ENDOGENOUS PARTY PLATFORMS: "STOCHASTIC" MEMBERSHIP

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Abstract. We analyze existence of divergent equilibria in a model of endogenous party platforms with stochastic membership. The parties proposals depend on their membership, while the membership depends both on the proposals of the parties and the unobserved idiosyncratic preferences of citizens over parties. A generalization of the previously established conditions for divergent equilibrium existence in the non-stochastic case is provided.

Keywords: political parties;stochastic membership;divergent equilibria

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1. Introduction

The issue of party platform formation has been a subject of substantial attention in political economy. The major idea in this literature is that platforms of political parties are formed in response to preferences of their members, whereas the memberships themselves are, at least in part, determined by the platforms. Thus, in equilibrium the party platforms should respond to the preferences of the members attracted by them. An early paper putting forward a political competition framework to define an equilibrium concept in which party ideology and its membership are endogenously determined was Baron (1993). His equilibrium concept was related to the one used in the "voting with one's feet" models developed in the study of local public goods (see Caplin and Nalebuff (1997) for an abstract framework that covers both the political economy and public finance applications). In related work Aldrich (1983a,b), Gerber and Ortuño Ortín (1998), and Poutvaara (2003) have considered the interrelationship between partisan policy platforms and political activism.

A major objective in this literature has been establishing conditions for existence of *divergent* equilibria, in which parties propose different policies and attract members with different policy preferences. In a deterministic model of this type (such as Ortuño-Ortín and Roemer (1998) or Gomberg et al. (2004)) such and equilibrium, if it exists, involves a full sorting of agents in terms of their preferences over the policy space: even minute policy differences between parties induce a unique

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party choice by almost all citizens (in the party activist literature, along the lines of Aldrich (1983a), where there is a third possibility - that of non-participation - it is still normally assumed that those actually actively taking part in partisan activities do it in the ideologically closest party). However, such perfect sorting is not commonly observed in reality: even ideologically identical people may frequently find themselves in different parties based on idiosyncratic non-policy considerations (perhaps, historical esthetical or personal). These non-policy issues might not even be observable by an outsider, making the observed policy preferences only stochastic predictors of individual party choice. This is, of course, not a new idea in political science, where the study of stochastic models of voting have been widespread for a long time (see Coughlin (1992) for a survey). Our stochastic preference model of endogenous membership follows the same intuition. Our focus is, however, somewhat different. In particular, rather than considering the vote-maximizing parties in an electoral context we restrict our attention to parties aggregating members' preferences and try to establish to which extent the results of an older deterministic models (such as our own Gomberg et al. (2004))extend to this new setting. In modeling parties as aggregating preferences of their members, while membership is, in turn, determined in part (but not fully) by party policy positions our paper is related to the work by Roemer (2001). Our approach, however, is different in such crucial aspects as, among others, our more explicit modeling of membership decisions, the nature of intra-party decision rules (which in Roemer's case discriminate among members of different ideologies based on belonging to a "partisan core"). Our objective is likewise distinct: we want to establish to which extent the results for the stochastic membership model may be viewed as an extension of those for the older deterministic model.

The approach incorporating the stochastic party preference has, indeed, been one of those proposed already by Caplin and Nalebuff (1997). They, however, found it unsatisfactory, since, in their opinion, such equilibria could, in many cases, be fully determined by the stochastic preference component and not by the observables. Furthermore, they believed, it could not guarantee existence of equilibria exhibiting policy divergence, for, as they note "there is the possibility that as the noise approaches zero, two institutions' positions will approach each other." They further conjectured that "whenever there is the non-existence of an equilibrium without probabilistic choice it must be the case that the group positions approach each other in the probabilistic choice model as the noise goes to zero." Remarkably, as we show in this paper, the converse is also true: whenever conditions for existence of divergent equilibria in a "non-stochastic" model, such as those in Caplin and Nalebuff (1997) and Gomberg et al. (2004) hold, such equilibria will also exist if we introduce a stochastic component in individual preferences, as long as the latter is sufficiently small. This, of course, implies that, contrary to what Caplin and Nalebuff (1997) conjectured noise is not going to fully determine the equilibrium policy positions of parties.

A seemingly major difficulty in this extension is that the studies of the deterministic model have used the sorting nature of equilibrium to derive the results from the properties of the space of sorting partitions (see Caplin and Nalebuff (1997) and Gomberg et al. (2004); in the context of local public goods this approach goes back to Westhoff (2005)). In the absence of perfect sorting this approach is, of course, not feasible. However, the crucial feature of the deterministic model is, in fact, not the sorting *per se*, but the instability of the convergent equilibrium in the following sense. Suppose there are two parties which propose the same policy. Then the entire population is indifferent and the vote splits in such a way that a convergent equilibrium obtains. We require now that even minor policy perturbations result in full population sorting and sharply divergent policies. It turns out that, if that is the case, the existence of (at least one) sorting equilibrium is, in fact, guaranteed. In this paper we show that this intuition, in part, extends to the stochastic context: if the convergent (or "almost " convergent) equilibrium exists but is unstable to small policy perturbations, it may be used to detect existence of divergent equilibria. In fact, as the addition of the stochastic component adds continuity to the model, in a sense the results become, in fact, more transparent in this setting. In particular, in a context of a "generalized example" in our framework, we show that when parties are perceived by voters to be very similar in non-policy terms, so that the observed randomness of individual partisan choice is relatively small, the results of the deterministic model extend to the stochastic case.

For the moment (and for simplicity) we abstract from possible strategic electoral competition by parties (in the terminology of Caplin and Nalebuff (1997) our parties are "membership-based"). The main reason here is methodological: we believe that the issue of endogenizing party membership is distinct from the issue of strategic behavior by party leaders in a democratic election. Our main concern here is the former, and we want to consider it separately. This assumption may be viewed as appropriate for either a model of parties in a setting without commitment (*e.g.,* when voters would not believe a party, once in office, can implement policies not supported by its membership) or in a setting without true electoral competition (*e.g.*, if parties' share of the office is determined through non-electoral means). Of course, we do intend to explore extending our results to cover the case of possible strategic interactions between parties.

We assume that a political party is characterized by a policy position and by some exogenously fixed idiosyncratic non-policy position or characteristic. The policy position will be endogenously determined by the membership of the party. Thus, parties are represented by positions in a multidimensional space with a fixed position in one dimension (the non-policy dimension) and a variable position in the other dimensions. Krasa and Polborn (2010) study equilibrium in a multidimensional model in which each candidate is exogenously fixed on some dimension. A difference with our approach is that in their model policy dimensions are binary whereas in our case we assume a continuum policy space. Moreover, they develop a model that can be seen as a combination of the Downsian model (parties can choose any policy) and the citizen candidate model (candidates can only propose their ideal policies). In addition, their equilibrium concept is quite different from the one we consider here, where parties are ideological and the ideology is endogenously determined. Dziubiński and Roy (2011) consider a model of electoral competition in a two-dimensional policy space where the position in one of the dimensions is fixed. They analyze existence of convergent and divergent Nash equilibria. A difference with our model is that their parties are Downsian (non-ideological, but concerned with election) whereas our main goal is to analyze the endogenous formation of parties ideologies.

Our model generates interesting predictions on the relation between the policy proposals of parties and their idiosyncratic non-policy characteristics. It is often claimed that when ideological parties strongly differ in non-policy characteristics (that are exogenously given) they have more incentives to propose divergent policies (see Roemer (2011)). This is due to the fact that proposing a more "radical" policy is not that costly to a party since voters decision are very much influenced by the large differences in the non-policy variable. In our model, however, this does not need to be the case. If agents' preferences over the policy variables are independent of their preferences over the non-policy characteristic of parties, increasing the differences between those non-policy characteristics might yield convergence of the policy proposals[1]. The intuition is clear: If parties are very different in their non-policy characteristics, their membership is basically determined by such non-policy characteristic. Hence, unless preferences in the policy and non-policy dimensions are correlated (in which case sorting in the non-policy dimension would by itself impose policy divergence), members of the two parties will be quite similar regarding their preferences on the policy variables. And, since parties just aggregate the preferences of their members, their policy proposals will be very similar.

The rest of this paper is organized as follows. Section 2 presents the model and develops a general existence result, section 3 presents the results for the mean and median voter rules in a single dimension of issue space and section 4 concludes. The Appendix provides a few stylized empirical facts to support some of our assumptions.

