Endogenous Aggregate Beliefs: Equity Trading under Heterogeneity in Ambiguity Aversion

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Abstract

We examine the potential importance of *heterogeneity* in consumers' ambiguity aversion for asset pricing, portfolio allocation, and the wealth distribution. Ambiguity aversion, which is a way of formalizing preferences that are consistent with the Ellsberg paradox, features first-order effects on utility even in an economy with a small amount of randomness. Thus ambiguity aversion contrasts the standard model, where risk aversion leads to second-order effects on utility, and it has sharp implications for portfolio demand and for equilibrium asset returns. In the context of a simple, Mehra-Prescott-style endowment economy, we consider two types of agents whose ambiguity aversions differ: some agents display ambiguity aversion while others do not. We show that the equilibrium "belief" of the ambiguity-averse consumer will evolve endogenously and nontrivially over time as a result of the equilibrium interaction. Moreover, we show that "standard agents" will dominate in the pricing of the assets in the long run (but much less so in the short run), unless there is a recurrent influx of new ambiguity-averse consumers. Also, given the heterogeneity, the ambiguity-averse agents become (almost) non-participants in the stock market over time; thus we obtain endogenous limited participation. This occurs as the ambiguity-averse consumers will see their relative wealth decline over time.

1 Introduction

What is the role for heterogeneity in consumer tastes for understanding macroeconomic phenomena? Indeed, individual decision-making is a key building block in micro-founded macroeconomic theorizing, though most of this theorizing relies on the "representative-agent" construct. Thus, the analysis relies on the premise either that, for the question at hand, an aggregation theorem holds, or that any quantitative departure from it does not significantly

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alter the conclusions of the analysis. Some abstractions from heterogeneity are surely easy to motivate, but others, we think, do deserve scrutiny, especially when the taste heterogeneity impacts importantly on the decisions macroeconomists build their models around. In this paper the focus is on understanding asset prices and wealth inequality. If different portfolios deliver different return characteristics, and if these characteristics have sufficiently different average returns, then in scenarios when different consumers have different portfolios of assets—a presumption that appears very much in line with data—one might well obtain sharp implications both for asset prices and for the long-run distribution of wealth. In this paper, we consider one possible determinant of different portfolio preferences, modeled through taste differences, and we look at its long-run effects on asset prices and wealth inequality: differences in the degree of *ambiguity aversion*.

Ambiguity aversion is a way of thinking more generally about "belief formation", and it has been used recently for understanding how risky assets might also be perceived as ambiguous and thus command lower prices and higher returns. Heterogeneity in ambiguity aversion is therefore a way of thinking about the possibility that consumers, despite their being rational, might have different beliefs about asset returns, thus leading to differences in their chosen portfolios. Since the ambiguity aversion we consider here is firmly based on axioms (see below), we are thus studying how differences in tastes might help us understand aggregates. Other approaches to understanding the observed heterogeneity in consumers' portfolios include (i) heterogeneous needs for risk sharing, (ii) heterogeneity in consumers' costs of accessing different financial instruments, and (iii) heterogeneity in information. Surely heterogeneity in risk sharing needs and transactions costs are important; the present analysis should be viewed as a complement in that regard. As for differences in information across people, they are likely very important too, and they are closely related to the object of study here, since they too lead to differences in consumer beliefs. However, they tend to be quite difficult to study in general equilibrium, since once needs to provide an analysis of why the price system does not reveal all privately held information. When it comes to the long-run effects on wealth inequality, in particular, we know of no study of it that is based on market interaction with partially revealing prices. Thus, one can perhaps view our current work as an alternative, and more tractable, way of dealing with belief differences in general equilibrium. Moreover, unlike in most representative-agent studies of ambiguity aversion, where equilibrium beliefs end up being determined very directly based on the modeler's choice of the typical parameter for ambiguity aversion ("a" below), in a model with heterogeneity in ambiguity aversion beliefs evolve endogenously and entirely nontrivially as a function of the market interaction, as we show in this paper.

Ambiguity aversion is a way of formalizing preferences that are consistent with the Ellsberg paradox, and it captures a form of violation of Savage's axioms of subjective probability. Instead, consumers behave as if a range of probability distributions are possible and as if they are averse toward the "unknown". With the typical parameterized representation of ambiguity aversion, consumers have minmax preferences, thus maximizing utility based on the worst possible belief within some given set of feasible beliefs. Thus, in an economy with a small amount of randomness, there are first-order effects on utility if there is ambiguity about this randomness. Thus, ambiguity aversion is in contrast to the standard model, where risk aversion leads to second-order effects on utility.

Previous papers, such as our own work, have analyzed the role of ambiguity aversion for

asset prices, but in representative-agent settings. Alonso and Prado (2008) looks at asset pricing in a simple Mehra-Prescott-style endowment economy. There, we demonstrate how larger equity premia can be obtained by assuming ambiguity aversion, along with low riskfree rates. The key parameter in the model is the amount of ambiguity aversion, but it interacts nonlinearly with other parameters, such as the coefficient of relative risk aversion. We show a range of calibrations that match the average returns on risky and riskless assets and calibrate ambiguity aversion by using the volatility of short-term return on bonds and the return on the long-term bonds.[1] A question we thus ask with the present paper is to what extent the results in this previous work are robust to introducing heterogeneity in ambiguity aversion.

We consider a simple model and assume that half of the agents display a given amount of ambiguity aversion while the rest (the "standard agents") do not. In order to make the analysis as simple as possible, we specialize to a logarithmic period utility function and *iid* and symmetric shocks. For this particular case, we are able to show that the standard agents will increasingly dominate in the pricing of the assets over time. Furthermore, with this heterogeneity, the most ambiguity-averse agents become (almost) non-participants in the stock market over time; thus, we obtain endogenous limited participation. In conclusion, although ambiguity aversion shows great potential in providing new asset-pricing implications and in allowing us to think of a reason why the elimination of aggregate fluctuations might be quite costly, heterogeneity in the degree of ambiguity aversion will tend to limit these implications and mainly have effects on wealth distribution and differences in portfolios across consumers. The implications for wealth inequality, however, are very striking and deserve further study. The observed inequality in wealth is difficult to explain with standard versions of calibrated macroeconomic models, and our present work seems to be one avenue for understanding why some consumers are so rich and others so poor.

Finally, we note that our model has rather extreme long-run implications (both for asset prices and for wealth) to a large extent because (i) agents live forever and (ii) asset markets are complete. In this paper, we do not consider asset-market incompleteness (though it would seem fruitful to examine it). However, we study an extension of the setting above, which features "overlapping generations": each period, there is an influx of ambiguity-averse consumers. Such an economy allows ambiguity to play a role also in the limit and we characterize its long-run effects.

2 The economy

This is an infinite-horizon exchange economy. Production is exogenous: the economy has a tree that pays dividends every period. The dividend grows at a random rate, which has a two-state support given by (λ_1, λ_2) and follows a first-order Markov process. The transition probabilities are given by $\phi_{ss'}$ – the probability of going to state s' if today's state is s, with $s, s' = 1, 2$.

When the consumer is ambiguous about these probabilities, he perceives them to be

$$\Phi(v) = \begin{pmatrix} \phi_{11} - v_1 & \phi_{12} + v_1 \\ \phi_{21} - v_2 & \phi_{22} + v_2 \end{pmatrix}, \tag{1}$$

[1] For a survey on the equity premium puzzle, see Kocherlakota (1996).

where $v_s \in [-a, a]$ $(s = 1, 2)$ with restrictions on a such that all probabilities are in $[0, 1]$. Parameter a measures the amount of ambiguity in the economy.

Preferences are given by the maxmin formulation

$$V_t(s^t) = u\left[c(s^t)\right] + \beta \min_{\pi \in \Pi_{s^t}} E_\pi V_{t+1}(s^{t+1}), \tag{2}$$

where c is consumption, $u(c)$ is the period utility function, and Π_{s^t} is a set of transition probability laws given the history s^t today.

Aversion to ambiguity is captured by the "minimization" part in the utility formulation above: the consumer behaves with pessimism, i.e., he assumes the worst possible probability distribution. For an axiomatic foundation for this preference formulation, see Gilboa and Schmeidler (1989) for the static setting and Epstein and Wang (1994) and Epstein and Schneider (2003) for a multiperiod setting.

We consider two types of agents whose ambiguity aversions differ. We look at a general planning problem first, and then focus on the case with *iid* shocks. Later, we look at the case of serial correlation in more detail (NOT DONE YET).

