* $\varphi_i(\succsim^6) \succsim_i \{i,j,k\}$. Then, $\varphi(\succsim^6) = \{i,j,k\}$.

Finally, let $\succsim^7 \in \mathcal{D}^*$ be such that $\succsim_{N\setminus\{j\}}^7 = \succsim_{N\setminus\{j\}}^6$ and $\succsim_j^7 = \succsim_j^4$. Then

$\succsim_i^7$:	$\succsim_j^7$:	$\succsim_k^7$:
$\{i,k\}$	$\{i,j\}$	$\{i,j,k\}$
$\{i,j,k\}$	$\{i,j,k\}$	$\{j,k\}$
$\{i\}$	$\{j\}$	$\{i,k\}$
$\{i,j\}$	$\{j,k\}$	$\{k\}$

Note that the only difference between $\succsim^4$ and $\succsim^7$ consists of k's preference. By *strategy-proofness,* $\varphi_j(\succsim^7) \succsim_j^7 \varphi_j(\succsim^6) = \{i,j,k\}$. By *individual rationality,* if $j \in \varphi_i(\succsim^7)$, then $\varphi_i(\succsim^7) = \{i,j,k\}$. Hence, $\varphi(\succsim^7) = \{i,j,k\}$. However, $\varphi_k(\succsim^7) \succ_k^4 \varphi_k(\succsim^4)$, which violates *strategy-proofness.*

Cases $(a.1)$ and $(a.2)$ exhaust (up to relabelling the agents) all the possibilities. Then, F^φ satisfies *Condition (a)*, which concludes the proof of Claim 1.

Claim 2. Let $n = 3$, then F^φ satisfies *Condition (b)* of the single-lapping property.

Assume, to the contrary, that F^φ does not satisfy *Condition (b).* Then, there is a list of coalitions $\{C_1, \ldots, C_m\} \subset F^\varphi$, with $m \geq 3$ such that for each $t = 1, \ldots, m$, $\#(C_t \cap C_{t+1}) \geq 1$ and for no $i \in N$, $(C_t \cap C_{t+1}) = \{i\}$. As, by Claim 1, φ satisfies *Condition (a)*, we have $F^\varphi = \{\{i\},\{j\},\{k\},\{i,j\},\{j,k\},\{i,k\}\}$. Then, for each $\succsim \in \mathcal{D}^*$, there is $i' \in \{i,j,k\}$ such that

$$\varphi_{i'}(\succsim) = \{i'\} \tag{*}$$

Let $\succsim \in \mathcal{D}^*$, be such that $\{i,j\} \succ_i \{i,k\} \succ_i \{i\}$, $\{j,k\} \succ_j \{i,j\} \succ_j \{j\}$, while $\{i,k\} \succ_k \{j,k\} \succ_k \{k\}$. Let $\succsim_i' \in \mathcal{D}_i^*$ be such that $\text{top}(F^\varphi, \succsim_i') = \{i,k\}$. By *top-coalition,*

$\varphi(\succsim_{N\setminus\{i\}}, \succsim_i') = (\{i,k\},\{j\})$. By *strategy-proofness*, $\varphi_i(\succsim) \succ_i \varphi(\succsim_{N\setminus\{i\}}, \succsim_i')$. Then, we have that $\varphi_i(\succsim) \neq \{i\}$. Using parallel arguments, we get $\varphi_j(\succsim) \neq \{j\}$ and $\varphi_k(\succsim) \neq \{k\}$, which contradicts (*) and concludes the proof of Claim 2.

Now, we extend the result to arbitrary finite societies.

Induction Step. There is $m \geq 3$ such that for each $n \leq m$, if the n-agent rule φ satisfies *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility*, then F^φ satisfies the single-lapping property. Now we prove this is true for $n = m+1$.

By Claims 1 and 2, the induction hypothesis is true for $n \leq 3$. (If $n = 1, 2$ the single-lapping property trivially holds.) Let $n = m+1$. Assume that φ satisfies *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility*. First, we prove two facts.