2. Model

There are two parties. Party $j = 1, 2$ proposes a policy vector $x_j \in X$, where X is a non-empty compact and convex subset of $\mathbb{R}^n$ with non-empty interior. In addition to a policy x_j, party $j = 1, 2$ is characterized by a non-policy variable $y_j \in Y$. The set Y is assumed to be a closed interval of $\mathbb{R}$. It may be interpreted as reflecting currently fixed or intrinsic characteristics that matter for individual preferences. Without loss of generality we shall assume that $y_1 \leq y_2$.[2]

There is a continuum of agent types with preferences over both policy and non-policy characteristics of parties. Specifically, each agent of type $(\alpha, \beta) \in A \times B \subset \mathbb{R}^n \times \mathbb{R}$ has Euclidean preferences represented by the utility function

$$u(x, y; \alpha, \beta) = -||(x, y) - (\alpha, \beta)||$$

where $x \in X$ is the policy platform adopted by the party and $y \in Y$ is the intrinsic characteristic of the party. We shall take $A = X$ and $Y = B$. Thus, for fixed $y \in Y$, an agent of type (α, β) may be identified with his/her ideal policy.

There is a measure space of agents (citizens) $(A \times B, \mathcal{A} \times \mathcal{B}, \eta)$, where $\mathcal{A}$ is the Borel σ-algebra on A, $\mathcal{B}$ is the Borel σ-algebra on B and η is a measure on $\mathcal{A} \times \mathcal{B}$ such that $\eta(A \times B) = 1$. We denote the distribution function of η as $F(\alpha, \beta)$.

Assumption 1. The measure η is associated to a continuous density function $f(\alpha, \beta)$, which is equivalent to Lebesgue measure, with $f_1(\alpha)$ denoting the unconditional density over A and $f_2(\beta)$ the unconditional density over B.

[1]Dziubiński and Roy (2011) provide a somehow related result. In their case, if parties strongly differ in their fixed policies in a given dimension, there is full convergence in the other dimension. In our case, however, there is no, in general, full convergence and the explanation and logic behind our result is very different from theirs.

[2]The case $y_1 = y_2$ being the one we considered in Gomberg et al. (2004).

Citizens play a twofold role. Each agent is a voter and a member of the party.[3] Given the parties policy and the intrinsic characteristics (x_1, y_1) and (x_2, y_2), citizens join the party they like the most. Thus, the individual party choice is unambiguous. However, from the point of view of the parties, the second coordinate of individual type (α, β) is unobservable. Thus, for the parties the observable individual preferences over policies (given by α) may serve only as an imperfect predictor of individual party choice. Therefore, from the perspective of the party, the citizen's decision which party to join appears stochastic.

Ignoring zero-measure sets, a party membership is observed as a finite measure ν_j on A. We shall restrict ourselves to measures on A which induce a continuous population density $g_j(\alpha)$ defined on A. Thus, the set of possible party memberships will be identified with a subset of the set $\bar{\Sigma}$ consisting of all continuous functions $g_j : A \to \mathbb{R}_+$. The set $\bar{\Sigma}$ endowed with an L^p norm is a Banach space. For our purposes it will be more convenient to work with the subspace $\Sigma = \{\delta \in \bar{\Sigma} : \delta(A) \neq 0\} \subset \bar{\Sigma}$. A population partition $\nu = (\nu_1, \nu_2)$ shall be considered *admissible* if $\nu_1, \nu_2 \in \Sigma$ and for every $S \in \mathcal{A}$, we have that $\nu_1(S) + \nu_2(S) = \eta(S \times B)$. We shall denote the set of all such admissible partitions as Σ^2.

A political party $j = 1, 2$ chooses its policy by aggregating the observed policy preferences of its members, according to some fixed rule P_j, defined for non-null subsets of X and which we shall call its statute. In the following we will consider only non-null subsets of X.

As parties do not observe β the aggregation applies only to α. We shall denote the profile of party statutes as $P = (P_1, P_2)$. The mapping $P : \Sigma^2 \to X \times X$ assigns to every admissible population partition ν a policy profile $P(\nu) = (P_1(\nu), P_2(\nu))$.

Assumption 2 (Scale Invariance)**.** For $j = 1, 2$, the function $P_j : \Sigma^2 \to X$ is scale invariant. That is $P_j(t\nu) = P_j(\nu)$ for every $t \in \mathbb{R}_+$ and every $\nu \in \Sigma$.

Assumption 3 (Regularity)**.** For $j = 1, 2$, the function $P_j : \Sigma^2 \to X$ is continuous in L^1 and Fréchet differentiable.[4]

As an example of such rule we may consider the mean (respectively, the median voter rule, defined only for $n = 1$) which assigns to each party the ideal policy of its mean voter (respectively, median voter). These two aggregation rules are studied in Sections 3.1 and 3.3 below. In particular, the mean voter rule, Q, assigns to each admissible population partition $\nu = (\nu_1, \nu_2) \in \Sigma \times \Sigma$ its mean,

$$Q_i(\nu) = \frac{1}{\nu_i(A)} \int_A \alpha \, d\nu_i(\alpha)$$

It clearly satisfies Assumption 2. Consider the exogenous idiosyncratic party characteristic parameters y_1 and y_2. Each policy proposal profile $x = (x_1, x_2) \in X \times X$ induces a party membership

$$\begin{aligned} A_1(x) &= \{(\alpha, \beta) \in A \times B : u(x_1, y_1; \alpha, \beta) \geq u(x_2, y_2; \alpha, \beta)\} \\ A_2(x) &= \{(\alpha, \beta) \in A \times B : u(x_2, y_2; \alpha, \beta) \geq u(x_1, y_1; \alpha, \beta)\} \end{aligned}$$

[3] In most real cases, only a small fraction of the population is a member of a party. Our results remain true if we assume that the set of citizens who become members of parties is a random sample of the whole population.

[4] see Luenberger (1969) for definitions of differentiability of mappings from and into function spaces.

Since we only consider population densities which are absolutely continuous with respect to Lebesgue measure, the line segment $A_1(x) \cap A_2(x)$ has measure zero. Ignoring this zero-measure set, we will think of $A_1(x)$ and $A_2(x)$ as a partition of $A \times B$.

We define next the mapping $\sigma : X \times X \to \Sigma \times \Sigma$. We write $\sigma(x) = (\sigma_1(x), \sigma_2(x))$. Given a proposal $x = (x_1, x_2)$, the induced party memberships $A_1(x)$ and $A_2(x)$ determine measures $\sigma_1(x)$ and $\sigma_2(x)$ whose associated density functions are

$$g_i(\alpha; x) = \int_{\{\beta \in B : (\alpha,\beta) \in A_i(x)\}} f(\alpha, \beta) d\beta, \quad i = 1, 2$$

That is, for each Lebesgue measurable set $S \subset A$, its measure induced by x is

$$\sigma_i(x)(S) = \int_S g_i(\alpha; x)\, d\alpha$$

For $n = 1$ we can provide a more explicit formulation of the densities $g_1(\alpha; x), g_2(\alpha; x)$. Assume first that $y_1 < y_2$ and let $x = (x_1, x_2) \in X \times X$. Agents affiliate to one or the other party, depending on which side of the line

$$z(t; x) = \frac{x_1 - x_2}{y_2 - y_1} t + \frac{y_1 + y_2}{2} - \frac{x_1^2 - x_2^2}{2(y_2 - y_1)} \tag{1}$$

they lie. That is,

$$g_1(\alpha; x) = \int_{-\infty}^{z(\alpha;x)} f(\alpha, \beta)\, d\beta \qquad g_2(\alpha; x) = \int_{z(\alpha;x)}^{+\infty} f(\alpha, \beta)\, d\beta$$

We remark that even though

$$\int_A \left(g_1(\alpha, x) + g_2(\alpha, x)\right)\, d\alpha = 1$$

the measures $\sigma_1(x)$ and $\sigma_2(x)$ do not necessarily have disjoint supports. That is, the product $g_1(\alpha, x) g_2(\alpha, x)$ does not necessarily vanish. Thus, the observable characteristic $\alpha \in A$ does not completely describe the behavior of the voters. Even though, agents are fully rational, their behavior is stochastic from the point of view of the parties. Note that if $x_1 = x_2$ and $y_1 < y_2$, we have that

$$\begin{aligned} g_1(\alpha; x) &= g_1(\alpha) = \int_{-\infty}^{\frac{y_1+y_2}{2}} f(\alpha, \beta) d\beta \\ g_2(\alpha; x) &= g_2(\alpha) = \int_{\frac{y_1+y_2}{2}}^{+\infty} f(\alpha, \beta) d\beta \end{aligned} \tag{2}$$

does not depend on the specific value of $x_1 = x_2$. Note also that, if $y_1 = y_2$ and $x_1 < x_2$, the induced population densities are

$$g_1(\alpha; x) = \begin{cases} \int_B f(\alpha, \beta) : d\beta & \text{if } \alpha < \frac{x_1+x_2}{2} \\ \frac{1}{2} \int_B f(\alpha, \beta) : d\beta & \text{if } \alpha = \frac{x_1+x_2}{2} \\ 0 & \text{if } \alpha > \frac{x_1+x_2}{2} \end{cases} \qquad g_2(\alpha; x) = \begin{cases} \int_B f(\alpha, \beta) :\ d\beta & \text{if } \alpha > \frac{x_1+x_2}{2} \\ \frac{1}{2} \int_B f(\alpha, \beta) :\ d\beta & \text{if } \alpha = \frac{x_1+x_2}{2} \\ 0 & \text{if } \alpha < \frac{x_1+x_2}{2} \end{cases}$$

Thus, whenever $y_1 = y_2 = y$ and $x_1 < x_2$ the measures $\sigma_1(x)$ and $\sigma_2(x)$ do have disjoint supports. There is full sorting of the parties' members into two disjoint sets and we recover the 'non-stochastic' case.