In section 3 we consider a model with both ambiguity-averse agents and "standard" agents who do not view the economy as ambiguous. In section 4 we conclude.

3 Heterogeneity in ambiguity aversion

3.1 The planner's problem

We first consider agents with a logarithmic period utility function. The state vector is (d, θ, s): today's dividend, the weight the planner puts on consumer 1, and today's shock. The planner solves the problem

$$V_s(d, \theta) = \max_{c_1, c_2, z_{1s'}, z_{2s'}} \theta \log c_1 + (1 - \theta) \log c_2 +$$

$$+\beta \left\{ \min_{v_1 \in [-a_1, a_1]} \theta \sum_{s'=1}^{2} \phi_{ss'}(v_1)\, z_{1s'} + \min_{v_2 \in [-a_2, a_2]} (1 - \theta) \sum_{s'=1}^{2} \phi_{ss'}(v_2) z_{2s'} \right\}$$

subject to

$$\min_{\theta'_{s'}} V_{s'}\left(d\lambda_{s'}, \theta'_{s'}\right) - \left[\theta'_{s'} z_{1s'} + \left(1 - \theta'_{s'}\right) z_{2s'}\right] \geq 0, \tag{3}$$

and

$$c_1 + c_2 = d, \tag{4}$$

where c_i is agent i's consumption, $i = 1, 2$, z_i is next period's present-value utility for agent i, $\phi_{s1}(v_i) = \phi_{s1} - v_i$, and $\phi_{s2}(v_i) = \phi_{s2} + v_i$. The first constraint (3) makes the problem recursive and the second constraint (4) is the resource constraint. This formulation which is based on Lucas and Stokey (1984) is also used in Alonso (2007).

Taking FOCs with respect to the consumption of agents 1 and 2, we have

$$c_1 = \theta d, \tag{5}$$

and

$$c_2 = (1 - \theta) d, \tag{6}$$

with respect to $z_1(1)$ and $z_2(1)$, we obtain

$$\frac{\theta_1'}{1-\theta_1'} = \frac{\theta(\phi_{s1} - v_1)}{(1-\theta)(\phi_{s1} - v_2)}, \tag{7}$$

and similarly with respect to $z_1(2)$ and $z_2(2)$ we have

$$\frac{\theta_2'}{1-\theta_2'} = \frac{\theta(\phi_{s2} + v_1)}{(1-\theta)(\phi_{s2} + v_2)}. \tag{8}$$

After some algebra where we use the law of motion for $\theta'(s)$ (equations 7 and 8), we can rewrite the planner's problem as

$$V_s(d,\theta) = \max_{c_1,c_2} \theta \log c_1 + (1-\theta)\log c_2 +$$

$$+\beta \min_{v_1,v_2} \left\{ \sum_{s'=1}^{2} \phi_{ss'} \left[\theta v_1 + (1-\theta)v_2\right] V_{s'}(d\lambda_{s'}, \theta_{s'}') \right\}$$

subject to

$$\theta_{s'}' = \theta \frac{\phi_{ss'}(v_1)}{\phi_{ss'}\left[\theta v_1 + (1-\theta)v_2\right]}, \tag{9}$$

and

$$c_1 + c_2 = d. \tag{10}$$

Note that $\phi_{s1}\left[\theta v_1 + (1-\theta)v_2\right] = \phi_{s1} - \theta v_1 - (1-\theta)v_2$ and $\phi_{s2}\left[\theta v_1 + (1-\theta)v_2\right] = \phi_{s2} + \theta v_1 + (1-\theta)v_2$.

3.1.1 A special case: no serial correlation and $a_2 = 0$

In the simpler case where shocks are *iid* and symmetric and consumer 2 is not ambiguity-averse ($v_2 = a_2 = 0$), the planner's problem becomes[2]:

$$V(d,\theta) = \max_{c_1,c_2,\theta_{s'}'} \theta \log c_1 + (1-\theta)\log c_2 +$$

$$+\beta \min_{v\in[-a,a]} \left\{ \sum_{s'=1}^{2} \phi_{s'}(\theta v) V(d\lambda_{s'}, \theta_{s'}') \right\}$$

subject to

$$\theta_{s'}' = \theta \frac{\phi_{s'}(v)}{\phi_{s'}(\theta v)}, \tag{11}$$

and

$$c_1 + c_2 = d. \tag{12}$$

[2]From now on, we drop the subscript on v and a, since it should be clear that they refer only to consumer 1.

Using the FOCs for consumption, we obtain $c_1 = \theta d$ and $c_2 = (1-\theta)d$, so we obtain

$$V(d,\theta) = \log d + \log \theta^{\theta}(1-\theta)^{1-\theta} +$$
$$+\beta \min_{v} \left[(\phi - \theta v)V(d\lambda_1, \theta_1') + (1-\phi+\theta v)V(d\lambda_2, \theta_2')\right],$$

with

$$\theta_1' = \theta \frac{\phi - v}{\phi - \theta v}, \tag{13}$$

and

$$\theta_2' = \theta \frac{1-\phi+v}{1-\phi+\theta v}. \tag{14}$$

Here, we conjecture that $V(d,\theta)$ takes the form $A\log d + W(\theta)$. This guess delivers

$$A\log d + W(\theta) = \log d + \log \theta^{\theta}(1-\theta)^{1-\theta} +$$
$$+\beta \min_{v} \left\{(\phi - \theta v)\left[A\log(d\lambda_1) + W(\theta_1')\right] + (1-\phi+\theta v)\left[A\log(d\lambda_2) + W(\theta_2')\right]\right\}.$$

Inspecting this functional equation, it can be seen that $A = \frac{1}{1-\beta}$ works and we can express $W(\theta)$ as

$$W(\theta) = \log \theta^{\theta}(1-\theta)^{1-\theta} +$$
$$+\beta \min_{v\in[-a,a]} \left\{(\phi - \theta v)\left[\frac{\log\lambda_1}{1-\beta} + W\left(\theta\frac{\phi-v}{\phi-\theta v}\right)\right] + (1-\phi+\theta v)\left[\frac{\log\lambda_2}{1-\beta} + W\left(\theta\frac{1-\phi+v}{1-\phi+\theta v}\right)\right]\right\}.$$

This is a one-dimensional dynamic programming problem delivering optimal v as a function of θ and hence, a law of motion for θ. The variable θ also corresponds to the fraction of the total wealth—the current dividend plus the value of the tree—owned by agent 1 in a complete-markets equilibrium. The following figures for $W(\theta)$ and $v(\theta)$ below assume the following values for the parameters: $\lambda_1 = 1.02$, $\lambda_2 = 1.01$, $\beta = 0.98$, $\phi = 0.5$, and $a = 0.2$.

Figure 1, for $W(\theta)$, reveals a shape similar to $\log \theta^{\theta}(1-\theta)^{1-\theta}$, which is the (constant) flow utility of a planner in a two-type economy where no consumer has ambiguity aversion.

Figure 2, for the optimal choice of v, shows that v is close to zero and interior at first (for small θ's), and then it increases monotonically in θ and reaches the upper bound a for a value of θ a little above 0.9. We will interpret these findings in more detail in the following sections.