Fact 1. For each $C, C' \in F^\varphi$ such that $C \cup C' \neq N$, $\#(C \cap C') = 1$.

Let $C, C' \in F^\varphi$ such that $(C \cup C') \neq N$. Let $j \in N \setminus (C \cup C')$. Let $\bar{\succsim}_j \in \mathcal{D}_j^*$ be such that for each $C \in \mathcal{C}_j$, $C \neq \{j\}$, $\{j\} \bar{\succ}_j C$. Let $\Sigma_{N\setminus\{j\}}$ denote all the partitions of the reduced society $N \setminus \{j\}$. Define the rule $\bar{\varphi}^{N\setminus\{j\}} : \mathcal{D}^*_{N\setminus\{j\}} \to \Sigma_{N\setminus\{j\}}$. Let $\bar{\varphi}^{N\setminus\{j\}}$ be such that for each $\succsim_{N\setminus\{j\}}$, $(\bar{\varphi}^{N\setminus\{j\}}(\succsim_{N\setminus\{j\}}), \{j\}) \equiv \varphi(\succsim_{N\setminus\{j\}}, \bar{\succsim}_j)$. By φ's *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility*, $\bar{\varphi}^{N\setminus\{j\}}$ satisfies *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility*. By the induction hypothesis, $F^{\bar{\varphi}^{N\setminus\{j\}}}$ satisfies the *single-lapping* property. By φ's *flexibility*, $C, C' \in F^{\bar{\varphi}^{N\setminus\{j\}}}$, then $\#(C \cap C') = 1$.

With similar arguments, we can also prove the following fact.

Fact 2. For each $\{C_1, \ldots, C_m\} \subseteq \Pi$ with $m \geq 3$, $\cup_{t=1}^m C_t \neq N$, and for each $t = 1, \ldots, m$, $\#(C_t \cap C_{t+1}) \geq 1$ (where $m+1 = 1$), there is $i \in N$ such that for each $t = 1, \ldots, m$, $C_t \cap C_{t+1} = \{i\}$.

Claim 1′. F^φ satisfies *Condition (a)*.

Assume, to the contrary, that there are $C, C' \in F^\varphi$ such that $(C \cup C') = N$, and $\#(C \cap C') \geq 2$. There are three cases:

Case $(a.0')$ Let $C, C' \neq N$.

By Fact 1, either $F^{\varphi} = \{[N], C, C'\}$, or $F^{\varphi} = \{[N], C, C', N\}$. Let $\bar{\succsim}_{N\setminus(C\cap C')} \in \mathcal{D}^*_{N\setminus(C\cap C')}$ be such that for each $j \in (C \setminus C')$, $\text{top}(F^{\varphi}, \succsim_j) = C$, whereas for each $k \in (C' \setminus C)$, $\text{top}(F^{\varphi}, \succsim_k) = C'$. Define the rule $\bar{\varphi}^{C\cap C'} : \mathcal{D}^*_{C\cap C'} \to \Sigma$ in such a way that for each $\succsim_{C\cap C'} \in \mathcal{D}^*_{C\cap C'}$, $\bar{\varphi}^{C\cap C'}(\succsim_{C\cap C'}) \equiv \varphi(\succsim_{C\cap C'}, \bar{\succsim}_{N\setminus(C\cap C')})$. As φ is *strategy-proof*, $\bar{\varphi}^{C\cap C'}$ is *strategy-proof*. Moreover, by *top-coalition*, $R^{\bar{\varphi}^{C\cap C'}} = \{[N], (C, [C' \setminus C]), (C', [C \setminus C'])\}$. Note that, richness of the domain $\mathcal{D}^*$ does not impose any restriction on the preferences of the agents in $(C \cap C')$ over the partitions in $R^{\bar{\varphi}^{C\cap C'}}$. By the Gibbard-Satterthwaite Theorem, $\varphi^{C\cap C'}$ is *dictatorial*. Let $i \in (C\cap C')$ be a dictator for $\varphi^{C\cap C'}$. Let $\succsim_{C\cap C'} \in \mathcal{D}^*_{C\cap C'}$ be such that $\text{top}(F^{\bar{\varphi}^{C\cap C'}}, \succsim_i) = C'$, while for each $j \in (C\cap C')\setminus\{i\}$, $\text{top}(F^{\bar{\varphi}^{C\cap C'}}, \succsim_j) = \{j\}$. Then, $\varphi(\succsim_{C\cap C'}, \bar{\succsim}_{N\setminus(C\cup C')}) = (C', [C \setminus C'])$, which violates *individual rationality*.