The measures $g_1(\alpha; x)$ and $g_1(\alpha; x)$ can be defined for $y_1 = y_2 = y$ and $x_1 = x_2 = t$ as a limiting case. However, this limit depends on how (x_1, x_2, y_1, y_2) approaches

(t, t, y, y). To avoid ambiguities, from now on and without loss of generality, we will assume that $y_1 < y_2$.

The following lemmas show that the mapping $\sigma : X \times X \to \Sigma^2$ is well defined. That is, the functions $g_1(\alpha; x)$ and $g_2(\alpha; x)$ are continuous on $\alpha \in A$. The proofs are provided in the Appendix.

Lemma 4. If $y_1 \neq y_2$, the mapping $\sigma_i : X \times X \to \Sigma$ is Fréchet differentiable for each $i = 1, 2$.

As a consequence, we have the following.

Corollary 5. If $y_1 \neq y_2$, the mapping $\sigma_i : X \times X \to \Sigma$ is continuous for each $i = 1, 2$.

2.1. **Equilibrium.** Throughout this subsection, unless explicitly stated, we assume $y_1 < y_2$. Consider the party population densities in the ideological space that would obtain if each person joined the party which s/he prefers solely on the unobservable y dimension (as would obtain if the parties adopted identical policies).

$$\begin{aligned} g_1(\alpha) &= \int_{-\infty}^{\frac{y_1+y_2}{2}} f(\alpha, \beta) d\beta \\ g_2(\alpha) &= \int_{\frac{y_1+y_2}{2}}^{+\infty} f(\alpha, \beta) d\beta \end{aligned} \tag{3}$$

Denoting the corresponding population partition as $(\bar{v}_1, \bar{v}_2) = \bar{v}$ the induced policy proposals will be $P(\bar{v}) = (P_1(\bar{v}), P_2(\bar{v}))$. This is the profile of policy proposals that would emerge if citizens chose parties purely for non-policy reasons. Of course, unless $P_1(\bar{v}) = P_2(\bar{v})$, the policy profile $P(\bar{v})$ will, in general, induce a very different population partition.

Our notion of equilibrium assumes free mobility of the electorate across parties. Thus, our equilibrium concept requires that: i) the proposals of the parties are determined by their respective membership; and (ii) given the party's proposal, all the members of each party prefer their own party to the alternative.

Definition 6. Given the profile of party statutes P and the idiosyncratic party characteristics y_1, y_2 we say that $(x^*, \nu^*) \in X \times X \times \Sigma^2$ is a **multi–party equilibrium** if:

1. $x^* = P(\nu^*)$
2. $\nu^* = \sigma(x^*)$

Furthermore, the equilibrium is **divergent** if $x_1^* \neq x_2^*$. Otherwise, we say that the equilibrium is **convergent**.

The above definition captures the idea that, in equilibrium, the platforms of the parties are consistent with the preferences of the members they attract. If $P_1(\bar{v}) = P_2(\bar{v})$, there is trivially a convergent equilibrium (Proposition 10 below). It turns out that as long as the distance between them in the y dimension is not too large, there will always exist a convergent equilibrium in which policy proposals are closer to each other than would have been the case if voters sorted between parties based solely on their non-policy positions. We will focus on the existence of divergent equilibria. This will be shown using the instability of the convergent equilibrium.

Consider the mapping $\phi : X \times X \to X \times X$ defined by $\phi(x) = P(\sigma(x))$. Clearly, an equilibrium is just a fixed point of this mapping.

Proposition 7. Suppose that $y_1 \neq y_2$ and Assumption 3 holds. Then, there exists an equilibrium.

Proof. Consider the mapping $\phi : X \times X \to X \times X$ defined by $\phi(x) = P(\sigma(x))$. The fixed points of this mapping correspond to equilibria of the model. The mapping is clearly defined on the entire $X \times X$ and continuity follows from Assumption 3. As $X \times X$ is compact and convex, by Brouwer's Fixed Point Theorem there must exist at least one such fixed point (possibly convergent). □

Note that if $y_1 = y_2$, so that individual membership is fully determined by policy positions, this model becomes deterministic and fully falls into the framework posed by Caplin and Nalebuff (1997). Therefore, Proposition 7 shows that, as long as what we are interested in is merely existence of multi-party equilibria, the present model still fits the approach of Caplin and Nalebuff (1997) and Gomberg et al. (2004): basic continuity and minimal internal support assumptions on the statutes (as in Gomberg et al. (2004)), together with the exogenously imposed difference between parties yields existence of equilibrium.

The novel case for us in this paper is $y_1 \neq y_2$. Under this assumption, we trivially obtain full sorting of the agents in the $A \times B$ space. However, this sorting may be entirely caused by the difference in the y's. In particular, if the observed policy positions of the parties (that is, their projections onto the A component) coincide, then this equilibrium would still be convergent.

Hence, the previous results are silent on the existence of divergent equilibria. It is establishing conditions for existence of this last equilibrium type that concerns us here. Consider the following simple example.

Example 8. Let $n = 1$ and suppose the population is distributed over $[0, 1] \times [0, 1]$. Suppose that the both parties use the mean voter rule Q so that

$$Q_i(\nu) = \frac{1}{\nu_i(A)} \int_A \alpha \, d\nu_i(\alpha)$$

Consider two possible population distributions over this domain: the uniform distribution over the whole of $[0, 1] \times [0, 1]$, so that individual preferences over the two dimensions are entirely uncorrelated; and the uniform over the diagonal $\Delta = \{(\alpha, \beta) \in [0, 1] \times [0, 1] : \alpha = \beta\}$, in which there is perfect correlation between the individual ideal points in the two dimensions. Note that in both cases the unconditional distribution of ideal points in the policy space is uniform: $f_1(\alpha) = 1$. Assume, for simplicity, that $y_2 = 1 - y_1$. Consider the case when $y_1 = y_2 = \frac{1}{2}$. As the citizen preference in this case depends only on $f_1(\alpha)$, for either of the two distributions there are the same three equilibria: $x_1 = x_2 = 1/2$; $x_1 = \frac{1}{4}$ $x_2 = \frac{3}{4}$; and $x_1 = \frac{3}{4}$ $x_2 = \frac{1}{4}$.[5] As we increase the difference between the idiosyncratic positions y_1 and y_2, the two cases will become increasingly different. As we show in

[5]Note that the convergent equilibrium in this example relies on full indifference of every agent between the two parties, requiring a slight modification of the definitions along the lins of Gomberg et al. (2004). In the uncorrelated case this equilibrium is properly defined for any $y_1 < \frac{1}{2}$. In the correlated case we rely on complete population indifference in the centrist equilibrium even if $y_1 < \frac{1}{2}$. However, as long as the correlation is imperfect, so that the population distribution is not concentrated on the hyperplane, this would no longer be the case, while, as we show below, the structure of equilibrium set remains the same.

Section 4, when the individual preferences in the two dimensions are uncorrelated $x_1 = x_2 = \frac{1}{2}$ remains an equilibrium always, while the other two equilibria converge to it, so that the symmetric equilibrium becomes the only one as long as $y_1 \leq \frac{1}{4}$. In contrast, in the case when preferences in the two dimensions are perfectly correlated, as long as $y_1 > \frac{1}{4}$ it is the divergent equilibria $x_1 = \frac{1}{4}$ $x_2 = \frac{3}{4}$ and $x_1 = \frac{3}{4}$ $x_2 = \frac{1}{4}$ that remain unchanged, while the convergent equilibrium gradually shifts: $x_1 = y_2$, $x_2 = y_1$. When $y_1 \leq \frac{1}{4}$ so that the parties are far apart, the only surviving equilibrium in this case is $x_1 = \frac{1}{4}$ $x_2 = \frac{3}{4}$: the population partition is now fully determined by the party difference in the y dimension.

2.1.1. *Equilibrium divergence.* Let

$$\lambda_i(x) = \sigma_i(x)(A) = \int_A g_i(\alpha; x)\, d\alpha$$

be the vote share of party $i = 1, 2$. Let $\Delta = \left\{(x_1, x_2) \in X^2 : x_1 = x_2\right\}$ denote the diagonal of the policy space.