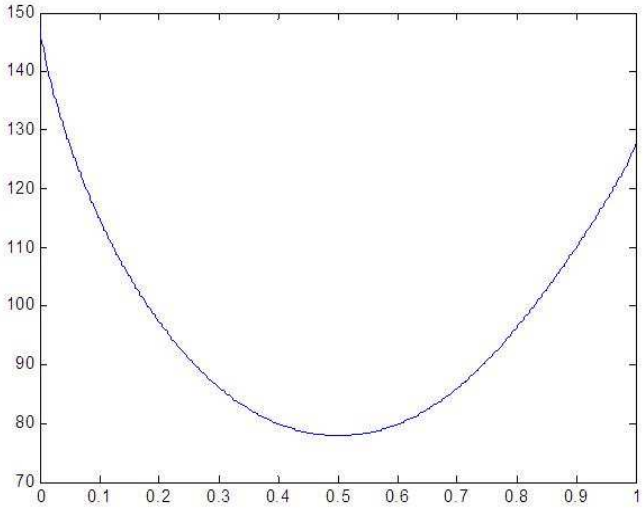

Figure 1: $W(\theta)$

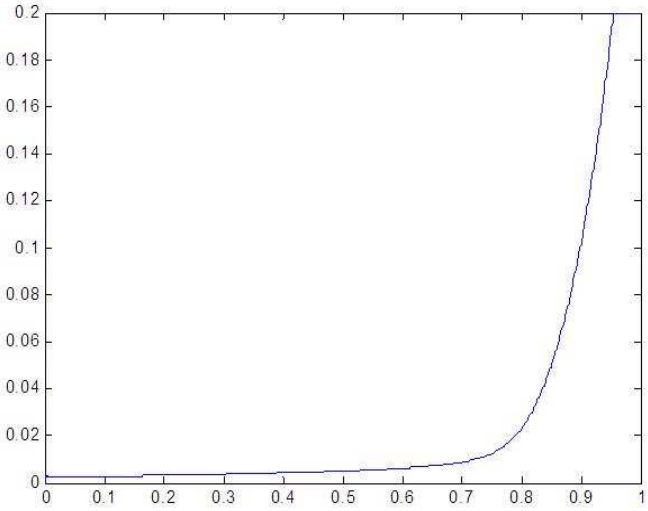

Figure 2: $v(\theta)$

3.2 The special (*iid*) case: the decentralized economy

Markets are complete and consumers trade in equity shares of the tree and in a riskless bond that is in zero net supply. We denote the consumer's bond and equity holdings b and e, respectively.

We assume that $\lambda_1 > \lambda_2$ so that the bad outcome is state 2 – the outcome where the dividend is low. The ambiguity-averse consumer puts a higher weight on the bad outcome than what is warranted by the objective probability; that is, he becomes pessimistic because he is worried about that outcome and does not know its probability. The objective probability of this state is $1-\phi$ but he chooses the belief in the bad state. His belief is $\phi(v) = 1-\phi+v$ and he chooses v from the set $v \in [-a, a]$. The higher is a, the more ambiguity there is in the economy.

The consumer's problem is given recursively by

$$V(d, w, \theta) = \max_{c,b,e} \left\{ \log c + \beta \min_{v} \sum_{s'=1}^{2} \phi_{s'}(v) V(\lambda_{s'} d, w'_{s'}, \theta'_{s'}) \right\},$$

subject to the budget constraint

$$c + p(d, \theta)e + q(d, \theta)b = w, \tag{15}$$

$$w'_{s'} = b + e\left[\lambda_{s'} d + p(\lambda_{s'} d, \theta'_{s'})\right], \tag{16}$$

and the law of motion for $\theta'_{s'}$ given by

$$\theta'_{s'} = g_{s'}(d, \theta), \tag{17}$$

where (d, w, θ) is the state vector, w is the consumer's wealth today, p is the price of equity, e is the fraction of the equity share held by the consumer, q is the price today of a bond that pays one unit of the consumption good next period, and b is the holdings of the bond. (The argument d is included for g only for completeness; it will not be there under the log assumption.)

The consumers' decision rules for all (d, w, θ) are

$$c_i(d, w, \theta) \tag{18}$$

$$b_i(d, w, \theta) \tag{19}$$

$$e_i(d, w, \theta) \tag{20}$$

for $i \in \{1, 2\}$.

Total wealth in the economy when the state variable is (d, θ) is $d + p(d, \theta)$. Thus, market clearing requires, for all values of the arguments,

$$c_1(d, \theta\left[d + p(d, \theta)\right], \theta) + c_2(d, (1-\theta)\left[d + p(d, \theta)\right], \theta) = d, \tag{21}$$

$$b_1(d, \theta\left[d + p(d, \theta)\right], \theta) + b_2(d, (1-\theta)\left[d + p(d, \theta)\right], \theta) = 0, \tag{22}$$

and

$$e_1(d, \theta\left[d + p(d, \theta)\right], \theta) + e_2(d, (1-\theta)\left[d + p(d, \theta)\right], \theta) = 1. \tag{23}$$

The relative wealth dynamics, finally, is given by

$$g_{s'}(d,\theta) = \frac{w'_{1s'}(d,\theta)}{w'_{1s'}(d,\theta) + w'_{2s'}(d,\theta)}, \tag{24}$$

where

$$w'_{1s'}(d,\theta) \equiv b_1(d,\theta\left[d + p(d,\theta)\right],\theta) + e_1(d,\theta\left[d + p(d,\theta)\right],\theta)(d\lambda_{s'} + p\left[d\lambda_{s'}, g_{s'}(d,\theta)\right]),$$

and

$$w'_{2s'}(d,\theta) \equiv b_2(d,(1-\theta)\left[d + p(d,\theta)\right],\theta) + e_2(d,(1-\theta)\left[d + p(d,\theta)\right],\theta)(d\lambda_{s'} + p\left[d\lambda_{s'}, g_{s'}(d,\theta)\right]).$$

Now we will show how to find prices and portfolio allocations in this economy. We use the planning problem and we identify the θ in that problem with the corresponding variable here: the planning weight on agent 1 equals the relative fraction of total wealth held in equilibrium by this agent.

To derive the prices for bonds and equity, we take first-order conditions from the consumer's problem with respect to b and e.

The price of bonds, $q(d,\theta)$, then becomes

$$q(d,\theta) = \beta\left[\frac{\phi}{\lambda_1} + \frac{1-\phi}{\lambda_2} + \theta v\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)\right] \equiv \hat{q}(\theta). \tag{25}$$

The price of the bond is increasing in v. In addition here, the price of bonds is increasing in θ. We show below that the ambiguity-averse agents demand the bond. The bond is more valuable when marginal utility of consumption is high (which occurs in the bad state). As θ increases, there is a higher demand for the bond, so its price goes up.

And the price of equity, $p(d,\theta)$, is given by

$$p(d,\theta) = \beta\left\{\frac{(\phi - \theta v)\left[\lambda_1 d + p(\lambda_1 d, \theta'_1)\right]}{\lambda_1} + \frac{(1 - \phi + \theta v)\left[\lambda_2 d + p(\lambda_2 d, \theta'_2)\right]}{\lambda_2}\right\}, \tag{26}$$

where we recall that

$$\theta'_1 = \frac{\theta(\phi - v)}{\phi - \theta v} \leq \theta, \tag{27}$$

and

$$\theta'_2 = \frac{\theta(1 - \phi + v)}{1 - \phi + \theta v} \geq \theta, \tag{28}$$

from the planning problem. (The inequalities above follow since $v \geq 0$.)

The latter laws of motion reveal that the ambiguity-averse agent gains in relative wealth when the state is bad and loses when it is good: his probability "beliefs" are tilted toward the bad state.

We see that $p(d,\theta) = d\hat{p}(\theta)$ solves this equation, delivering

$$\hat{p}(\theta) = \beta\left\{(\phi - \theta v)\left[1 + \hat{p}(\theta'_1)\right] + (1 - \phi + \theta v)\left[1 + \hat{p}(\theta'_2)\right]\right\}. \tag{29}$$

This is a functional equation: it holds for all θ (recall that v may also depend on θ). The solution to this functional equation is

$$\hat{p}(\theta) = \frac{\beta}{1-\beta}, \tag{30}$$

and

$$p(d,\theta) = d\frac{\beta}{1-\beta}. \tag{31}$$

So the price of equity does not depend on θ.

The equilibrium holdings of equity of consumer 1, which can be obtained by using the expression for future wealth, $w'_{1s'} = b_1 + e_1(\lambda_{s'}d + p'_{s'})$, together with the equilibrium condition that $w'_{1s'} = \theta'_{s'}(d\lambda_{s'} + p'_{s'})$, are given by

$$e_1(d,\theta) = \frac{\theta'_1\lambda_1 - \theta'_2\lambda_2}{\lambda_1 - \lambda_2} \equiv \hat{e}_1(\theta). \tag{32}$$

Thus, the equity holdings of agent 1 are independent of the level of d. We see that if $v = 0$, in which case $\theta'_1 = \theta'_2 = \theta$, then $\hat{e}_1(\theta) = \theta$: the consumer's share of the tree equals his initial share of total wealth.

On the other hand, when $v > 0$ (recall that wlog we use $\lambda_1 > \lambda_2$), we know that $\theta'_1 < \theta < \theta'_2$, which makes the holdings of equity lower as compared to the case when $v = 0$. That is, the ambiguity-averse agent will have a smaller share of equity holdings than his overall wealth would otherwise prescribe: this is a *portfolio composition* effect. How much his portfolio composition will be changed must be numerically examined.