Case $(a.1')$ Let $C' = N$, and for no $i \in C$ there is $j \in N \setminus C$, $C'' \subset N$, such that $\{i, j\} \subseteq C'' \in F^{\varphi}$.
Let $\succsim_{N\setminus C} \in \mathcal{D}^*_{N\setminus C}$ be such that for each $j \in (N \setminus C)$, $\text{top}(F^{\varphi}, \bar{\succ}_j) = N$. Define now the rule $\bar{\varphi}^C : \mathcal{D}^*_C \to \Sigma$ in the following way. For each $\succsim_C \in \mathcal{D}^C_*$, $\bar{\varphi}^C(\succ_C) \equiv \varphi(\succsim_C, \bar{\succsim}_{N\setminus C})$. Clearly, $\bar{\varphi}^C$ satisfies *strategy-proofness*. Moreover, by *top-coalition*, $R^{\varphi} = R^{\bar{\varphi}^C}$. Richness of the domain of preferences does not impose any restriction on the order in which agents in C may compare partitions in $R^{\bar{\varphi}^C}$. Then, by the Gibbard-Satterthwaite Theorem, $\bar{\varphi}^C$ is *dictatorial*, which, by an already familiar argument, violates φ's *individual rationality*.

Case $(a.1')$ Let $C' = N$, and for some $i \in C$ there is $j \in N \setminus C$, $C'' \subset N$, such that $\{i, j\} \subseteq C'' \in F^{\varphi}$.

Note first that, by Fact 1, for each $C'' \in (F^{\varphi} \setminus N)$, $\#(C \cap C'') \leq 1$. Moreover, by Fact 2, there is no cycle of three coalitions in F^{φ} that does not involve the grand coalition N. Let $T \equiv N \setminus C$. Let $j \in C$ be such that there is $\bar{T} \subseteq T$ such that $\bar{T} \cup \{j\} \in F^{\varphi}$. Note that by Fact 1, for each $T' \subseteq T$ with $(T' \cup \{j\}) \in F^{\varphi}$, $(\bar{T} \cap T') = \{\varnothing\}$. Let $\bar{C} \equiv (C \setminus \{j\})$.

Assume that $C, N \in F^{\varphi}$, and also there is $j \in C$ such that for some $T \subseteq N \setminus C$, $T \cup \{j\} \in F^{\varphi}$. Note that, by Fact 1, if $T' \in F^{\varphi} \setminus C$, then neither there is $i \in C \setminus \{j\}$, such that $\{i, j\} \subseteq T'$, nor there is $k \in T$ such that $\{i, k\} \subseteq T'$. Let $\bar{C} \equiv C \setminus \{j\}$.

In this step, we simply replicate the arguments of Case $(a.2)$. Let $\succsim^1 \in \mathcal{D}^*$ be such that for each $i \in \bar{C}$, there is $P^1_i \in \mathcal{P}$ with $j = \max(N, P^1_i)$, $N^+_i(P) = C$, and $\succsim^1_i = \succsim^-_i (P^1_i)$, for j there is $P^1_j \in \mathcal{P}$ with $N^+_j(P) = C$, and $\succsim^1_j = \succsim^-_j (P^1_j)$, while for each $k \in N \setminus C$, there is $P_k^1 \in \mathcal{P}$ with $N_k^+(P) = \{j\} \cup \{k\}$, and $\succsim_k^1 = \succsim_k^+ (P_k^1)$. By *top-coalition*, for each $i \in C$, $\varphi_i(\succsim^1) = C$.