Remark 9. If the party proposals x_1 and x_2 are such that $x = (x_1, x_2) \in \Delta$, then $g_i(\alpha; x) = g_i(\alpha)$ for every $\alpha \in A$ and $i = 1, 2$. Hence, for $x \in \Delta$,

(1) $\lambda_i(x) = \lambda_i$, $i = 1, 2$ does not depend on x.
(2) the induced partition of the population is described by the density functions

$$\frac{1}{\lambda_1} g_1(\alpha) = \frac{1}{\lambda_2} g_2(\alpha) = g(\alpha)$$

Under the conditions assumed in Remark 9, it should be noted that the vote shares are themselves independent of the common ideological position of the parties and are entirely determined by y_1 and y_2. In particular, σ is constant on the diagonal $\Delta \subset X \times X$. It follows that there can be at most a unique convergent equilibrium policy profile x^*.

Lemma 10. There is, at most, one convergent equilibrium $x^* \in \Delta$.

Proof. Consider a policy profile $x \in \Delta$. By remark 9, we know that every such policy profile it is mapped into a population partition determined by

$$\frac{1}{\lambda_1} g_1(\alpha) = \frac{1}{\lambda_2} g_2(\alpha) = g(\alpha)$$

with g_1 and g_2 given by Equation (3). In other words, the entire diagonal of the policy space Δ is mapped into a single population partition $\sigma \in \Sigma$, which, in turn, gets mapped into a single point $P(\sigma) \in X \times X$. Therefore, there can exists, at most, one convergent equilibrium $x^* \in \Delta$. □

Of course, the convergent equilibrium, if it exists, may turn out to be the only one, since no more is guaranteed by Proposition 7. Our question in this paper shall be to find out the conditions for existence of a divergent equilibrium.

In order to do this and following Caplin and Nalebuff (1997) and Gomberg et al. (2004) we shall postulate, in addition to the two previously introduced assumptions, two extra ones on policy rules. The first one ensures that parties choose interior ideological positions. For this purpose, we consider the following property.

Assumption 11 (Minimal Internal Support)**.** There is $\delta > 0$ such that the following holds: for each proposal $x \in X \times X$, any other policy $t \in X$ and any $j = 1, 2$, the induced party memberships satisfy the following

$$\int_{\{(\alpha,\beta)\in A_j(x):u(P_j(\sigma(x)),y_j;\alpha,\beta)>u(t,y_j;\alpha,\beta)\}} g_j(\alpha,x)d\alpha > \delta\lambda_j(x)$$

In a deterministic version of the model, Caplin and Nalebuff (1997) have postulated the assumption that the party policy rules would never result in identical policies if party populations have opposing preference, in the sense of being divided by a hyperplane in the ideological space. Even in their model, this assumption is problematic, unless the policy rules are just aggregating intra-party preferences ("membership-based" in their terminology). And since, in the present model, the sorting is not perfect, it is entirely inapplicable here. Fortunately, it turns out that what was driving the Caplin and Nalebuff (1997) result was not this, but a weaker condition: instability of the convergent equilibrium under adjustment dynamics. We shall assume the following.

Assumption 12 (Instability of Pooling)**.** There exists an open neighborhood $O \subset X \times X$ containing the diagonal Δ such that for any boundary point $x \in \partial O \backslash \Delta$, we have that $\phi(x) \notin O \cup \partial O$.

Intuitively, this states that once a convergent policy profile is perturbed, the induced population partition induces further policy divergence. It is not difficult to check that, in a deterministic model, the requirement of distinct policies from sorting partitions imposed in earlier work implies this divergence. Any minute policy difference in a deterministic model induces a full population sorting which, in turn, induces distinct policies, which, by continuity of policy rules, cannot be close to the diagonal. This weaker assumption is, in fact, sufficient for our next result.

Proposition 13. If the dimension n of the policy space is odd and Assumptions 2, 3, 11 and 12 hold then, there exists a divergent equilibrium.

Proof. If no convergent equilibrium exists, the result follows trivially from Proposition 7. Hence, assume there exists a convergent equilibrium x^*. The equilibrium x^* is *stable* along the diagonal, Δ, since any policy profile on Δ is mapped directly into x^*. Furthermore, this equilibrium is either isolated or else, there is at least a divergent equilibrium. Thus, we may assume this equilibrium is an isolated fixed point.[6]

We will show that x^* cannot be the unique[7] equilibrium. In fact, suppose x^* is the unique equilibrium. We argue that the boundary of $X \times X$ is unstable. Let $x = (x_1, x_2) \in \partial X \times X$. By making x'_j arbitrarily close to x_j, and using Assumption 11, we can make an arbitrarily large proportion of party members strictly prefer x'_j. Hence, $\phi(x) \in \text{int}(X \times X)$ and the boundary of $X \times X$ is unstable.

As ϕ is a mapping from the compact and convex set $X \times X$ to itself, and as x^* is assumed to be non-degenerate, by the Lefschetz Fixed Point Theorem (see

[6]Even if it is not, 12 ensures that the component of the fixed point set it belongs to will be inside the open neighborhood O, so that the basic argument for equilibria outside of O follows in any case.

[7]That is, there are other non-convergent equilibria.

Guillemin and Pollack (2010), pp. 119-130) the total sum of the indices of the fixed points x^* must be equal to 1 (the Euler characteristic of $X \times X$). Recall, that the index ind(x^*) of a non-degenerate fixed point x^*may be calculated as $(-1)^d$, where d is the dimension of the unstable manifold of x^*. By Assumption 12, the equilibrium x^* is unstable off diagonal. As the co-dimension of the diagonal Δ is 1, the index of the diagonal fixed point equals $(-1)^n$, which implies it cannot be unique if n is odd. Hence, a divergent equilibrium must exist. □

2.2. **Divergence of similar parties.** The previous result relies on us being able to support identical policies $x_1 = x_2$ in equilibrium. This may not be the case, in particular, if the distribution of individual preferences in the policy and non-policy dimensions is not independent. In particular, if α is correlated with β then, even if parties use the same internal policy-setting rule $P_1 = P_2$ this may not be the case, which means that equilibrium divergence emerges straightforwardly. That introducing *ex ante* difference between parties may effectively impose policy divergence exogenously has been a particular concern in earlier work of Caplin and Nalebuff (1997), who found a similar approach to be unsatisfactory for this very reason. It turns out, however, that, as long as the basic assumptions of our model hold, the structure of equilibrium set remains the same. Indeed, as we have seen in example 8, despite the perfect correlation between ideal positions on the two dimensions, even as the exogenous difference between the political parties in the y dimension is small, there will remain a properly divergent equilibrium, in which party policy positions will stay apart even as $|y_2 - y_1| \to 0$.

Thus, for any population distribution $f(\alpha, \beta)$, party policy rule profile P and the non-policy party positions $y_1 < y_2$ we may bring the latter together by setting $y_1^t = \frac{1}{2}(y_1 + y_2 + t(y_1 - y_2))$ and $y_2^t = \frac{1}{2}(y_1 + y_2 + t(y_2 - y_1))$. This allows us to define, for each $t \in (0,1)$ a corresponding $\phi^t(x)$. As before, for each t the fixed points x^{t*} of $\phi^t(x)$ will be equilibria of our model. Consider now two distributions over $A \times B, \eta$ and $\bar{\eta}$ which induce the same density of ideal points over the policy space $f_1(\alpha)$, even though their corresponding densities over the entire domain may differ $f(\alpha, \beta) \neq \bar{f}(\alpha, \beta)$. In particular, let $\bar{f}(\alpha, \beta) = f_1(\alpha) f_2(\beta)$ (that is, under $\bar{f}$ the distribution of the ideal points is uncorrelated across dimensions).

Proposition 14. Let the parties use the same policy-setting rule $P_1 = P_2$. If the dimension of the policy space n is odd, and assumptions 2, 3, 11, and 12 hold for an uncorrelated distribution $\bar{f}(\alpha, \beta) = f_1(\alpha) f_2(\beta)$. Then, for any $f(\alpha, \beta)$ with the same marginal distributions as $\bar{f}(\alpha, \beta)$ there exists a sequence of equilibria x^{t*} corresponding to fixed points of $\phi^t(x)$ which, as $t \to 0$, does not converge to the diagonal Δ.

Proof. Fix a pair of $x_1 \neq x_2$ and for any $t \in [0,1]$ define $y_1^t = y_1 + t(y_2 - y_1)$. Note, that as $t \to 0$ the induced partition hyperplane will converge to perfect sorting in the policy dimension (in fact, the convergence of the induced population partitions $\nu^t \in \Sigma$ can be seen to be uniform in L^1). This, of course, implies that the induced population partitions for $f(\alpha, \beta)$ and $\bar{f}(\alpha, \beta)$ converge uniformly to one another, and, in the limit, depend only on $f_1(\alpha)$. Hence, the corresponding mappings $\phi(x)$ and $\bar{\phi}(x)$ from the proposition 7, will converge pointwise for any $x \notin \Delta$ as $t \to 0$ (note that this convergence is not uniform away from the diagonal, as, in fact, the functions will not converge to each other on Δ).

