Jurisdictions on the Plane

Jacques Drèze* Michel Le Breton† Alexei Savvateev‡
Shlomo Weber§

December 2005

Abstract

We consider a jurisdiction formation problem on the plane uniformly populated by a continuum of agents. This could be interpreted either as a real two-dimensional space where these agents live, or alternatively as a space of pairs of two parameters of a public good on which the agents may have horizontally differentiated preferences, in which case any agent is identified with the point of his best variety, out of the space considered.

We study jurisdiction formation under transferable utility, and the main focus is on contribution schemes which lie in the core of a corresponding cooperative game. The proper core turns to be empty, and we consider the minimal ε-core. We show that it essentially contains only one allocation, which is the *egalitarian* (or *Rawlsian*) contribution scheme under which all agents are left with one and the same level of utility. (In fact, as shown in (Ballobas and Stern 1972), the minimal ε is extremely small — approximately, 0.0018).

Keywords: secession-proofness, optimal jurisdictions, rawlsian allocations, geometric structure, (minimal) ε-core.

JEL Classification Numbers: D70, H20, H73.

*CORE, Catholic University of Louvain, Louvain-La-Neuve, Belgium.

†Université de Toulouse I, GREMAQ and IDEI, Toulouse, France.

‡Central Economics and Mathematics Institute, Moscow; Institute for Theoretical and Experimental Physics, Moscow; New Economic School, Moscow; CORE, Catholic University of Louvain, Louvain-La-Neuve, Belgium. Financial support through grants R98-0631 from the Economic Education and Research Consortium, # NSh-1939.2003.6 School Support, Russian Foundation for Basic Research No. 04-02-17227, and Russian Science Support Foundation is gratefully acknowledged.

§Southern Methodist University, Dallas, USA, and CORE, Catholic University of Louvain, Louvain-La-Neuve, Belgium.

1 Introduction

We consider a heterogeneous society whose population has to make selections from a two-dimensional space $\Re^2$ of horizontally[1] differentiated public projects of Mas-Colell (1983). We assume that members of a society have heterogeneous preferences over $\Re^2$, and that these preference relations all are single-peaked, with respect to the standard Euclidean metric. This could be interpreted as the existence of transportation costs. We identify each individual with the location of his best project on $\Re^2$, and assume *uniform distribution* of (best projects of) citizens over $\Re^2$.

For instance, the population of a big city has to decide locations of several libraries, to be built in the city. The overall number of these libraries is also a matter of choice. Each citizen then will be assigned to one and only one library. All the individuals assigned to one and the same library form a subset called "a jurisdiction"; as a result, we have a "jurisdiction structure" which is a partition of the society (or, equivalently, of the space $\Re^2$) into pairwise disjoint jurisdictions.

Following the choice of a jurisdiction structure, including a selection of libraries in all jurisdictions, the contribution scheme towards the costs of libraries must be chosen. In addition to these monetary costs, each citizen bears his personalized costs of being "far" from the library to which he is assigned. We assume that the monetary expression of a transportation cost is just the distance between the location of a citizen and the location of the library to which he is assigned.

Thus, the group choice of the locational problem described here consists of three items:

• - *jurisdiction structure*, which is a partition P of the space $\Re^2$ into subsets of individuals, *jurisdictions*, assigned to the same library;

• - *libraries locations* in each jurisdiction, and

[1]For the analysis of vertically differentiated projects, see Guesnerie (1995), Guesnerie and Oddou (1981,1988), Greenberg and Weber (1986), Jehiel and Scotchmer (2001), Weber and Zamir (1985), Westhoff (1977), Wooders (1978,1980).

• - *sharing rule*, that is a choice of a contribution scheme in order to cover the total cost of libraries in all jurisdictions.

Our approach is that of *transferable utility (TU)*, assumption.[2]. The first question of the analysis is that of the core, i.e. under which jurisdiction structures and subsequent contribution schemes, the society is stable, in a sense that no group of its members would wish to secede and form a new jurisdiction, decreasing their overall costs, including integrated personalized costs and monetary costs?

It turns out that the core is empty, i.e. there are no secession-proof partitions[3] and contribution schemes. Then, a natural question arises of how far we are from the core? Namely, if the government of the city intervenes to compensate a certain share of costs to citizens *in case* the efficient partition of $\Re^2$ is implemented, then which is the minimal possible intervention such that secession-proofness is reinforced?