We can also examine the portfolio effect by looking at the amount of bonds purchased by agent 1. Her equilibrium holdings of bonds are obtained as

$$b_1(d,\theta) = \frac{d\lambda_1}{1-\beta}\left[\theta'_1 - \hat{e}_1(\theta)\right]. \tag{33}$$

It is interesting to note here that bond holdings are proportional to d. Naturally, they are zero in the special case $v = 0$, when $e_1 = \theta$ and $\theta' = \theta$. Moreover,

$$\theta'_1 - \hat{e}_1(\theta) = \theta'_1 - \frac{\theta'_1\lambda_1 - \theta'_2\lambda_2}{\lambda_1 - \lambda_2} =$$

$$= \theta'_1\left(1 - \frac{\lambda_1 - \frac{\theta'_2}{\theta'_1}\lambda_2}{\lambda_1 - \lambda_2}\right) > 0, \tag{34}$$

since $\theta'_2 > \theta'_1$, and thus we conclude, consistently with the above insights regarding equity holdings, that the ambiguity-averse agent *increases* his bond holdings relative to the $v = 0$ zero-bonds case: his portfolio composition moves away from equity and into bonds because he is more pessimistic than person 2 in his perception of the return (performance) of equity.

The consumer is exposed to two sources of uncertainty in this economy: (i) the payoff of equity and (ii) the price of the bond. The price of the bond depends on θ, the relative wealth of consumer 1, and this variable is random. In particular, since $\theta'_2 > \theta > \theta'_1$, the price of the bond, q, increases if state 2 occurs and it decreases if state 1 occurs.

Below we numerically compute solutions for $v(\theta)$, $\theta_1'(\theta)$, $\theta_2'(\theta)$, $e(\theta)$, $\frac{p(\theta)e(\theta)}{p(\theta)e(\theta)+q(\theta)b(\theta)}$, $q(\theta)$, and $b(\theta)$ for agent 1. Once more, the parameter values are $\lambda_1 = 1.02$, $\lambda_2 = 1.01$, $\beta = 0.98$, $\phi = 0.5$, and $a = 0.2$.

As we see from the graphs in figure 3, the ambiguity-averse consumer short-sells equity for most values of θ. The reason for this is the following. State 2 is bad for the ambiguity-averse consumer for two reasons: (i) the payoff from equity is low and (ii) the price of the bond increases so that it makes the good next period more expensive (this consumer does not own any goods next period). Therefore, to provide protection against the former type of uncertainty, the ambiguity-averse consumer buys bonds and to provide protection against the latter type of uncertainty, the ambiguity-averse consumer sells equity short. We shall analyze the details of the graphs in figure 3 in what follows.

Outcomes in the two extreme cases: $\theta = 0$ and $\theta = 1$:

When θ is 0 – the standard agent has all the wealth – asset pricing is standard because all assets are priced by the standard agent. In this case, there is *voluntary non-participation in equity*: at $\theta = 0$, a measure-zero ambiguity-averse agent chooses not to participate in equity markets. He chooses $v = 0$ and holds only bonds. Why? If he had positive equity, he would need to choose a $v > 0$, because his portfolio would do well in good times, so he would be pessimistic with regard to that state. But such beliefs would imply that selling equity short is optimal which is a contradiction. Similarly, negative equity implies $v < 0$, which implies that positive equity is optimal which again is a contradiction. So the only possibility for the ambiguity-averse agent at $\theta = 0$ is to hold zero equity, hold bonds only, choose $v = 0$, and have constant consumption.

When θ is 1 – the ambiguity-averse agent has all the wealth – asset pricing is non-standard. In particular, there is a high equity premium: since the representative agent, who must be holding all the equity and no bonds, now is ambiguity-averse, v must be positive and high (at its bound, if a is small). This means pessimism toward the good state, a very high bond price, and a low risk-free rate. Thus, the equity premium becomes very high. With CRRA and more than log curvature, the price of equity would fall and the equity premium would be even higher. (No closed-form solutions for the heterogeneous-agent economy here, though.) The case with a representative agent with ambiguity aversion is treated in Appendix

Intermediate values: $\theta > 0$ but not too high

Unless θ is quite high (in the numerical example), the ambiguity-averse agent is still at an interior solution: the present value utility next period is the same for each value of λ.

When θ is positive but not too large, the ambiguity-averse agent short-sells the stock. Why? State 2 (the bad state) is now bad because it makes θ fall (equity does poorly, so the standard agent loses in relative terms). The price of the bond must then increase since ambiguity-averse agents become richer in relative terms and demand more of it.

A very small amount of pessimism rationalizes their choice, i.e., liking state 2 makes you sell equity short. Thus, you are still indifferent between which state occurs: if state 1 occurs, the bond price is low (good, since they demand it); if state 2 occurs, their portfolio does well. v is set so that the present value utility is equal across the two outcomes.

Intermediate values: nonlinearity

If θ is positive but small, changes in θ do not have any considerable effects on q, so the randomness in q is not so important. Then, ambiguity-averse consumers mainly hold bonds

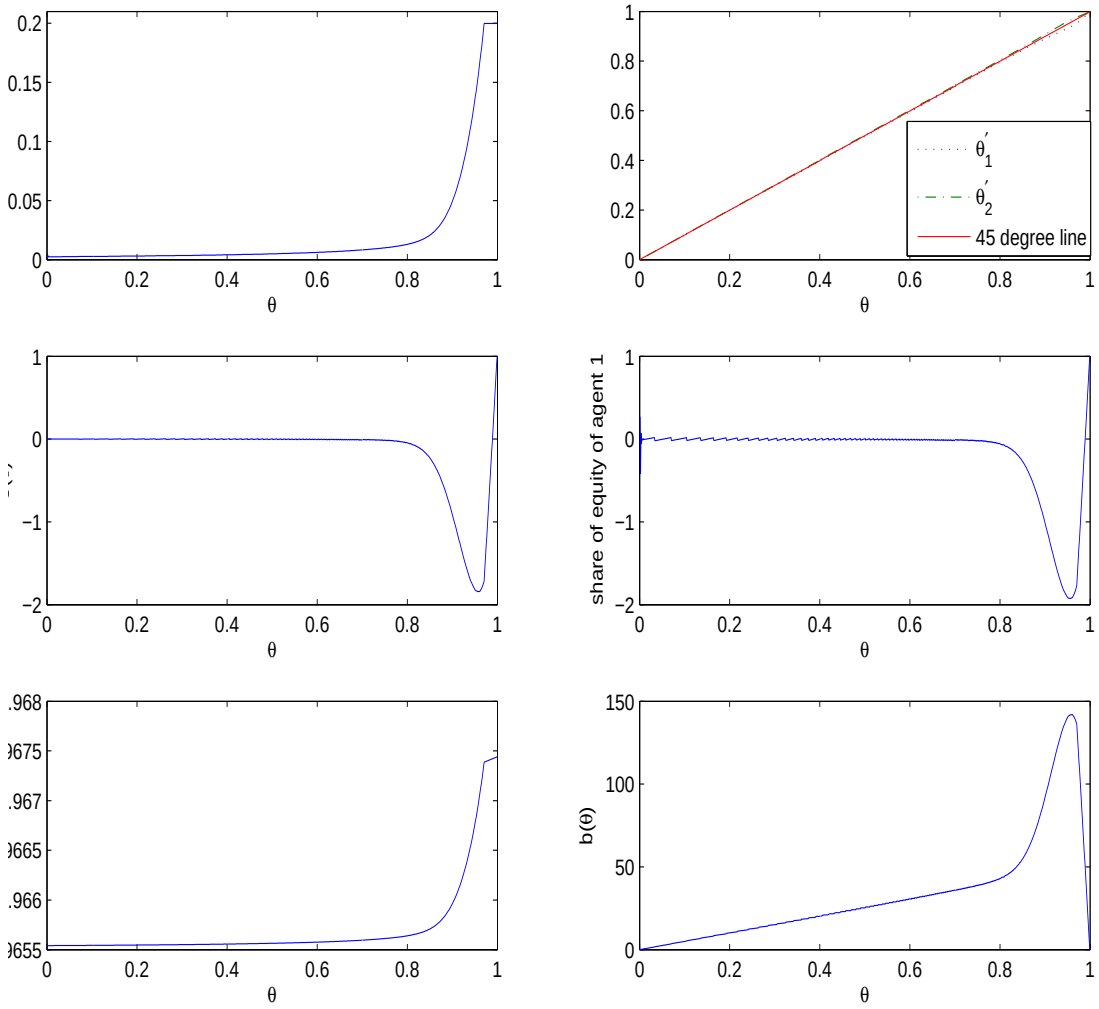

Figure 3: From left to right from top to bottom: (a) $v(\theta)$, (b) law of motion for θ (θ_1 below the 45 degree line; θ_2 above the 45 degree line), (c) $e_1(\theta)$, (d) share of equity of agent 1, (e) $q(\theta)$, (f) $b(\theta)$

and short-sell equity somewhat to protect against the uncertainty in p. This asset choice makes

$$V(w_1', \theta_1') = V(w_2', \theta_2') \tag{35}$$

for a small value of the belief v; that is, v is still an interior solution.