Consider now the unstable open neighborhood O of Δ from 12. Its complement $B = X \times X \backslash O$ is compact. Hence, there exists $d > 0$ such that for any $x \in \Delta$ $||x_1 - x_2|| \geq d$. Hence, the slope of the partition hyperplane is bounded from below for every t so that its convergence to the line orthogonal to $||x_1 - x_2||$ is uniform and, since $x \in X \times X$ - compact and the population measures are both equivalent to Lebesgue with continuous densities this implies the uniform convergence of the corresponding $\sigma(x)$ and $\bar{\sigma}(x)$ on B. Since P is continuous in L^1 we therefore have uniform convergence of $\phi(x)$ and $\bar{\phi}(x)$ on B. Hence, for t close enough to 0 the index of all the fixed points in B is the same for the two maps (that is, as we know from the proof of the previous proposition, not zero), so that, by the by the Lefschetz Fixed Point Theorem there must exist a fixed point x^* of $\phi(x)$ on B, so that $||x_1^* - x_2^*|| \geq d > 0$.

□

Of course, the last two propositions, on their own, are of limited interest: unless we can show that Assumption 12 holds for cases where $y_1 \neq y_2$ the result is vacuous. Fortunately, as we show below, in many cases, including when parties use the mean and the median voter rule**,** we can show that whenever $|y_2 - y_1|$ is sufficiently small, the assumption does hold, as we will show in the following section.

Note also that, under the conditions of Propositions 13 and 14, the only stable equilibria would be bounded away from convergence. Therefore, in the situation described here, we may expect to observe only divergent equilibria.

3. Mean and Median

In this section we show that the conditions of the previous section, particularly 12, apply to the case of two common voting rules: the mean and the median - at least as long as the difference $|y_2 - y_1|$ is sufficiently small . While the results of the previous section guarantee existence of divergent equilibria for parties that are sufficiently similar in the y dimension, our approach actually allows us to compute explicit bounds on the exogenous interparty difference $|y_2 - y_1|$ which guarantees policy divergence. For simplicity, in computing these bounds we stick to the case in which the distribution of ideal policies of citizens is uncorrelated across dimensions and the two parties use the same internal rule, so that $P_1(\bar{v}) = P_2(\bar{v})$, implying that the trivially convergent equilibrium exists. Of course, even if $P_1(\bar{v}) \neq P_2(\bar{v})$ and no such equilibrium exists, the last proposition of our previous section guarantees us that equilibrium policies of parties will not converge to each other even as $|y_2 - y_1| \to 0$.

Throughout this section we assume the unidimensional policy space $n = 1$.

3.1. **The Mean Voter Rule.**

The mean voter rule, Q, assigns to each admissible population partition $\nu = (\nu_1, \nu_2) \in \Sigma \times \Sigma$ its mean,

$$Q_i(\nu) = \frac{1}{\nu_i(A)} \int_A \alpha \, d\nu_i(\alpha)$$

One checks easily that this aggregation rule satisfies Assumption 2. That it also satisfies Assumption 3 is proved in Appendix 3.

Lemma 15. The map $Q : \Sigma \times \Sigma \to A \times A$ is continuous and differentiable.

We now consider the case in which parties use the mean voter rule, $Q_1 = Q_2 = Q$. Recall that, for each $x \in X \times X$, the induced population partition $\sigma_j(x)$ is represented by the density $g_j(\alpha; x)$. That is, each party chooses

$$Q_j(\sigma(x)) = \frac{\int_A \alpha g_j(\alpha; x) d\alpha}{\int_A g_j(\alpha; x) d\alpha} = \frac{\int_A \alpha g_j(\alpha; x) d\alpha}{\sigma_j(x)(A)}$$

Recall that an equilibrium is a fixed point of the mapping $\phi : X \times X \to X \times X$ defined by $\phi(x) = (Q_1(\sigma(x)), Q_2(\sigma(x)))$. Therefore, (x_1^*, x_2^*) is an equilibrium if

$$x_j^* = \frac{\int_A \alpha g_j(\alpha; x^*) d\alpha}{\int_A g_j(\alpha; x^*) d\alpha} \quad j = 1, 2$$

As a consequence of Lemma 15, we have the following.

Corollary 16. The map $\phi : X \times X \to X \times X$, defined by $\phi(x) = Q(\sigma(x))$, is continuous and differentiable.

Note that, for the mean voter rule, if parties are *ex ante* identical and their proposals satisfy $x \in \Delta$, then $Q_1(\sigma_1(x)) = Q_2(\sigma_2(x)) = \mu$, the observed mean of the overall population on A. Therefore, (μ, μ) is a fixed point of $\phi(x)$. We want to determine conditions under which the fixed point (μ, μ) is unstable off diagonal. As the stability along the diagonal is immediate from the definition of the rule (from anywhere on Δ the function ϕ immediately maps to (μ, μ)) we should simply need that the eigenvalues of the matrix[8]

$$B(x) = \begin{pmatrix} \partial_1 \phi_1(x) - 1 & \partial_2 \phi_1(x) \\ \partial_1 \phi_2(x) & \partial_2 \phi_2(x) - 1 \end{pmatrix}$$

have different signs around (μ, μ). In other words, the necessary condition for Assumption 12 to hold is that

$$\lim_{x \to (\mu,\mu)} \det(B(x)) < 0 \tag{4}$$

It turns out that as $|y_2 - y_1|$ gets smaller (but remains strictly positive), existence of a divergent equilibrium can be assured.

Proposition 17. Let $n = 1$. Suppose that the parties are *ex ante* identical and use the mean voter rule. Let $x \in \Delta$ and $\mu = Q_1(\sigma_1(x)) = Q_2(\sigma_2(x))$. If

$$|y_2 - y_1| < \frac{1}{\lambda_1 \lambda_2} \int_A (\alpha - \mu)^2 f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha \tag{5}$$

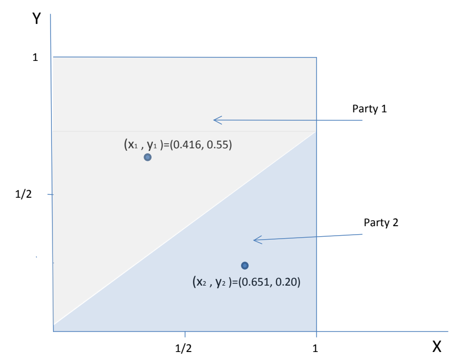
then, there exists a divergent equilibrium.

The proof is provided in the Appendix. In there, we establish that $\det(B(x)) < 0$. The boundary established by Proposition 17 depends only on two bits of population statistics: the variance of ideal points in the observable ideological space of those citizens who would be indifferent between parties in the absence of ideological differences between them, and the relative size λ_1 of the part of the population that exogenously prefers one party to another when there is no ideological difference between them.

While, strictly speaking, exceeding the boundary does not guarantee the uniqueness of the convergent equilibrium, examples of the latter are not hard to find.

[8]By ∂_i we denote the partial derivative with respect to x_i.

FIGURE 1. The unique divergent equilibrium



Example 18. Let $n = 1$ and the population distribution be uniform on $[0, 1] \times [0, 1]$ so that $f(\alpha, \beta) = 1$. Suppose that both parties use the mean voter rule Q and the idiosyncratic variables are such that $0 < |y_2 - y_1| < 1/(12\lambda_1(1 - \lambda_1)) = 1/3$. By our proposition there exists a divergent equilibrium. As an example satisfying this inequality take $y_1 = 0.55$ and $y_2 = 0.20$. It is not difficult to show that the unique divergent equilibrium has $x_1 = 0.416$, $x_2 = 0.651$. See Figure 1.

Since small exogenous differences between the parties implies that the citizens' membership decision is mostly determined by the observed policy differences (in particular, if we let $y_2 = y_1$ and $x_1 \neq x_2$, we are reduced to the deterministic model), this result shows that the deterministic case is not isolated. Rather, in this case there is continuity: a small amount of uncertainty about individual membership decision does not affect the existence of a divergent equilibrium.

3.2. **Other 'Mean'-type rules.** It should be noted that the continuity result of Proposition 17 can be extended to policy rules other than the mean voter rule, though the precise boundary would be different. In particular, suppose that, instead of choosing the ideal point of the mean of its voter distribution, parties propose policies according to a different rule

$$S_j(\eta)) = \frac{\int_A h(\alpha) d\eta}{\eta(A)}$$

where $h : A \to A$ is a non-constant continuous mapping.

Suppose the parties are *ex ante* identical. Let $x \in \Delta$ (so $g(\alpha) = g_1(\alpha; x) = g_2(\alpha; x)$ does not depend on x). Denote the policy society as a whole would adopt as $\chi = \int_A h(\alpha) g(\alpha) d\alpha \in \text{int}(A)$. Following the steps of the proof of Proposition 17 we shall easily obtain the following boundary on the exogenous difference between parties that guarantees existence of divergent equilibria:

$$0 < |y_2 - y_1| < \frac{1}{\lambda_1(1 - \lambda_1)} \int_A (h(\alpha) - \chi)(\alpha - \chi) f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha$$

which implies that, as long as $\int_A (h(\alpha) - \chi)(\alpha - \chi) f\left(\alpha, \frac{y_1+y_2}{2}\right) d\alpha > 0$ for these rules, likewise, sufficiently small uncertainty about individual membership choices leads to the existence of divergent equilibria.