We show that: (i) The minimal possible intervention is remarkably negligible: this is sufficient to cover less than 0.002 per capita cost to reinsure stability, and (ii) Rawlsian (i.e. egalitarian) allocation, where everyone ends up with the same total costs (monetary plus transportation), plays a major part. Namely, the compensation scheme leading to the Rawlsian allocation under the efficient partition of $\Re^2$, is the only secession-proof scheme under the minimal possible intervention. This provides an additional justification for egalitarism, together with (Bogomolnaia et al 2005a).

Results obtained here generalize (Le Breton et al 2004) to the simplest multi-dimensional case, which is the case of two dimensions. When the dimensionality is greater than 1, we encounter a problem of *inner geometry* of the space under consideration. Namely, we observe that the form of the optimal jurisdiction is not consistent with the space $\Re^2$ itself, in the sense that we cannot partition $\Re^2$ into most efficient jurisdictions (in fact, balls). This very fact is responsible for the empty core and, as a result, there is a case for exogenous (government)

[2]For a treatment of NTU-case in the uni-dimensional setting, see (Bogomolnaia et al 2005).

[3]We will use terms *jurisdiction structure* and *partition* interchangeably, throughout the paper.

intervention.

2 The Model

Let us specify a formal model. We assume that individuals are located uniformly over the whole (unbounded) space $\Re^2$. The volume of an arbitrary measurable[4] subset S will be denoted by $\lambda[S]$, or simply by $|S|$:

$$\lambda[S] = |S| = \int\limits_S dt, \tag{1}$$

which is a real number from the closed segment $[0, +\infty]$.[5] Here, $t = (t_1, t_2)$ is a two-dimentional coordinate on $\Re^2$.

The cost of every library is given by a positive parameter g. We normalize to $g = 1$. The transportation cost incurred by individual[6] t, assigned to a library located at point p, is given by the cost function $d(t, p) = \sqrt{|t_1 - p_1|^2 + |t_2 - p_2|^2}$ which is the Euclidean distance on $\Re^2$.

Let us now introduce a concept of n-partition of an arbitrary measurable subset $S \subset \Re^2$ to an arbitrary positive or countable number n of its parts, *jurisdictions*:

Definition 1: An n-partition $P = (S_i)_{1 \le i \le n}$ is a jurisdiction structure that consists of n bounded measurable sets of a positive finite measure, pairwise disjoint up to a null-set, the union of which being equal to the entire subset S.[7] The set of all n-partitions P of

[4] An arbitrary subset is measurable if and only if its intersection with every measurable subset of a finite measure is measurable; hence, we allow for infinite-measured measurable subsets.

[5] Throughout the paper, the following agreement will be made. Namely, when we calculate average values of functions over the whole space $\Re^2$, or over its Cartesian powers, or over an infinite-measured subset, we often write them as ratios of the two infinite integrals. This always makes sense in our story, since we impose sufficiently rigorous restrictions on such functions; essentially, we require *periodicity* of allocations, sharing rules etc. so that we interpret these ratios as evaluated on the (finite-measured) periodicity generating set.

[6] We will not distinguish between individual t and an individual located at point $t \in \Re^2$.

[7] Restrictions on the measure and size of possible jurisdictions are imposed with the aim of having costs of all citizens uniformly bounded from above. If one allowed for null-set jurisdictions, then *all* its members would incur infinitely high costs; as for unbounded jurisdictions (including those of infinite measure), there is always the case that costs of members of such jurisdictions are unbounded from above.

a subset S will be denoted by $\mathcal{P}_n(S)$; the set of all partitions of S is denoted by $\mathcal{P}(S)$:

$$\mathcal{P}(S) = \bigcup_{n=1}^{+\infty} \mathcal{P}_n(S) \bigcup \mathcal{P}_\infty(S). \tag{2}$$

When we consider partitions of $S = \Re^2$, we will omit subscripts, writing $\mathcal{P}_n$ and $\mathcal{P}$, respectively. For $P \in \mathcal{P}(S)$ we denote by $N(P)$ the number of jurisdictions in P; we have $N(P) \in \mathcal{N} \bigcup +\infty$. We always have $P \in \mathcal{P}_{N(P)}(S)$ tautologically for any $P \in \mathcal{P}(S)$.