The higher is θ, the more the bond price reacts: the ambiguity-averse agents now want to short-sell equity more, and an even higher v is needed to justify this behavior. Agents buy bonds and short-sell equity more heavily. To understand this in somewhat more detail, note that the value of v is larger, reflecting more pessimism about state 2. Since V is decreasing in θ and increasing in w (the former is true because q is increasing in θ), and since θ_1' is much larger than θ_2', w_2' needs to be much larger than w_1' in order to equate $V(w_1', \theta_1')$ and $V(w_2', \theta_2')$ – and hence still make v an interior solution. This is achieved by short-selling equity even more heavily. Thus, a high θ and a high v are reinforcing; as can be seen, θ and v appear multiplicatively in the bond-price formula above! I.e., we have a nonlinearity in the portfolio pattern. So when θ is high, ambiguity aversion makes the fluctuations in q very large.

At some point, v hits the bound a. Beyond this bound, as the the ambiguity-averse agent has a higher fraction of total wealth, he will decrease the bond holding and increase the equity holding, simply for market-clearing reasons (recall: at $\theta = 1$ he holds only equity). Over this range, the equity premium increases sharply, to motivate the ambiguity-averse agent to hold more equity and less bonds. Close to $\theta = 1$, the fluctuations in θ have become very small, and the uncertainty resulting from changes in q is therefore also very small and ambiguity-averse agents consequently do not need to short-sell the stock.

3.3 Relative consumption and wealth in the long run

We can analytically show[3] that

$$E(\theta'|\theta) < \theta, \tag{36}$$

i.e., that over time, the relative wealth of the ambiguity-averse agents decreases toward zero: these agents disappear, economically speaking.

However, it can also be shown that

$$E\left(\frac{\theta'}{\theta}\right) \to_{\theta \to 0} 1, \tag{37}$$

so the rate at which they disappear goes to zero: they remain with positive wealth for a long, long time. [to be continued]

4 Overlapping generations:an influx of ambiguity-averse consumers

[to be completed]

[3]The proofs of expressions (36) and (37) are in the appendix. This result and the following discussion are reminiscent of the analysis in Coen-Pirani (2004).

5 Conclusion

[to be completed]

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Appendix

A1 Heterogeneity in ambiguity aversion

A1.1 The planning problem

We restate the planning problem:

$$V_s(d,\theta) = \max_{c_1,c_2} \theta \log c_1 + (1-\theta)\log c_2 +$$

$$+\beta \min_{v_1,v_2} \left\{ \sum_{s'=1}^{2} \phi_{ss'} \left[\theta v_1 + (1-\theta)v_2\right] V_{s'}(d\lambda_{s'}, \theta'_{s'}) \right\}$$

subject to

$$\theta'_{s'} = \theta \frac{\phi_{ss'}(v_1)}{\phi_{ss'}\left[\theta v_1 + (1-\theta)v_2\right]}, \tag{38}$$

and

$$c_1 + c_2 = d. \tag{39}$$

Taking FOCs, we obtain $c_1 = \theta d$ and $c_2 = (1-\theta)d$. The rewritten problem becomes:

$$V_s(d,\theta) = \log d + \log \theta^{\theta}(1-\theta)^{1-\theta} +$$

$$+\beta \min_{v_1,v_2} \left\{ \left[\phi_{s1} - \theta v_1 - (1-\theta)v_2\right] V_1(d\lambda_1, \theta'_1) + \left[\phi_{s2} + \theta v_1 + (1-\theta)v_2\right] V_2(d\lambda_2, \theta'_2) \right\}$$

subject to

$$\theta'_{s'} = \theta \frac{\phi_{ss'}(v_1)}{\phi_{ss'}\left[\theta v_1 + (1-\theta)v_2\right]}. \tag{40}$$

We conjecture that $V_s(d,\theta)$ takes the form $A\log d + W_s(\theta)$. This guess delivers

$$A\log d + W_s(\theta) = \log d + \log \theta^{\theta}(1-\theta)^{1-\theta} +$$

$$+\beta \min_{v_1,v_2} \left\{ \left[\phi_{s1} - \theta v_1 - (1-\theta)v_2\right] \left[A\log d\lambda_1 + W_1(\theta'_1)\right] + \left[\phi_{s2} + \theta v_1 + (1-\theta)v_2\right] V_2(d\lambda_2, \theta'_2) \right\}$$

A1.1.1 Special case: serial correlation and $\mathbf{a}_2 = 0$

In this case, we have

$$V_s(d,\theta) = \max_{c_1,c_2,\theta'_{s'}} \theta \log c_1 + (1-\theta)\log c_2 +$$

$$+\beta \min_{v_s} \left\{ \sum_{s'=1}^{2} \phi_{ss'}(\theta v) V_{s'}(d\lambda_{s'}, \theta'_{s'}) \right\}$$

subject to

$$\theta'_{s'} = \theta \frac{\phi_{ss'}(v_s)}{\phi_{ss'}(\theta v_s)}, \tag{41}$$

and

$$c_1 + c_2 = d. \tag{42}$$

Using the FOCs for consumption, we obtain $c_1 = \theta d$ and $c_2 = (1-\theta)d$ so we obtain

$$V_s(d,\theta) = \log d + \log \theta^\theta (1-\theta)^{1-\theta} +$$
$$+\beta \min_{v_s} \left[(\phi_{s1} - \theta v_s) V_1(d\lambda_1, \theta_1') + (\phi_{s2} + \theta v_s) V_2(d\lambda_2, \theta_2') \right]$$

with

$$\theta_1' = \theta \frac{\phi_{s1} - v_s}{\phi_{s1} - \theta v_s}, \tag{43}$$

and

$$\theta_2' = \theta \frac{\phi_{s2} + v_s}{\phi_{s2} + \theta v_s}. \tag{44}$$

Here, we conjecture that $V_s(d,\theta)$ takes the form $A \log d + W_s(\theta)$. This guess delivers

$$A \log d + W_s(\theta) = \log d + \log \theta^\theta (1-\theta)^{1-\theta} +$$
$$+\beta \min_{v_s} \left((\phi_{s1} - \theta v_s)(A \log(d\lambda_1) + W_1(\theta_1')) + (\phi_{s2} + \theta v_s)(A \log(d\lambda_2) + W_2(\theta_2')) \right).$$

Inspecting the above expression, it can be seen that $A = \frac{1}{1-\beta}$ works and leaves

$$W_s(\theta) = \log \theta^\theta (1-\theta)^{1-\theta} +$$
$$+\beta \min_{v_s} \left\{ (\phi_{s1} - \theta v_s) \left[\frac{\log \lambda_1}{1-\beta} + W_1 \left(\theta \frac{\phi_{s1} - v_s}{\phi_{s1} - \theta v_s} \right) \right] + (\phi_{s2} + \theta v_s) \left[\frac{\log \lambda_2}{1-\beta} + W_2 \left(\theta \frac{\phi_{s2} + v_s}{\phi_{s2} + \theta v_s} \right) \right] \right\}$$

for $s = 1, 2$. This is a two-dimensional dynamic programming problem that delivers optimal v_s, $s = 1, 2$, as a function of θ, and hence a law of motion for θ.