3.3. **The Median Voter Rule.** It is possible to use the same techniques to establish similar bounds for other rules, that do not belong to the class described above. For example, suppose $n = 1$ (so that X is a compact interval) and parties use the median voter rule. That is, for each $\mu \in \Sigma$ the mapping T assigns $T(\mu)$ defined implicitly by the following equation.

$$\mu\left(\{\alpha \in A : \alpha \leq T(\mu)\}\right) = \frac{\mu(A)}{2}$$

We assume now that parties use the aggregation rule $T_1 = T_2 = T$. As before, we define $\phi : X \times X \to X \times X$ by $\phi = (\phi_1, \phi_2) = T \circ \sigma$. Clearly, what we are after is finding the fixed points of the mappings $\phi_j : X \to X$ given implicitly by the equations

$$\int_{-\infty}^{\phi_j(x)} g_j(\alpha; x) d\alpha = \frac{\lambda_j}{2}, \quad j = 1, 2 \tag{6}$$

Lemma 19. The map $\phi : X \times X \to X \times X$ is continuous and differentiable.

Suppose the two parties are *ex ante* identical. Then, the median m of the population distributions of the parties is the same for both parties. That is, if $g_1(\alpha)$ and $g_2(\alpha)$ are defined by (3), then

$$\int_{\{\alpha \in A : \alpha \leq m\}} g_j(\alpha) d\alpha = \frac{\lambda_j}{2} \quad j = 1, 2$$

It is straightforward to check that the point $(m, m) \in \Delta$ is a fixed point of the mappings ϕ_j, $j = 1, 2$. That is, (m, m) is a convergent equilibrium.

As before, we would like to establish the (in)stability of ϕ around the convergent fixed point. That is, we would like to establish conditions under which inequality (4) holds. Let

$$f_1(m) = \int_{-\infty}^{\frac{y_1 + y_2}{2}} f(m, \beta)\, d\beta$$

Proposition 20. Let $n = 1$. Suppose the parties are *ex ante* identical and use the median voter rule. If

$$\frac{1}{f_1(m)} \left(\int_{-\infty}^{\infty} (\alpha - m) f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha - 2 \int_{-\infty}^{m} (\alpha - m) f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha \right) > |y_2 - y_1| > 0$$

then, there exists a divergent equilibrium.

It should be noted that when the density of citizens's ideological viewpoints at the median point of the whole distribution is $f_1(m) = 0$, then the bound on $|y_2 - y_1|$ explodes, as the minor changes of policies cause the intra-party medians to move at an infinite rate.[9]

Another interesting observation is, that, at least for the uniform distribution of citizens over $A \times B = [0, 1] \times [0, 1]$, the boundary for $|y_2 - y_1|$ implied by the median voter rule (which, in this case is easily computed to be equal to $\frac{1}{2}$ for *ex ante* identical parties with $\lambda_1 = \lambda_2 = \frac{1}{2}$) is weaker than that for the mean voter rule (for which it is $\frac{1}{3}$). Therefore, as the policy difference between parties induces

[9]Though, strictly speaking, the function ϕ is not differentiable (not even necessarily continuous) in this case, the instability of the convergent equilibrium and the consequent existence of a divergent equilibrium can be easily shown using standard approximation techniques.

ideologically skewed memberships within each party, the medians move towards the edges faster than the means.

Finally, it should be noted that the divergent equilibria exist even when parties use distinct rules. Of course, the result is trivially true if there is no convergent equilibrium. Thus, if party 1 uses the mean voter rule, while party 2 uses the median voter rule a convergent equilibrium only exists if the mean equals the median for the overall population distribution $f_1(\alpha)$, i.e. if $m = \mu$. Still, even for the case when $m = \mu$ we can guarantee existence of a divergent equilibria for $|y_2 - y_1|$ small enough. In fact, using inequality 4, we can establish that such equilibria exist in this case whenever the following condition holds.

$$\begin{aligned} 0 \quad &< \quad |y_2 - y_1| < \frac{1}{\lambda_1}\int_A (\alpha-\mu)^2 f\left(\alpha, \frac{y_1+y_2}{2}\right) d\alpha + \\ &+ \quad \frac{1}{1-\lambda_1}\frac{1}{f_1(\mu)}\int_{\{\alpha:\alpha\leq\mu\}} (\mu-\alpha) f\left(\alpha, \frac{y_1+y_2}{2}\right) d\alpha \end{aligned}$$

4. Policy divergence and non-ideological characteristics

In this section we provide robust numerical and theoretical examples showing that, if parties are very different in their non-policy characteristics, then, unless there is correlation of individual preferences across ideological and non-ideological dimensions, their policy proposals will turn out to be very similar. This is most likely to happen if agents preferences over the policy variables are independent of their preferences over the non-policy characteristic of parties.

4.1. **The Mean Rule.** Let us examine first the mean voter rule of Section 3.1. Let $A = [0,1]$, $B = [-b_0, b_0]$. Let $f : A \times B \to \mathbb{R}_+$ be a continuous density function. Let M be the supremum of f on the set $A \times B$. We require that parties are *ex ante* identical. For simplicity, we assume $y_1 = -\frac{u}{2}, \quad y_2 = \frac{u}{2}$. Hence, $\frac{y_1+y_2}{2} = 0$ and

$$\int_A\int_{-\infty}^0 f(\alpha,\beta)\,d\alpha\,d\beta = \lambda_1, \qquad \int_A\int_0^\infty f(\alpha,\beta)\,d\alpha\,d\beta = \lambda_2$$

We will write

$$z(t;x;u) = z(t;x) = \frac{(x_1-x_2)(2\mathrm{t}-x_1-x_2)}{2u}$$

$\phi_j(x;u) = \phi_j(x)$ and, $g_i(\alpha;x;u) = g_i(\alpha;x)$, $i = 1,2$, to make explicit the dependence on u. Note that $|z(t;x;u)| \leq \frac{1}{2u}$ for every $x = (x_1,x_2) \in X \times X$. Hence $\lim_{u\to\infty} z(t;x;u) = 0$ uniformly on $x \in X \times X$. Since parties are *ex ante* identical,

$$\lim_{u\to\infty}\int_A g_1(\alpha;x;u)\,d\alpha = \int_A\int_{-\infty}^0 f(\alpha,\beta)\,d\alpha\,d\beta = \lambda_1$$

and

$$\lim_{u\to\infty}\int_A g_1(\alpha;x;u)\,d\alpha = \int_A\int_0^\infty f(\alpha,\beta)\,d\alpha\,d\beta = \lambda_2$$

uniformly on $x \in A$. There is a real number $b_1 > 0$ such that if $u \geq b_1$, then

$$\int_A g_i(\alpha;x;u)\,d\alpha \geq \frac{\lambda_i}{2}$$

for every $x \in X \times X$. From the proof of Lemma 15, we see that

$$|\partial_i g_j(\alpha;x;u)| = f(\alpha, z(\alpha;x))\frac{|\alpha-x_i|}{|y_2-y_1|} = f(\alpha,z(\alpha;x))\frac{|\alpha-x_i|}{u} \leq M\frac{|\alpha-x_i|}{u}$$

And, since $\alpha, \phi_j(x;u)$ and x_i belong to $A = X$ we have that $|\alpha - \phi_j(x;u)| \le 1$ and $|\alpha - x_i| \le 1$. Hence,

$$|\partial_i \phi_j(x;u)| \le \frac{1}{\left(\int_A g_j(\alpha;x) d\alpha\right)} \int_A |\alpha - \phi_j(x)| \, |\partial_i g_j(\alpha;x)\, d\alpha| \le \frac{2M}{u\lambda_j}$$

for every $x \in X \times X$. It follows that there is b_2 such that $|\partial_i \phi_j(x;u)| \le 1/4$ for every $u \ge b_2$ and for every $x \in X \times X$. We can now proof the following.

Proposition 21. Suppose the distribution of agents is such that $b_0 > b_2$. Then, there is a unique equilibrium. This equilibrium is convergent.

Proof. The proof of Lemma 15 shows that

$$\det(B(x)) = 1 + \partial_1\phi_1(x;u)\partial_2\phi_2(x;u) - \partial_1\phi_2(x;u)\partial_2\phi_2(x;u) - \partial_1\phi_2(x;u) - \partial_2\phi_2(x;u) > 0$$

Since the sum of the indices of the fixed points must add to the Euler Characteristics of the rectangle $A \times B$, which is $+1$, there can be at most a fixed point. And this unique fixed point coincides with the convergent equilibrium.