For the analysis of the location problem at hand, we construct a *TU*-game with the set of players coinciding with $\Re^2$, and the class of coalitions coinciding with all measurable sets, either finite- or infinite-measured. This is done in two steps. As a first step, for each bounded measurable subset S of $\Re^2$ of a positive measure (which is a *possible* jurisdiction), denote by $D[S]$ the value of a following minimization problem:

$$D[S] := \min_{m \in \Re^2} \int_S d(t, m) dt. \tag{3}$$

This is called "MAT(S)" in Mathematical Programming, which is Minimal Aggregate Transportation of the set S. For obvious reasons, solution(s) to this problem exist (the integral in (3) is continuous in m, and for $m \to \infty$ the value of a program goes to $+\infty$). Any solution to (3) is called a *median location* of S, and we denote the set of all solutions to this program by $M(S)$ (by analogy with the uni-dimentional case where

$$M(S) = \left\{ p \in I : \lambda\left(\{t \in S : t \leq p\}\right) = \lambda\left(\{t \in S : t \geq p\}\right) = \frac{1}{2}\lambda(S) \right\} \tag{4}$$

is the set of median locations). Once we have a possible jurisdiction S, its members would like to minimize transportation costs by placing their library to one of the points in $M(S)$.

Now, at the second step, for an arbitrary *coalition* $S \subset \Re^2$, which is a measurable subset of a positive (maybe infinite) measure, we will define its *per capita characteristic function* (in terms of costs) in the following way:

$$c[S] := \left(\frac{1}{|S|}\right) \inf_{P \in \mathcal{P}(S)} \left(gN(P) + \sum_{S' \in P} D[S'] \right). \tag{5}$$

The solution to program (5) is always finite.[8] Any partition solving (4) is called *an efficient partition* of S.

So, we defined a characteristic function on the class of measurable subsets in $\Re^2$, including those of infinite measure. Note that, by allowing any group S to partition itself in an efficient way and not just to function as a unique jurisdiction, we automatically get super-additivity: the union of the two disjoint coalitions, S and S' at least can partition itself as the union of the two partitions, hence we observe that

$$c[S \bigcup S'] \leq \frac{|S|c[S] + |S'|c[S']}{|S| + |S'|}, \tag{6}$$

a manifestation of super-additivity for games in the *per capita* characteristic form.

Denote $\bar{c} = c[\Re^2]$, and by $\bar{P}$ — any efficient partition of $\Re^2$. We have the following result. For the proof and discussions around, see (Ballobas and Stern 1972; Haimovich and Magnanti 1988).

Theorem 1: Given the (uniformly populated) society $\Re^2$ with some g as fixed cost of a library, we have:

(i) $\bar{c} = c[\Re^2] = c[H]$, where H is a hexagon of an optimal size;

(ii) Among efficient partitions $\bar{P} \in \mathcal{P}$, there are partitions of $\Re^2$ into hexagons of the optimal size.

Now, we are going to define a core of this game, following the standard definition of the core. Fix an arbitrary efficient partition $\bar{P} \in \mathcal{P}$ that consists of hexagons of the optimal size. Denote any of these hexagons by H.

First, we introduce a concept of a *contribution scheme*, or equivalently of a *sharing rule* (these two terms are being used interchangibly, in what follows). Namely, a sharing rule, $x(t)$ describes the monetary contribution of each individual t towards the cost of the libraries in

[8]See footnote 7.

the partition $\bar{P}$. We assume that each jurisdiction in $\bar{P}$ balances its budget:[9]

$$\forall H \in \bar{P} \quad \int_{H} x(t)dt - g = 0. \tag{7}$$

An *allocation* corresponding to a sharing rule, $x(\cdot)$, is the distribution of total costs in a population, arising from the rule $x(\cdot)$:

$$c(t) = x(t) + d(t, m(H^t)), \tag{8}$$

where $H^t \in \bar{P}$ is the hexagonal jurisdiction containing t, and $m(H)$ is the center of the hexagon H (which is the location of a public good in a jurisdiction H).

Now, we say that a function $c(t)$ is a *feasible cost allocation* if there exists a sharing rule $x(\cdot)$ such that $c(\cdot)$ corresponds to $x(\cdot)$ via (8). These notions could be defined not only for $\Re^2$, but also for an arbitrary coalition $S \subset \Re^2$, in which case we denote them by $x_S(\cdot)$ and $c_S(\cdot)$. Sharing rules and costs allocations, as well as partitions, are defined up to a null-set (since we even cannot uniquely define the allocation corresponding to a given sharing rule on the (null-)set where different jurisdictions in a given partition intersect).