A1.2 The decentralized economy

The problem of the consumer is

$$V_s(d, w, \theta) = \max_{c,b,e} \left\{ \log c + \beta \min_{v_s} \sum_{s'=1}^{2} \phi_{ss'}(v_s) V_{s'}(\lambda_{s'} d, w_{s'}', \theta_{s'}') \right\}$$

subject to the budget constraint

$$c + p_s(d,\theta) e + q_s(d,\theta) b = w, \tag{45}$$

$$w_{s'}' = b + e \left[\lambda_{s'} d + p_{s'}(d\lambda_{s'}, \theta_{s'}') \right], \tag{46}$$

and the law of motion for $\theta_{s'}'$ given by

$$\theta_{s'}' = g_{s'}(d, \theta, s) \tag{47}$$

where p is the price of equity, e is the fraction of the equity share held by the consumer, q is the price today of a bond that pays one unit of the consumption good next period, and b

is the holdings of the bond. (The argument d is included for g only for completeness; it will not be there under the log assumption.)

The consumers' decision rules for all (d, w, θ, s) are

$$c_{is}(d, w, \theta) \tag{48}$$

$$b_{is}(d, w, \theta) \tag{49}$$

$$e_{is}(d, w, \theta) \tag{50}$$

for $i \in \{1, 2\}$.

Total wealth in the economy when the state variable is (d, θ, s) is $d + p_s(d, \theta)$. Thus, market clearing requires, for all values of the arguments,

$$c_{1s}(d, \theta \left[d + p_s(d, \theta)\right], \theta) + c_{2s}(d, (1 - \theta) \left[d + p_s(d, \theta)\right], \theta) = d \tag{51}$$

$$b_{1s}(d, \theta \left[d + p_s(d, \theta)\right], \theta) + b_{2s}(d, (1 - \theta) \left[d + p_s(d, \theta)\right], \theta) = 0 \tag{52}$$

$$e_{1s}(d, \theta \left[d + p_s(d, \theta)\right], \theta) + e_{2s}(d, (1 - \theta) \left[d + p_s(d, \theta)\right], \theta) = 1, \tag{53}$$

The relative wealth dynamics, finally, are given by

$$g_{s'}(d, \theta, s) = \frac{w'_{1s'}(d, \theta, s)}{w'_{1s'}(d, \theta, s) + w'_{2s'}(d, \theta, s)}, \tag{54}$$

where

$$w'_{1s'}(d, \theta, s) \equiv b_{1s}(d, \theta \left[d + p_s(d, \theta)\right], \theta) + e_{1s}(d, \theta \left[d + p_s(d, \theta)\right], \theta)(d\lambda_{s'} + p_{s'} \left[d\lambda_{s'}, g_{s'}(d, \theta)\right])$$

and

$$w'_{2s'}(d, \theta, s) \equiv b_{2s}(d, (1-\theta) \left[d + p_s(d, \theta)\right], \theta) + e_{2s}(d, (1-\theta) \left[d + p_s(d, \theta)\right], \theta)(d\lambda_{s'} + p_{s'} \left[d\lambda_{s'}, g_{s'}(d, \theta)\right])$$

Now, we will show how to find prices and portfolio allocations in this economy. We use the planning problem and identify the θ in that problem with the corresponding variable here: the planning weight on agent 1 equals the relative fraction of total wealth held in equilibrium by this agent.

The Euler equation for bonds is given by

$$\frac{q_s(d, \theta)}{c_{is}(d, w, \theta)} = \beta \left(\phi_{s1}(v) \frac{1}{c_{i1}(\lambda_1 d, w'_{i1}, \theta'_1)} + \phi_{s2}(v) \frac{1}{c_{i2}(\lambda_2 d, w'_{i2}, \theta'_2)} \right), \tag{55}$$

and since $c_{1s}(d, w, \theta) = \theta d$ (and $c_{2s}(d, w, \theta) = (1 - \theta)d$), we have

$$q_s(d, \theta) = \beta\theta \left(\frac{\phi_{s1}(v)}{\theta'_1 \lambda_1} + \frac{\phi_{s2}(v)}{\theta'_2 \lambda_2} \right). \tag{56}$$

Simplifying further using the law of motion for $\theta'_{s'}$ and deriving the Euler equation for equity in a similarly fashion, the prices of bonds, $q_s(d, \theta)$, and of equity, $p_s(d, \theta)$, then become

$$q_s(d, \theta) = \beta \left[\phi_{s1}(v) \frac{\phi_{s1} - \theta v}{(\phi_{s1} - v)\lambda_1} + \phi_{s2}(v) \frac{\phi_{s2} + \theta v}{(\phi_{s2} + v)\lambda_2} \right] \equiv \hat{q}_s(\theta), \tag{57}$$

and

$$p_s(d,\theta) = \beta \left\{ \frac{(\phi_{s1} - \theta v)\left[\lambda_1 d + p_1(\lambda_1 d, \theta_1')\right]}{\lambda_1} + \frac{(\phi_{s2} + \theta v)\left[\lambda_2 d + p_2(\lambda_2 d, \theta_2')\right]}{\lambda_2} \right\} \tag{58}$$

We see that $p_s(d,\theta) = d\hat{p}_s(\theta)$ solves this equation, delivering

$$\hat{p}_s(\theta) = \beta \left\{ (\phi_{s1} - \theta v)\left[1 + \hat{p}_1(\theta_1')\right] + (\phi_{s2} + \theta v)\left[1 + \hat{p}_2(\theta_2')\right] \right\} \tag{59}$$

This is a system of two functional equations.

Asset holdings are the following. First, his equilibrium holdings of bonds are

$$b_{1s}(d,\theta) = \left[d\lambda_1 + p_1(\lambda_1 d, \theta_1')\right]\left[\theta_1' - e_{1s}(d,\theta)\right] = d\lambda_1\left[1 + \hat{p}_1(\theta_1')\right]\left[\theta_1' - \hat{e}_{1s}(\theta)\right]$$

It is interesting to note here that bond holdings are proportional to d. Naturally, they are zero in the special case $v = 0$, when $e = \theta$ and $\theta' = \theta$.

And his equilibrium holdings of equity are

$$e_{1s}(d,\theta) = \frac{\theta_1'\left[d\lambda_1 + p_1(\lambda_1 d, \theta_1')\right] - \theta_2'\left[d\lambda_2 + p_2(\lambda_2 d, \theta_2')\right]}{\left[d\lambda_1 + p_1(\lambda_1 d, \theta_1')\right] - \left[d\lambda_2 + p_2(\lambda_2 d, \theta_2')\right]} =$$

$$= \frac{\theta_1'\lambda_1\left[1 + \hat{p}_1(\theta_1')\right] - \theta_2'\lambda_2\left[1 + \hat{p}_2(\theta_2')\right]}{\lambda_1\left[1 + \hat{p}_1(\theta_1')\right] - \lambda_2\left[1 + \hat{p}_2(\theta_2')\right]} \equiv \hat{e}_{1s}(\theta) \tag{60}$$

This is once more a system of two functional equations.

Neither bond holdings nor equity holdings depend directly on s, but they do through the dependence of the θ's on s.

A1.3 The special case where $\theta = 0$

We solve the problem for an ambiguity-averse agent who is measure zero in the economy. This agent solves the problem

$$V(w,d) = \max_{c,b,e} u(c) + \min_{v} \beta\left[(\phi - v)V(w_1', d\lambda_1) + (1 - \phi + v)V(w_2', d\lambda_2)\right]$$

subject to

$$c + qb + pde = w \tag{61}$$

$$w_1' = b + (\lambda_1 d + p\lambda_1 d)e \tag{62}$$

$$w_2' = b + (\lambda_2 d + p\lambda_2 d)e \tag{63}$$

The FOCs with respect to b are

$$qu'(w - qb + pde) =$$

$$= \beta\left\{(\phi - v)u'\left[b + e\lambda_1 d(1+p) - qb' - pd\lambda_1 e'\right] + (1 - \phi + v)u'\left[b + e\lambda_2 d(1+p) - qb' - pd\lambda_2 e'\right]\right\}$$

and with respect to e, they are

$$pdu'(w - qb + pde) =$$

$$= \beta\{(\phi - v)u' \left[b + e\lambda_1 d(1+p) - qb' - pd\lambda_1 e'\right] \lambda_1 d(1+p) +$$
$$+(1-\phi+v)u' \left[b + e\lambda_2 d(1+p) - qb' - pd\lambda_2 e'\right] \lambda_2 d(1+p)\}$$

Using logarithmic utility, we see that these equations become

$$q = \beta \left[\frac{\phi - v}{b + e\lambda_1 d(1+p) - qb' - pd\lambda_1 e'} + \frac{1-\phi+v}{b + e\lambda_2 d(1+p) - qb' - pd\lambda_2 e'}\right](w - qb + pde)$$

$$\frac{p}{1+p} = \beta \left[\frac{(\phi - v)\lambda_1}{b + e\lambda_1 d(1+p) - qb' - pd\lambda_1 e'} + \frac{(1-\phi+v)\lambda_2}{b + e\lambda_2 d(1+p) - qb' - pd\lambda_2 e'}\right](w - qb + pde).$$