Next, let $\succsim^2 \in \mathcal{D}^*$ be such that $\succsim_{N\setminus\bar{C}}^1 = \succsim_{N\setminus\bar{C}}^2$, while for each $i \in \bar{C}$ there is $P_i^2 \in \mathcal{P}$ such that $j = \max(N, P_i^2)$, $N = N_i^+(P_i^2)$, and $\succsim_i^2 = \succsim_i^+ (P_i^2)$. Note that for each $i \in \bar{C}$, $N = \text{top}(F^\varphi, \succsim_i^2)$ and $C = \text{top}(F^\varphi \setminus N, \succsim_i^2)$. Let $i \in \bar{C}$, by *strategy-proofness*, $\varphi_i(\succsim_{N\setminus\{i\}}^1, \succsim_i^2) \succsim_i^2 \varphi_i(\succsim^1) = C$. By *individual rationality*, $\varphi_j(\succsim_{N\setminus\{i\}}^1, \succsim_i^2) \neq N$. Then, $\varphi_i(\succsim_{N\setminus\{i\}}^1, \succsim_i^2) = C$. By *non-bossiness*, $\varphi(\succsim_{N\setminus\{i\}}^1, \succsim_i^2) = \varphi(\succsim^1)$. Repeating the same argument iteratively with each $i \in \bar{C}$, we get $\varphi(\succsim^2) = \varphi(\succsim^1)$.

Let $\succsim^3 \in \mathcal{D}^*$ be such that $\succsim_{N\setminus\{j\}}^2 = \succsim_{N\setminus\{j\}}^3$ and $\succsim_j^3 = \succsim_j^+ (P_j^1)$. Note that $\text{top}(F^\varphi, \succsim_j^3) = C$. By *strategy-proofness*, $\varphi_j(\succsim^3) \succsim_j^3 \varphi(\succsim^2)$. Then, $\varphi_j(\succsim^3) = C$, and by *non-bossiness*, $\varphi(\succsim^3) = \varphi(\succsim^2)$.

Let $\succsim^4 \in \mathcal{D}^*$ be such that $\succsim_{N\setminus\bar{C}}^3 = \succsim_{N\setminus\bar{C}}^4$, while for each $i \in \bar{C}$ there is $P_i^4 \in \mathcal{P}$ such that for some $\bar{k} \in \bar{T}$ $\max(N, P_i^4) = \bar{k}$, $N_i^+(P_i^4) = \bar{T} \cup \{i\}$, and $\succsim_i^4 = \succsim_i^+ (P_i^4)$. Note that by Fact 2, and our assumptions on F^φ, for each $i \in \bar{C}$, $\text{top}(F^\varphi, P_i^4) = N$, and for each $C \in F^\varphi \cap \mathcal{C}_i$, if $C \neq N$, then $\{i\} \succsim_i^4 C$. Let $i \in \bar{C}$, by *strategy-proofness*, $\varphi_i(\succsim^3) \succsim_i^3 \varphi_i(\succsim_{N\setminus\{i\}}^3, \succsim_i^4)$. Hence, $\varphi_i(\succsim_{N\setminus\{i\}}^3, \succsim_i^4) \neq N$. Repeating the argument for each $i \in \bar{C}$, we obtain that $\varphi(\succsim^4) \neq N$. Then, by *individual rationality*, we have that $\varphi(\succsim^4) = [N]$.[10]

Consider now the profile $\succsim^5 \in \mathcal{D}^*$, such that for that for each $i \in \bar{C}$, $\succsim_i^5 = \succsim_i^4$, for some $P_j^5 \in \mathcal{P}$ such that there is $\bar{k} \in T$, with $\max(N, P_j^5) = \bar{k}$ and $N_j^+(P_j^5) = N$, and $\succsim_j^5 = \succsim_j^+ (P_j^5)$, while for each $k \in N \setminus C$, there is $P_k^5 \in \mathcal{P}$ such that $j = \max(N, P_k^5)$, $N = N_k^+(P_k^5)$, and $\succsim_k^5 = \succsim_k^+ (P_k^5)$. By *unanimity*, $\varphi(\succsim^5) = N$.