For any feasible allocation $c(\cdot)$ and an arbitrary measurable subset $S \subset \Re^2$ we define the average cost of members of S under the allocation $c(\cdot)$ by the formula

$$\bar{c}_S = \left(\frac{1}{|S|}\right) \int_{S} c(t)dt; \tag{9}$$

for the grand coalition $S = \Re^2$, we have the following identity:

$$\bar{c}_{\Re^2} = \bar{c}, \tag{10}$$

taking into account (7) and (8) — the average cost of all citizens under any feasible allocation is equal to the value of the per capita characteristic function on the grand coalition $\Re^2$.

Now, we are ready to define the core of our cooperative game. The definition below simply says that if for some coalition S we observe $\bar{c}_S > c[S]$, then the coalition S will secede and

[9]Since they all have the same form, there is no point in inter-jurisdictional transfers.

form its own efficient partition $P' \in \mathcal{P}_S$, economizing on total aggregate costs of its members. (Transferable utility paradygm then says that there exists a way to distribute costs $c[S]$ among members of S such that *everyone* in S becomes better off.)

Definition 2: Given the society $\Re^2$, we say that an allocation $c(\cdot)$ lies in the *core* (notation: $c(\cdot) \in C[\Re^2]$) if for any coalition $S \subset \Re^2$ we observe that

$$\bar{c}_S \leq c[S]. \tag{11}$$

Next definition introduces an allocation $r(\cdot)$ which plays a major part in what follows. This is the allocation for which the utility of the most disadvantaged individual is maximized (which allows us to name this allocation after Rawls). Under transferable utility, this implies equating costs, so this allocation could is also called *egalitarian*, and we use these two terms inter-changibly.

Definition 3: By the *Rawlsian*, or *egalitarian* allocation we mean the allocation $r(t) \equiv \bar{c}$.

3 The main result

Now we are ready to state and prove the main findings of our study. First, we demonstrate below that the core $C[\Re^2]$ is empty. This mere fact leaves us non-satisfied concerning questions "what then to do" and "what we expect to be the outcome of the game". When the core is empty, one is sometimes searching for a solution which is mostly close to being in the core. For instance, we may have assumed that there is a fixed per capita cost of a secession by a proper subgroup S; alternatively, one can consider the "government intervention" which compensates a certain fraction of a total cost to every citizen, in case they do not form seceding groups. Both approaches are essentially equivalent; using the latter approach, we come to a following definition of an ε-*core*:

Definition 4: Given the society $\Re^2$, we say that an allocation $c(\cdot)$ lies in the ε-*core* (notation: $C_\varepsilon[\Re^2]$), if for any measurable subset $S \subset \Re^2$ we have

$$(1-\varepsilon)\bar{c}_S \leq c[S]. \tag{12}$$

In other words, if people are following "the agreement" of the grand coalition, then the ε-part of the cost is covered "outside"; if, however, a certain coalition, S poses a threat for a secession, then their members are to bear costs on their own.

This definition relaxes the constraints which determine the core, and leaves us a hope that, for some values of ε we will reinforce non-emptiness. Formally, we consider the value $\hat{\varepsilon}$ such that

- $C_{\hat{\varepsilon}}[\Re^2] \neq \emptyset$;
- For every $\varepsilon < \hat{\varepsilon}$, we observe that $C_\varepsilon[\Re^2] = \emptyset$.

We will demonstrate that this value exists, and even more, we give its full characterization. For the $\hat{\varepsilon}$-core, we have yet another name: we call it a *minimal* ε-*core*, and denote by $\mathcal{C}$.

In order to formulate the main result of the analysis, it is left to introduce the notion of an optimal jurisdiction form. Consider the following minimization problem over the class of all measurable subsets $S \subset \Re^2$:

$$\min_{S \subset \Re^2} \frac{D[S]+g}{|S|}. \tag{13}$$

That is, we search for a jurisdiction(s) that minimize total per capita costs of its members. Denote any potential solution to (13) by $\hat{S}$. We have the following result.

Lemma 1: The median set $m(\hat{S})$ is a singleton: $m(\hat{S}) = \alpha$, for some $\alpha \in \Re^2$. Moreover, in fact, up to a null-set, $\hat{S}$ is a ball B^l_α centered at the point α with a certain, "optimal" radius l. (Therefore, all the solutions of (13) are parametrised by points in $\Re^2$, namely, centers of these balls: $\{B_\alpha\}_{\alpha\in\Re^2}$.)