We guess that

$$b = \alpha_b w \tag{64}$$

and

$$ed = \alpha_e w. \tag{65}$$

Then

$$q =$$
$$= \beta \left\{\frac{\phi - v}{\left[\alpha_b + \alpha_e\lambda_1(1+p)\right](1 - q\alpha_b - p\lambda_1\alpha_e)} + \frac{1-\phi+v}{\left[\alpha_b + \alpha_e\lambda_2(1+p)\right](1 - q\alpha_b - p\lambda_2\alpha_e)}\right\}(1 - q\alpha_b - p\alpha_e)$$
$$\frac{p}{1+p} =$$
$$= \beta \left\{\frac{(\phi - v)\lambda_1}{\left[\alpha_b + \alpha_e\lambda_1(1+p)\right](1 - q\alpha_b - p\lambda_1\alpha_e)} + \frac{(1-\phi+v)\lambda_2}{\left[\alpha_b + \alpha_e\lambda_2(1+p)\right](1 - q\alpha_b - p\lambda_2\alpha_e)}\right\}(1 - q\alpha_b - p\alpha_e)$$

The problem of the consumer can be rewritten as

$$V(w) = \max_{c,b,e} \log c + \min_{v} \beta\left[(\phi - v)V(w_1') + (1-\phi+v)V(w_2')\right]$$

subject to

$$c + qb + p\hat{e} = w \tag{66}$$
$$w_1' = b + \hat{e}\lambda_1(1+p) \tag{67}$$
$$w_2' = b + \hat{e}\lambda_2(1+p), \tag{68}$$

where $\hat{e} \equiv de$. The variable v will be chosen (due to the envelope theorem) so that $V(w_1') = V(w_2')$, i.e., so that $w_1' = w_2'$. That means that $b = \alpha_b w$ and $\hat{e} = 0$ – the agent does not hold equity – and from the FOC above, that

$$q = \beta \left[\frac{\phi - v}{\alpha_b(1 - q\alpha_b)} + \frac{1-\phi+v}{\alpha_b(1 - q\alpha_b)}\right](1 - q\alpha_b). \tag{69}$$

This expression simplifies to

$$\alpha_b q = \beta \tag{70}$$

and then consumption is given by

$$c = (1-\beta)w. \tag{71}$$

Since $\frac{p}{1+p} = \beta$ in the $\theta = 1$ case, this implies

$$\alpha_b = (\phi - v)\lambda_1 + (1 - \phi + v)\lambda_2. \tag{72}$$

Therefore,

$$v = \frac{\phi\lambda_1 + (1-\phi)\lambda_2 - \alpha_b}{\lambda_1 - \lambda_2} = \phi - \frac{\alpha_b - \lambda_2}{\lambda_1 - \lambda_2} \tag{73}$$

and

$$v = \phi - \frac{\alpha_b - \lambda_2}{\lambda_1 - \lambda_2}. \tag{74}$$

For $\phi = 0.5$, $\beta = 0.98$ $\lambda_1 = 1.02$, and $\lambda_2 = 1.01$, $\alpha_b = \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2}$, and

$$v = \phi - \frac{\lambda_2}{\lambda_1 + \lambda_2} = 0.00246. \tag{75}$$

A2 Proofs of subsection 5.3

We want to proof that

$$E(\theta' \mid \theta) < \theta. \tag{76}$$

Since

$$E(\theta' \mid \theta) = \phi\frac{\theta\,(\phi - v)}{\phi - \theta v} + (1-\phi)\,\frac{\theta\,(1 - \phi + v)}{1 - \phi + \theta v}, \tag{77}$$

expression (76) becomes:

$$\phi\frac{\theta\,(\phi - v)}{\phi - \theta v} + (1-\phi)\,\frac{\theta\,(1 - \phi + v)}{1 - \phi + \theta v} < \theta. \tag{78}$$

Simplifying (78) yields:

$$\theta^2 v^2 < \theta v^2. \tag{79}$$

Since $v \neq 0$,

$$\theta < 1. \tag{80}$$

And condition (80) is always true in the case we study, otherwise we would be back to the case of one agent.

The proof that $\lim_{\theta\to 0} E\left(\frac{\theta'}{\theta}\right) = 1$ is even simpler. First, consider expression for $E\left(\frac{\theta'}{\theta}\right)$:

$$E\left(\frac{\theta'}{\theta}\right) = \phi\frac{\phi - v}{\phi - \theta v} + (1-\phi)\,\frac{1 - \phi + v}{1 - \phi + \theta v}. \tag{81}$$

Then, the limit becomes:

$$\lim_{\theta\to 0} \phi\frac{\phi - v}{\phi - \theta v} + (1-\phi)\,\frac{1 - \phi + v}{1 - \phi + \theta v} =$$

$$\phi\frac{\phi - v}{\phi} + (1-\phi)\,\frac{1 - \phi + v}{1 - \phi} = \phi - v + 1 - \phi + v = 1. \tag{82}$$

A3 Representative-agent asset pricing

In this section and for simplicity, we first consider an ambiguity-averse representative agent with a logarithmic period utility function and discount factor β. In addition, we first assume that shocks are *iid* and symmetric, i.e., $\phi_{ss'} = 0.5$. After that, we consider a CRRA period utility function and assume serially correlated shocks. Then, we calibrate the economy and report the model's performance.

There is an equity share that is competitively traded and a riskless bond that is in zero net supply. We denote the consumer's bond and equity holdings b and e, respectively.

The representative agent holds the tree and thus, his consumption in every period is the dividend of the tree. A log-period utility function together with the assumption of *iid* shocks imply that p, the price of the tree, will be linear in d, the dividend, and independent of the state: $p(d) = \hat{p}d$.

The ambiguity-averse consumer puts a higher weight on the bad outcome than what is warranted by the objective probability; that is, he becomes pessimistic because he is worried about that outcome and does not know its probability.

We assume that $\lambda_1 > \lambda_2$ so that the bad outcome is state 2 – the outcome where the dividend is low. The objective probability of this state is 0.5, but he chooses the belief in the bad state. His belief is $\phi(v) = 0.5 - v$ and he chooses v from the set $v \in [-a, a]$. The higher is a, the more ambiguity there is in the economy.

The problem of the representative agent with wealth today given by w is

$$V(w) = \max_{e} \log\left[w - p(d)e\right] + \beta \min_{v\in[-a,a]} (\phi - v)V(w_1') + (1 - \phi + v)V(w_2')$$

subject to

$$w_1' = \left[\lambda_1 d + p(d\lambda_1)\right] e,$$

and

$$w_2' = \left[\lambda_2 d + p(d\lambda_2)\right] e.$$

Here, for ease of notation, we have excluded the bond (since bond holdings must be zero in equilibrium). Moreover, the budget constraint: $c + p(d)e + q(d)b = w$ where $w = \left[d + p(d)\right] e_{-1} + b_{-1}$ (e_{-1} and b_{-1} are equity and bond holdings chosen in the previous period) has been substituted away. The Euler equation for equity is

$$p(d)u'(d) =$$

$$\beta\left\{(\phi - a)[\lambda_1 d + p(\lambda_1 d)]u'(\lambda_1 d) + (1 - \phi + a)[\lambda_2 d + p(\lambda_2 d)]u'(\lambda_2 d)\right\}. \tag{83}$$

Clearly, p is linear in d (a constant times d), whenever $u'(c) = c^{-\sigma}$ (here, $\sigma = 1$). Since the period utility is logarithmic, the price of equity does not depend on beliefs because the payoff and the inverse of marginal utility (u') are proportional to λd so that the payoff times marginal utility is the same in both states. Thus, $p(d) = \frac{\beta}{1-\beta} d$ solves the Euler equation above: the price of equity is independent of ϕ and a.

Trivially here, since $e = 1$ in equilibrium, $w_1' = \frac{\lambda_1 d}{1-\beta}$, $w_2' = \frac{\lambda_2 d}{1-\beta}$, then $V(w_1') > V(w_2')$, so the solution for v is a corner, i.e., $v = a$. In section 5, we show that v can be an interior solution when the economy is populated by both ambiguity-averse and standard consumers.