Finally, let $\succsim^6 \in \mathcal{D}^*$, be such that for each $\succsim_C^7 = \succsim_C^4$, while $\succsim_{N\setminus C}^6 = \succsim_{N\setminus C}^5$. That is, we only change agent j's preferences with respect to the previous profile. By *strategy-proofness*, $\varphi_j(\succsim^6) \succsim^6 \varphi_6(\succsim^5 = N$. By *individual rationality*, for each $i \in \bar{C}$,if $j \in \varphi_i(\succsim^6)$, then $\varphi_i(\succsim^6) = N$. Hence, $\varphi(\succsim^6) = N$. Note now that $\succsim^6$ only differs from $\succsim^4$ in the preferences of the agents who belong to $N \setminus C$. Let $k \in N \setminus T$. By *strategy-proofness*, $\varphi_k(\succsim_{N\setminus\{k\}}^6, \succsim_k^4) \succsim_k^4 \varphi_k(\succsim^6) = N$. Then, $j \in \varphi_k(\succsim_{N\setminus\{k\}}^6, \succsim_k^4)$. By *individual rationality*, there is $i \in \bar{C}$ such that $i \in \varphi_j(\succsim_{N\setminus\{k\}}^6, \succsim_k^4)$. By Fact 1, and our assumptions over F^φ, $\varphi(\succsim_{N\setminus\{k\}}^6, \succsim_k^4) = N$. Repeating the argument as many times as necessary, we get that

[10]Note that for each $i \in \bar{C}$, N is the only coalition in F^φ that is preferred to staying on her own. On the other hand, for agent j, the coalitions that are preferred to staying alone include some member of $\bar{C}$. Thus, j also stays alone. Finally, each agent $k \in N \setminus C$ requires the presence of agent j in order to be happy about joining any given coalition.

$\varphi(\succsim^4) = N$, a contradiction.

Cases $(a.0')$, $(a.1')$ and $(a.2')$ exhaust all the possibilities. Then, this suffices to prove that φ satisfies *Condition (a).*

Claim $2'$**.** F^{φ} satisfies *Condition (b).*

Assume, to the contrary, that φ does not satisfy *Condition (b).* Then, there is a list of coalitions $\{C_1, \ldots, C_m\}$, with $k \geq 3$ such that for each $t = 1, \ldots, m$, $(m+1 = 1)$, $(C_t \cap C_{t+1}) \neq \{\varnothing\}$, and there is no $i \in N$ such that for each $t = 1, \ldots, m$, $\{i\} = (C_t \cap C_{t+1})$. As we have just proved that F^{φ} satisfies *Condition (a)* of the single-lapping property, for each $t = 1, \ldots, k$, $\#(C_t \cap C_{t+1}) = 1$. With similar arguments to those we use in Fact 1, we can prove that $\cup_{t=1}^{k} C_t = N$. Moreover, $F^{\varphi} = \{C_1, \ldots, C_k\} \cup [N]$.

For each $t = 1, \ldots, k$, let $i_t \equiv (C_t \cap C_{t+1})$. Let $\succsim \in \mathcal{D}^*$ be such that for each $t = 1, \ldots, k$ and each $j \in (C_t \setminus \{i_t, i_{t+1}\})$, $\text{top}(F^{\varphi}, \succsim_j) = C_t$. On the other hand, for each $t = 1, \ldots, k$, $\text{top}(F^{\varphi}, \succsim_{i_t}) = C_{t+1}$, and $C_t \succ_{i_t} \{i_t\}$. By *top-coalition* and the repeated application of *strategy-proofness*, for each $t = 1, \ldots, m$; $\varphi_{i_t}(\succsim) \succsim_{i_t} C_t$.

Assume first that m is odd, there is $t' \in \{1, \ldots, m\}$ such that $\varphi_{i_{t'}}(\succsim) = \{i_{t'}\}$, a contradiction with $\varphi_{i_t}(\succsim) \succsim_{i_t} C_t$ for each $t = 1, \ldots, m$.