□

The Assumption in Proposition 21 can be interpreted as saying that the population is not too much concentrated around the line $y = 0$. For example, Proposition 20 shows that, if b_0 is small enough, then in addition to the convergent equilibrium, there are other divergent equilibria. On the other hand, the following remark shows that if b_0 is large, then as $|y_2 - y_1|$ increases there is a unique equilibrium.

Remark 22. We assume, as before, that $A = [0,1]$, $B = [-b_0, b_0]$ with b_0 large enough. The particular value of b_0 is irrelevant for the computations below. Agents are uniformly distributed on $A \times B$. We require that parties are *ex ante* identical. For simplicity, we let

$$y_1 = -\frac{u}{2}, \quad y_2 = \frac{u}{2}$$

We claim that for $u > 2/(3b_0)$, there is a unique convergent equilibrium. Let $x = (x_1, x_2)$. Recall that

$$z(t;x) = \frac{2t(x_2 - x_1) + x_1^2 - x_2^2 + y_1^2 - y_2^2}{2(y_1 - y_2)} = \frac{(x_1 - x_2)(2\mathrm{t} - x_1 - x_2)}{2u}$$

After some straightforward computations we get that

$$g_1(t;x) = \frac{2tx_1 - 2tx_2 + u - x_1^2 + x_2^2}{2u}, \quad g_2(t;x) = \frac{-2tx_1 + 2tx_2 + u + x_1^2 - x_2^2}{2u} = 1 - g_1(t;x)$$

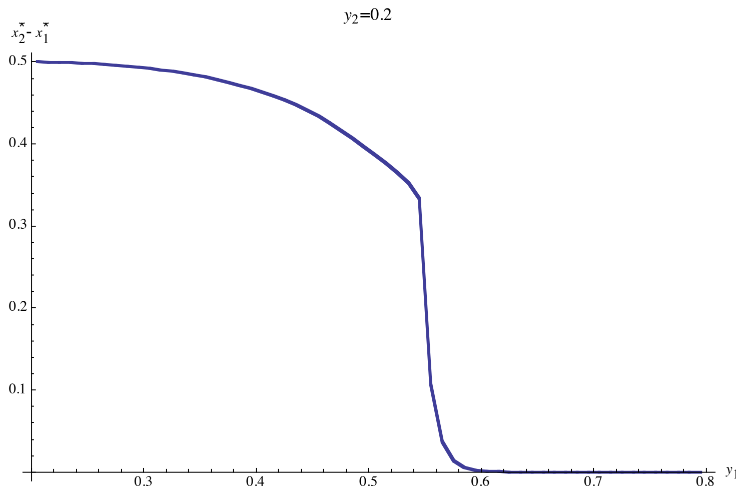
and

$$\phi_1(x) = \frac{1}{6}\left(\frac{x_1 - x_2}{2b_0u - (x_1 - x_2)(x_1 + x_2 - 1)} + 3\right), \quad \phi_2(x) = \frac{1}{6}\left(\frac{x_2 - x_1}{2b_0u + (x_1 - x_2)(x_1 + x_2 - 1)} + 3\right)$$

So, the fixed points of ϕ are determined by the equations $x_i = \phi_i(x)$, $i = 1, 2$. A straightforward computation shows that these equations are equivalent to the following system of equations

$$\begin{aligned} 0 &= 2b_0u(6b_0u(2x_1 - 1) - (x_1 - x_2)(3x_1(2x_1 + 2x_2 - 3) - 3x_2 + 4)) \\ 0 &= 2b_0u\left(6b_0u(2x_2 - 1) + (x_1 - x_2)\left(x_1(6x_2 - 3) + 6x_2^2 - 9x_2 + 4\right)\right) \end{aligned}$$

FIGURE 2. The unique divergent equilibrium. Mean Voter. 1



Adding and subtracting the above two equations we obtain

$$\begin{aligned} 0 &= 12b_0u(x_1+x_2-1)\left((x_1-x_2)^2-2b_0u\right) \\ 0 &= 4b_0u(x_1-x_2)\left(6b_0u-3\left(x_1^2+2x_1(x_2-1)+x_2^2\right)+6x_2-4\right) \end{aligned}$$

We see that $x_1 = x_2 = 1/2$ is a solution. If $x_1 \neq x_2$, then we must have that

$$u = \frac{3x_1^2+6x_1(x_2-1)+3(x_2-2)x_2+4}{6b_0}$$

It is now straightforward to check that, for $0 \le x_1, x_2 \le 1$; the function

$$\frac{1}{6}\left(3x_1^2+6x_1(x_2-1)+3(x_2-2)x_2+4\right)$$

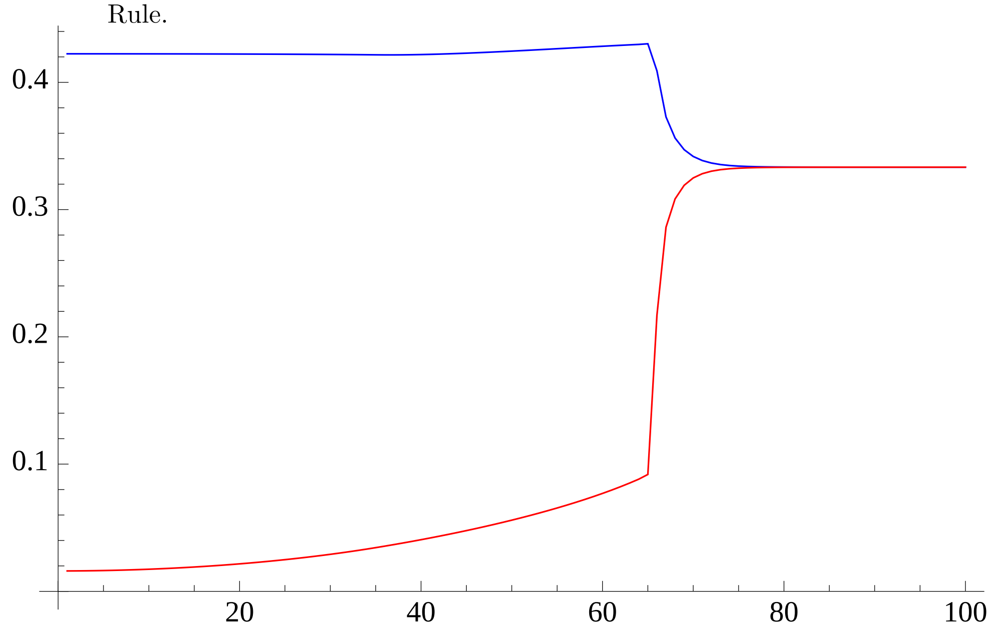
attains the maximum value 2/3 at the points $x_1 = x_2 = 0$ and $x_1 = x_2 = 1$. Thus, for $u > 2/(3b_0)$, the unique solution is $x_1 = x_2 = 1/2$.

The following simulation summarizes the above observations and shows how, as $|y_2 - y_1|$ becomes large, the two divergent equilibria converge to the unique convergent equilibrium which then becomes stable.

Example 23. Let the population distribution be uniform on $[0,1]\times[0,1]$, so that $f(\alpha,\beta) = 1$. Suppose the aggregation function $Q_j(\sigma_j(x))$ is given by the mean voter rule of Section 3.1. We show that the divergent equilibrium is such that the difference between the two policies, $|x_2^* - x_1^*|$, is decreasing in $|y_2 - y_1|$. We fix $y_2 = 0.2$ and take values of y_1 running from 0.2 to 0.8 in steps of 0.01.(We have repeated the exercise for many different values of y_2 obtaining always the same type of result). Figure 2 shows the result.

The ordinate shows the different values of y_1 and the abscissas the corresponding values of $|x_2^* - x_1^*|$. We see that there is a negative relation between the value of y_1 (which increases the value of the difference $|y_2 - y_1|$) and the difference $|x_2^* - x_1^*|$.

FIGURE 3. The unique divergent equilibrium. Weighted Median Rule.