The Euler equation for bonds similarly gives

$$q(d)u'(d) = \beta\left\{(\phi - a)u'(\lambda_1 d) + (1 - \phi + a)u'(\lambda_2 d)\right\}. \tag{84}$$

We see that q depends on beliefs:

$$q = \beta\left[\left(\frac{\phi}{\lambda_1} + \frac{1-\phi}{\lambda_2}\right) + a\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)\right]. \tag{85}$$

The higher is a – the more ambiguity aversion there is in the economy – the higher is the belief that the bad state will happen, and the higher is the present value of one unit tomorrow, since the probability weight placed on the state with a high marginal utility is higher.

The net expected return on equity, ER_e, is given by

$$ER_e = \frac{\phi\lambda_1 + (1-\phi)\lambda_2}{\beta} - 1, \tag{86}$$

and it is independent of the belief. The net return on bonds, R_b, decreases when ambiguity aversion increases, because $R_b = \frac{1}{q} - 1$.

The equity premium in this economy is

$$ER_e - R_b = \frac{\phi\lambda_1 + (1-\phi)\lambda_2}{\beta} - \frac{\lambda_1\lambda_2}{\beta\left[(1-\phi)\lambda_1 + \phi\lambda_2 + a(\lambda_1 - \lambda_2)\right]}. \tag{87}$$

If we make $\phi = 0.5$, then the equity premium in this economy is

$$ER_e - R_b = \frac{\lambda_1 + \lambda_2}{2\beta} - \frac{\lambda_1\lambda_2}{\beta\left[0.5(\lambda_1 + \lambda_2) + a(\lambda_1 - \lambda_2)\right]}. \tag{88}$$

When ambiguity is most extreme, i.e., when $a = 0.5$, the equity premium becomes

$$\frac{\lambda_1 - \lambda_2}{2\beta}.$$

Using $\lambda_1 = 1.02$, $\lambda_2 = 1.01$, and $\beta = 0.98$, the equity premium is 0.5%, which is 200 times larger than the equity premium for the same parameter values when $a = 0$ – the standard model. Although this is an example, and not a calibration, it illustrates that the effect of ambiguity on asset prices/returns can be substantial.

If the period utility is $u(d) = \frac{d^{1-\alpha}}{1-\alpha}$, the price of equity depends on beliefs. In fact:

$$\hat{p} = \frac{\beta\left[(\phi - a)\lambda_1^{1-\alpha} + (1 - \phi + a)\lambda_2^{1-\alpha}\right]}{1 - \beta\left[(\phi - a)\lambda_1^{1-\alpha} + (1 - \phi + a)\lambda_2^{1-\alpha}\right]}. \tag{89}$$

A3.1 Serial correlation

We now assume that the period utility is $u(c) = \frac{c^{1-\alpha}}{1-\alpha}$ and the shocks are serially correlated.

The problem of the representative agent with wealth today given by w and today's shock s is

$$V_s(w) = \max_e u\left[w - p_s(d)e\right] + \beta \min_{v_s \in [-a,a]} (\phi_{s1} - v_s)V_1(w_1') + (\phi_{s2} + v_s)V_2(w_2')$$

subject to

$$w_1' = [\lambda_1 d + p_1(d\lambda_1)]\, e,$$
$$w_2' = [\lambda_2 d + p_2(d\lambda_2)]\, e.$$

The Euler equation for equity is

$$p_s(d)u'(d) =$$
$$\beta\left\{(\phi_{s1} - v_s)[\lambda_1 d + p_1(\lambda_1 d)]u'(\lambda_1 d) + (\phi_{s2} + v_s)[\lambda_2 d + p_2(\lambda_2 d)]u'(\lambda_2 d)\right\} \tag{90}$$

The price of equity is still linear in d, and is now given by

$$p_s(d) = k_s d \tag{91}$$

where

$$k_s = \beta\left[(\phi_{s1} - v_s)\lambda_1^{1-\alpha}(1 + k_1) + (\phi_{s2} + v_s)\lambda_2^{1-\alpha}(1 + k_2)\right], \tag{92}$$

for $s = 1, 2$.

Explicitly solving for k_1 and k_2, we obtain:

$$k_1 = \frac{\beta(\phi_{11} - a)\lambda_1^{1-\alpha}\left[1 - \beta(\phi_{22} + a)\lambda_2^{1-\alpha}\right] + \beta(\phi_{12} + a)\lambda_2^{1-\alpha} + \beta^2(\phi_{12} + a)(\phi_{21} - a)(\lambda_1\lambda_2)^{1-\alpha}}{\left[1 - \beta(\phi_{22} + a)\lambda_2^{1-\alpha}\right]\left[1 - \beta(\phi_{11} - a)\lambda_1^{1-\alpha}\right] - \beta^2(\phi_{12} + a)(\phi_{21} - a)(\lambda_1\lambda_2)^{1-\alpha}}$$

and

$$k_2 = \frac{\beta(\phi_{22} + a)\lambda_2^{1-\alpha}\left[1 - \beta(\phi_{11} - a)\lambda_1^{1-\alpha}\right] + \beta(\phi_{21} - a)\lambda_1^{1-\alpha} + \beta^2(\phi_{21} - a)(\phi_{12} + a)(\lambda_1\lambda_2)^{1-\alpha}}{\left[1 - \beta(\phi_{22} + a)\lambda_2^{1-\alpha}\right]\left[1 - \beta(\phi_{11} - a)\lambda_1^{1-\alpha}\right] - \beta^2(\phi_{12} + a)(\phi_{21} - a)(\lambda_1\lambda_2)^{1-\alpha}}$$

Thus, wealth in the next period is:

$$w_1' = \lambda_1 d(1 + k_1), \tag{93}$$

and

$$w_2' = \lambda_2 d(1 + k_2). \tag{94}$$

The price of the bond is given by

$$q_s(d) = \beta\left[\phi_{s1}\frac{1}{\lambda_1^\alpha} + \phi_{s2}\frac{1}{\lambda_2^\alpha} + a\left(\frac{1}{\lambda_2^\alpha} - \frac{1}{\lambda_1^\alpha}\right)\right] \tag{95}$$

for $s = 1, 2$.

The conditional expected net return on equity is

$$ER_s^e = \frac{\phi_{s1}\left[\lambda_1 d + p_1(\lambda_1 d)\right] + \phi_{s2}\left[\lambda_2 d + p_2(\lambda_2 d)\right]}{p_s(d)} - 1 \tag{96}$$

for $s = 1, 2$, and the unconditional expected net return on equity ER^e, is

$$\pi ER_1^e + (1 - \pi)ER_2^e - 1$$

where the invariant probability π solves

$$\pi = \phi_{11}\pi + \phi_{21}(1 - \pi). \tag{97}$$

Therefore,

$$ER^e = \pi \frac{\phi_{11}\left[\lambda_1 d + p_1(\lambda_1 d)\right] + \phi_{12}\left[\lambda_2 d + p_2(\lambda_2 d)\right]}{p_1(d)} +$$

$$+(1-\pi)\frac{\phi_{21}\left[\lambda_1 d + p_1(\lambda_1 d)\right] + \phi_{22}\left[\lambda_2 d + p_2(\lambda_2 d)\right]}{p_2(d)} - 1 \tag{98}$$

$$ER^e = \pi \frac{\phi_{11}\lambda_1(1+k_1) + \phi_{12}\lambda_2(1+k_2)}{k_1} + (1-\pi)\frac{\phi_{21}\lambda_1(1+k_1) + \phi_{22}\lambda_2(1+k_2)}{k_2} - 1 \tag{99}$$

The expected net return on the bond, R^b, is given by

$$\pi\frac{1}{q_1} + (1-\pi)\frac{1}{q_2} - 1 =$$

$$\frac{1}{\beta}\left[\frac{\pi}{\phi_{11}\frac{1}{\lambda_1^\alpha} + \phi_{12}\frac{1}{\lambda_2^\alpha} + a\left(\frac{1}{\lambda_2^\alpha} - \frac{1}{\lambda_1^\alpha}\right)} + \frac{(1-\pi)}{\phi_{21}\frac{1}{\lambda_1^\alpha} + \phi_{22}\frac{1}{\lambda_2^\alpha} + a\left(\frac{1}{\lambda_2^\alpha} - \frac{1}{\lambda_1^\alpha}\right)}\right] - 1. \tag{100}$$

Finally, the equity premium is given by

$$ER^e - R^b.$$