Assume now that m is even. Without loss of generality, assume that for each t odd, $\varphi_{i_t}(\succsim) = C_{t+1}$ and for each t' even, $\varphi_{i_{t'}}(\succsim) = C_{t'}$. Let $\bar{t}$ be even and let $\succsim'_{i_{\bar{t}}} \in \mathcal{D}^*_{i_{\bar{t}}}$ be such that $\text{top}(F^{\varphi}, \succsim'_{i_{\bar{t}}}) = C_{\bar{t}+1}$ and for each $T \nsubseteq C_{\bar{t}+1}$, $\{i_{\bar{t}}\} \succ'_{i_{\bar{t}}} T$. By *individual rationality*, we obtain $\varphi_{i_{\bar{t}}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) \neq C_{\bar{t}}$. Let $\succsim'_{i_{\bar{t}-1}} \in \mathcal{D}^*_{i_{\bar{t}-1}}$ be such that $\text{top}(\Pi_{\varphi}, \succsim'_{i_{\bar{t}-1}}) = C_{\bar{t}-1}$. By *top-coalition*, $\varphi_{i_{\bar{t}-1}}(\succsim_{N\setminus\{i_{\bar{t}-1}, i_{\bar{t}}\}}, \succsim'_{\{i_{\bar{t}-1}, i_{\bar{t}}\}}) = C_{\bar{t}-1}$. By *strategy-proofness*, we obtain $\varphi_{i_{\bar{t}-1}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{\{i_{\bar{t}}\}}) \succsim_{i_{\bar{t}-1}} \varphi_{i_{\bar{t}-1}}(\succsim_{N\setminus\{i_{\bar{t}-1}, i_{\bar{t}}\}}, \succsim'_{\{i_{\bar{t}-1}, i_{\bar{t}}\}})$. Then, $\varphi_{i_{\bar{t}-1}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) = C_{\bar{t}-1}$, and $\varphi_{i_{\bar{t}-2}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) = C_{\bar{t}-1}$. Repeating the argument as many times as necessary, for each t odd, $\varphi_{i_t}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) = C_t$, while for each t' even $\varphi_{i_{t'}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) = C_{t'+1}$, and $\varphi_{i_{\bar{t}}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) = C_{\bar{t}+1}$. Then, we get $\varphi_{i_{\bar{t}}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) \succ_{i_{\bar{t}}} \varphi_{i_{\bar{t}}}(\succsim)$, which violates *strategy-proofness.* ■

Proof of Theorem 2. Let φ satisfy *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility.* By Lemma 2, φ satisfies *top-coalition.* Let $\succsim \in \mathcal{D}^*$. By Lemma 4, As F^{φ} satisfies the single-lapping property. Then , there is $C \in F^{\varphi}$ such that for each $i \in C$,

$\mathrm{top}(F^{\varphi}, \succsim_i) = C$. By *top-coalition*, for each $i \in C$, $\varphi_i(\succsim) = C$. Moreover, again by *top-coalition*, for each $\succsim' \in \mathcal{D}^*$ such that $\succsim_C = \succsim'_C$, for each $i \in C$, $\varphi_i(\succsim') = C$. Let $\Sigma_{N \setminus C}$ denote the set of all possible partitions of the reduced society $N \setminus C$. Define now the restricted social choice function $\bar{\varphi}^{N \setminus C} : \mathcal{D}^*_{N \setminus C} \to \Sigma_{N \setminus C}$, in such a way that for each $\succsim_{N \setminus C} \in \mathcal{D}^*_{N \setminus C}$, $(\bar{\varphi}^{N \setminus C}(\succsim_{N \setminus C}), C) \equiv \varphi(\succsim_{N \setminus C}, \succsim_C)$. Clearly, $\bar{\varphi}^{N \setminus C}$ satisfies *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility*. Moreover, $F^{\bar{\varphi}^{N \setminus C}} = \{C' \in F^{\varphi}, C \cap C' = \{\varnothing\}\}$, and $F^{\bar{\varphi}^{N \setminus C}}$ satisfies the single-lapping property. Repeating the same arguments as many times as necessary, we get $\varphi(\succsim) = \bar{\sigma}^{F^{\varphi}}(\succsim)$. ■

7 Applications to Matching Problems

In this section we present several applications of our results. Corollaries in this section consist of simple examples of rules whose sets of feasible coalitions do not fulfill the single-lapping property.