Proof of Lemma 1: Fix any $\alpha \in M(S)$ and omit subscripts in balls centered at α. Hence, any such ball is characterized by its radius, l. Now, consider $0 \leq l_1 \leq l_2 \leq +\infty$ such that $B^{l_1} \setminus \hat{S}$ is a null-set, $\hat{S} \setminus B^{l_2}$ is a null-set, and $\forall l \in (l_1, l_2)$ we have $B^l \setminus \hat{S}$ has a positive measure and $\hat{S} \setminus B^l$ has a positive measure. Such l_1 and l_2 trivially exist. We claim that they coincide, hence, $\hat{S} = B^{l^1} = B^{l_2}$, up to a null-set.

Indeed, if not, take $l_3 = (2l_1 + l_2)/3$ and $l_4 = (l_1 + 2l_2)/3$ and consider moving some positive mass μ from $\hat{S} \setminus B^{l_4}$ to $B^{l_3} \setminus \hat{S}$. This gives us new $\hat{S}'$ for which $D[\hat{S}']$ in (13) is strictly lower: the aggregate distance to the point α is already lower by at least $\mu(l_2 - l_1)/3$, a positive number, hence, MAT is lower by this number as well. But then $\hat{S}'$ gives strictly lower value to (13) than $\hat{S}$, a contradiction. $\square$

We will denote by $\hat{c}$ the value of the problem (13). We claim that $\hat{c} = c[\hat{S}]$, for any $\hat{S}$ solving the problem (13). This is not obvious at once, since there could have existed a *partition* P of $\hat{S}$ which were leading to lower value of $c[\hat{S}]$. But in this case, due to (6), in at least one of jurisdictions $S' \in P$ of this partition, $c[S']$ would be less than $\hat{c}$, which would mean that $\hat{S}$ was not a solution to (13).

We are prepared to state and prove the main result of our analysis.

Theorem 2: Given the (uniformly populated) society $\Re^2$, we have:

(i) $\hat{\varepsilon} = 1 - \dfrac{\hat{c}}{\bar{c}} \neq 0$;

(ii) $\mathcal{C} = \{r(\cdot)\}$ — a singleton, up to a null-set.

Summing this up, we state that the proper core is empty, and that the minimal ε-core is essentially single-valued.

Proof of Theorem 2: Consider $\varepsilon = 1 - \dfrac{\hat{c}}{\bar{c}}$, and the Rawlsian allocation, $r(\cdot)$. We claim that it is in the ε-core. Indeed, for this allocation and for an arbitrary measurable $S \subset \Re^2$, we have that $c_S = \bar{c}$, and so

$$(1 - \varepsilon)c_S = \frac{\hat{c}}{\bar{c}}\bar{c} = \hat{c} \leq c[S], \tag{14}$$

by the very definition of $\hat{c}$.

Now, we demonstrate that the ε-core is empty if

$$\varepsilon < 1 - \frac{\hat{c}}{\bar{c}}. \tag{15}$$

This will prove the (i) part, and that the Rawlsian allocation is in the minimal core. To do this, we should make use of the Fubini's theorem stating independence of the value of a repeated integral on the order of integration.[10] But before using this celebrated theorem, we need to make some arrangements.

Namely, consider a parametrized family $\{B^l_\alpha\}_{\alpha\in\Re^2}$ of balls of the optimal radius l, which is the set of all solutions to (13); denote by $\hat{\lambda}$ the volume of any ball from this family. From now on, we omit the superscript l when referring to any ball of the optimal size, thus denoting it by $\{B_\alpha$, parametrized only by its center.

Suppose that ε-core is nonempty under given ε and pick up any allocation $c(\cdot) \in C_\varepsilon[\Re^2]$. Using this allocation, we are going to demonstrate that $\varepsilon \geq 1 - \frac{\hat{c}}{\bar{c}}$.

Observe that, in the average over $\Re^2$, this allocation assigns costs of $\bar{c}$ to individuals, and if this allocation were Rawlsian, we would have picked any of the optimal form jurisdiction $\{B_\alpha$, and due to (15), we would have observed that this optimal jurisdiction would have seceded, unless the required inequality on ε is fulfilled. Hence, $r(\cdot) \notin C_\varepsilon[\Re^2]$, once $\varepsilon < 1 - \frac{\hat{c}}{\bar{c}}$.

The problem arises when the allocation is arbitrary. In this case, for a given B_α, it is possible that $(1-\varepsilon)c_{\hat{S}}$ is lower than $c[S]$, even when the required inequality on ε is violated. However, our intuition suggests that *the class of optimal jurisdictions,* $\{B_\alpha\}_{\alpha\in\Re^2}$ *is "uniformly distributed on* $\Re^2$*"*, and hence in this case, *in average* we should observe that $(1-\varepsilon)c_{\hat{S}} < c[S]$, and one can pick up at least one jurisdiction in this family characterized by the desired inequality.