7.1 Marriage and Roommate Problems

The marriage problems are a special class of coalition formation problems. There are two disjoint groups of agents.[11] These two sets are usually interpreted as a set of men and a set of women. Each man has preferences over women and remaining single, and each woman has preferences over men and remaining single. A coalition is feasible if it consists of a couple formed by a man and a woman, or it is formed by a single agent. Then, we say that a rule φ^m is defined over a subclass of marriage problems if, there are two disjoint sets of agents M, W, $M \cup W = N$ such that

$$F^{\varphi^m} = \{(m, w) \subset N, m \in M, w \in W\} \cup [N].$$

Clearly, if $\#M \geq 2$ and $\#W \geq 2$, the set of feasible coalitions of a rule defined on a subclass of marriage problems does not satisfy the single-lapping property. Note that, additive representability of agents' preferences does not introduce any restriction on the ranking of couples. Moreover, from Lemma 3, we know that the choice of a rule satisfying our axioms is not affected by changes on the preferences over unfeasible coalitions. Then, from Theorem 1, we can derive immediately the following corollary.

[11] See Roth and Sotomayor [14] for a comprehensive exposition of modeling and analysis of such problems.

Corollary 3. *There is no rule defined on the class of marriage problems with at least two men and two women that satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility.*[12]

The only possibility to avoid the impossibility consists of reducing the set of feasible coalitions, in such a way that not every couple is feasible. That would be the case when the set of men (or women) is a singleton.

A generalization of marriage problems is known as *roommate problems.* There is a set of agents that have to be organized in couples. For instance, there are a number of rooms available and we can assign either 1 or 2 persons to each room. (Some room may remain empty.) Then, we say that a rule φ^r is defined over a the class of roommate problems if

$$F^{\varphi^r} = \{C \in \mathcal{N},\ \#C \leq 2\}.$$

Corollary 4. *There is no rule defined on the class of roommate problems that satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility.*

Of course, Corollary 2 can be extended to problems for which larger coalitions are admissible. However, in that case, the result only holds if agents' preferences over roommate are additively representable.

7.2 College Admission Problems when Students Care about Classmates

Another generalization of the marriage problem is known as the college admission problem. There are two disjoint sets of agents, a set of colleges C, and a set of new students S. Each college $c \in C$ may admit up to q_c new students. Colleges have additively representable preferences over new students. New students have additively representable preferences over colleges and classmates. A coalition is feasible if and only if either is a singleton or it contains exactly one college and the number of students assigned to each college is not larger than its respective quota q_c. Dutta and Massó [9] have shown that core-stable partitions may fail to exist in such problems when students may care about the identity of their classmates. We say a rule φ^c is defined on the college admission problem where

[12] Alcalde and Barberà [1], Roth [13], and Sönmez [17] show that for the marriage problem, no rule satisfies *strategy-proofness*, *individual rationality*, *flexibility*, and *Pareto efficiency.*

students care about classmates if there are two disjoint sets, C, S, $C \cup S = N$, and a list of quotas $\{q_c\}_{c \in C}$, such that

$$F^{\varphi^c} = \{(c, S_c),\ c \in C,\ S_c \subseteq S,\ \text{and}\ \#S_c \leq q_c\} \cup [N].$$

Corollary 5. *Assume that $S \geq 2$, and either $\#C \geq 2$ or if $C = \{c\}$, $q_c \geq 2$. Then, there is no rule defined on the class of college admission problems where students care about their classmates that satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility.*

Note that the previous corollary only holds if students care about their classmates. In fact, Sönmez [16] shows that if students only care about the college they attend, and each college has an unlimited number of slots, there is always a unique core-stable partition, and the rule that selects that partition satisfies *strategy-proofness*, *individual rationality*, and *Pareto efficient.*

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