[10] It is easy to see that this basic theorem holds for our environment as described in Footnote 5.

To reinforce this intuition in a precise way, define the function $\Phi(t,\alpha)$ on $\Re^2 \times \Re^2$ by the formula

$$\Phi(t,\alpha) = \begin{cases} c(t), & \text{if we have} \quad t \in B_\alpha; \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

In other words, this function gives us costs of t in the allocation under consideration, if the individual t belongs to the optimal ball B_α centered in the point α; otherwise, this function gives just zero.

Using Fubini's theorem, we will perform its integration first over t, next over α, and after that vise-versa. We have:

$$\frac{1}{\hat{\lambda}|\Re^2|}\int\limits_{\Re^2}\int\limits_{\Re^2}\Phi(t,\alpha)dtd\alpha =$$

$$\frac{1}{|\Re^2|}\int\limits_{\Re^2}\left[\frac{1}{\hat{\lambda}}\int\limits_{B_\alpha}c(t)dt\right]d\alpha = \tag{17}$$

$$\frac{1}{|\Re^2|}\int\limits_{\Re^2}c_{B_\alpha}d\alpha,$$

by the definition of c_S for a measurable $S \subset \Re^2$ and an allocation $c(\cdot)$. Hence, this double integral expresses the average of total per capita costs within balls centered in various points in $\Re^2$.

Before applying Fubini's theorem in the opposite direction, let us state the following result. It expresses a duality property of an arbitrary ball B_α.

Assertion: For every $t \in \Re^2$, we have that

$$\{\alpha | t \in B_\alpha\} \equiv B_t. \tag{18}$$

Indeed, this is just due to the symmetry property of a distance $d(t,\alpha)$ as a function of the two arguments, and the definition of a ball B_t as a set of α such that $d(\alpha,t) = d(t,\alpha) \leq l$. $\square$

Now, we are ready to integrate in the opposite direction. We use the fact that all the sets B_α have volume $\hat{\lambda}$, hence, the set B_t as well. In the middle of the following calculations, we make use of the assertion just made.

$$
\begin{aligned}
&\frac{1}{\hat{\lambda}}\frac{1}{|\Re^2|}\int\limits_{\Re^2}\int\limits_{\Re^2}\Phi(t,\alpha)d\alpha dt = \\
&\frac{1}{|\Re^2|}\int\limits_{\Re^2}\frac{c(t)}{\hat{\lambda}}\int\limits_{\{\alpha:t\in B_\alpha\}} d\alpha dt = \\
&\frac{1}{|\Re^2|}\int\limits_{\Re^2}\frac{c(t)}{\hat{\lambda}}\int\limits_{B_t} d\alpha dt = \\
&\frac{1}{|\Re^2|}\int\limits_{\Re^2} c(t)dt = \bar{c},
\end{aligned}
\tag{19}
$$

by its very definition.

Summing up and using Fubini's theorem, we obtain the following equality:

$$
\bar{c} = \frac{1}{|\Re^2|}\int\limits_{\Re^2} c_{B_\alpha} d\alpha. \tag{20}
$$

Now, if $c(\cdot) \in C_\varepsilon[\Re^2]$, then by definition of an ε-core, we must have that any B_α is secession-proof, hence,

$$
\forall\alpha \in \Re^2 \quad (1-\varepsilon)c_{B_\alpha} \leq \hat{c}. \tag{21}
$$

But in this case we have

$$
\begin{aligned}
\bar{c} &= \frac{1}{|\Re^2|}\int\limits_{\Re^2} c_{B_\alpha} d\alpha \leq \\
&\frac{\hat{c}}{1-\varepsilon}\frac{1}{|\Re^2|}\int\limits_{\Re^2} d\alpha = \\
&= \frac{\hat{c}}{1-\varepsilon}.
\end{aligned}
\tag{22}
$$

But $\bar{c} \leq \dfrac{\hat{c}}{1-\varepsilon} \Leftrightarrow \varepsilon \geq 1 - \dfrac{\hat{c}}{\bar{c}}$.

To prove the main theorem, we are left to demonstrate that, up to measure 0, there are no other allocations in the minimal core but the Rawlsian one. In fact, the proof is almost identical to that presented in (Le Breton at al 2004), and therefore is skipped. □

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