4.2. **Weighted Mean Rules.** Next we show a simulation for the aggregation rule

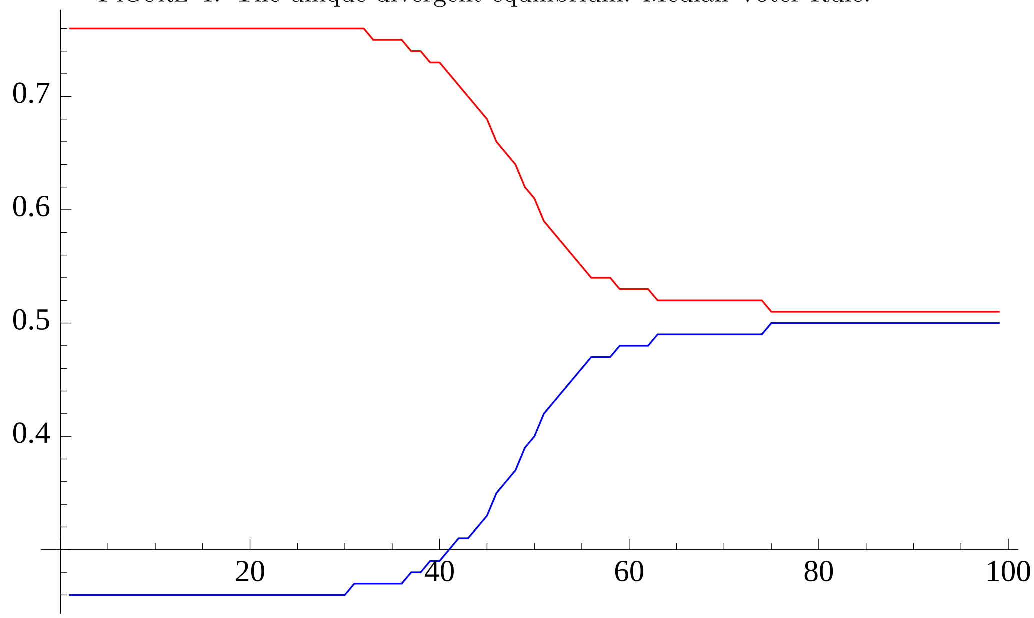
$$S_j(\eta)) = \frac{\int_A \alpha^2 \, d\eta}{\eta(A)} \quad j = 1, 2$$

The population distribution is uniform on $[0, 1] \times [0, 1]$, so that $f(\alpha, \beta) = 1$ and the variables α and β are independent. Figure 3 graphs the fixed points corresponding to the idiosyncratic issues $y_1 = 1/2 - n/400$, $y_2 = 1/2 + n/400$ as the values of n range from $n = 1$ to 100. The horizontal coordinate, say t corresponds to the values $y_1 = 1/2 - t/400$, $y_2 = 1/2 + t/400$. The blue and red lines represent the values of the equilibria, x_1 (blue) and x_2 (red).

4.3. **The median voter rule.** Consider now the median voter rule T of Section 3.3. Let the population distribution be uniform on $[0, 1] \times [0, 1]$. Notice that in this case the variables α and β are independent. Figure 4 graphs the fixed points corresponding to the idiosyncratic issues $y_1 = 1/2 - n/200$, $y_2 = 1/2 + n/200$ as the values of n range from $n = 1$ to 100. The horizontal coordinate, say t corresponds to the values $y_1 = 1/2 - t/200$, $y_2 = 1/2 + t/200$. The blue and red lines represent the values of the equilibria, x_1 (blue) and x_2 (red).

The intuition for the above results is clear. If the non-ideological characteristics y_1 and y_2 are very different, the memberships of parties are basically determined by such characteristics. In this case, and given that the distributions of α and β are independent, both parties will have a similar mean value of α. Example 18 in Section 3 provides an extreme case of this type of situation where the equilibrium policy characteristics of both parties coincide (see Figure 4).

FIGURE 4. The unique divergent equilibrium. Median Voter Rule.



5. CONCLUSIONS AND FURTHER RESEARCH

In this paper we have introduced a model in which the citizens' party choice is determined both by the ideological difference between the parties and the unobserved non-ideological attitudes. As the membership choice is only incompletely determined by the observed policy proposals, it may be interpreted as stochastic from the point of view of the parties. The party membership, in turn, determines party policy stances by means of intra-party preference aggregation rules.

We have especially focussed on the important cases in which parties aggregate preferences by choosing ideal points of their mean or median voters. Parties are perceived by citizens as 'similar' in the sense that the non-policy difference is small compared to the mean of the agents' preferences in the ideological space. In this context and with two parties we show that we are guaranteed existence of divergent equilibria, even in an *ex ante* symmetric model. In this sense, the present stochastic model shows continuity with the deterministic endogenous platform model studied earlier in Gomberg et al. (2004).

It should be noted that a similar stochastic preference model has been previously considered in Caplin and Nalebuff (1997). However, the authors of that paper believed that approach was, in a certain sense, simply imposing policy divergence exogenously, rather than having it emerge endogenously in the model. In fact, they were worried that as the stochastic preference component would become smaller, equilibrium policy divergence would itself disappear. This observation, in fact, motivated their "index theory" approach, which we have also utilized since. What we show in this paper, however, is that, under the assumptions of the model, properly divergent equilibria are guaranteed to exist even as the stochastic preference component becomes weaker.

The model provides what we believe are original insights pertaining to the level of party polarization. Under certain conditions on the distribution of preferences, if parties are very different in their non-policy characteristics, their policy proposals will be very similar.

It remains to consider how the results extend to increasing the number of parties, as well as considering different party decision-making rules (including, possible strategic interaction in a democratic context).

6. Appendix 1

The model assumes that party $j = 1, 2$ chooses its policy by aggregating the preferences of its members according to some fixed rule P_j. This aggregation applies only to the observed variable α. Here, we provide some empirical evidence suggesting that this might be a realistic assumption. In particular, we consider the possibility that the aggregating rule P_j is the average or median. Since there is no data on party activists we consider the ideology of their supporters. For a selection of countries, we analyze the political platforms of the main parties and the average ideal policy of their supporters. We find that, in countries with only two major parties (US, UK), the political platform of the party and the average ideal policy of its supporters are strikingly similar. This is not the case in other countries with more than two major parties or in a clear unstable political period.

We assume that the policy space X is one-dimensional and can be identified with the Left-Right ideological position. The information about the ideological position of supporters of each party is obtained from the World Values Survey Group et al. (1994). While the information on parties' platforms comes from the Manifesto Project (MP). See Volkens et al. (2010). The WVS provides information about the respondents self-reported ideological position on the (1-10) Left-Right scale. It also reports which party the respondent would vote for. Thus, we can compute the average ideological position of the supporter of each party. The MP measures the policy position of the parties in many issues in the electoral period for a series of democratic countries. In particular, it provides the Left-Right position of parties (see Budge et al. (2001) and Klingemann et al. (2006) for an explanation of the methodology used to obtain such positions). Since the information in the WVS and in the MP often cover different years we select for each year reported in the MP the closest earlier year in the WVS, if the difference between the two is less than three years. Table 1 reports the positions of the major parties and the average position of their supporters, for three electoral periods in Great Britain. For example, according to the WVS, in year 1990 the average ideological position of the individuals supporting the Labour Party was 4.28. Whereas, the average position for the supporters of the Conservative Party was 6.74. On the other hand, according to the MP, the position of the Labour Party in the 1992 electoral period was 4.13. And the position of the Conservative party was 6.76. Notice that the standard deviation of the ideological position of individuals is high (column 5 in the table), which makes even more surprising this congruence between individuals ideology and parties platforms. Table 2 shows for the US the same type of information for the five electoral periods for which the required data is available. The closeness between the two numbers is even clearer than in the case of Great Britain.

However, it would be wrong to expect that those values are so similar in all countries. Our model tries to capture the situation in two-party systems during 'stable' periods. Table 3 shows the results for the case of Portugal during the 90's. Even tough, the two parties reported in the table (Social democrat and Socialist) obtained together close to 80% of the vote–so that the two-party system can be a reasonable assumption here– the position of the Social democrat Party is quite different from the average position of its supporters. This suggests that either an 'equilibrium' has not been reached yet or that the aggregating rule P_j is quite different from the mean or median rule. Other countries like, for example, Belgium, Finland, France and Spain also present a big difference between the ideological position of the main parties and the average (or median) position of their supporters. The tables with this information for a series of 20 countries is available from the authors upon request.

Even tough we do not carry out a rigorous statistical empirical analysis, the data provided here suggest that, in certain countries, the proposals of parties might be very close to the average, or median, positions of their respective supporters (Dalton (1985), Holmberg (1999) and Mattila and Raunio (2006) also show a high congruence between some parties and their voters in Europe)

Table 1. Left-Right Position. Great Britain

Party	WVS	Manifesto	WVS average	SD	Manifesto	Sample SizeWVS
Conservative	1981	1983	7.69	1.87	6.81	99
	1990	1992	6.74	1.62	6,76	490
	1999	2001	6.37	1.69	6.17	163
average			6.79	1.71	6.58	752
Labour	1981	1983	4	1.99	3.74	91
	1990	1992	4.28	1.80	4.13	475
	1999	2001	4.53	1.60	5.75	282
average			4.34	1.76	4.54	848

Source: World Values Survey and the Manifesto Project (MRG/CMP/MARPOR). The original values from the Manifesto Project have been transformed to the 1-10 scale.

Table 2. Left-Right Position. US

Party	WVS	Manifesto	WVS average	SD	Manifesto	Sample SizeWVS
Republican	1982	1984	6.7	1.87	7.01	451
	1990	1992	6.23	1.73	6.87	561
	1995	1996	6.57	1.86	6.59	548
	1999	2000	6.73	1.95	7	350
	2006	2008	6.9	1.81	6.63	352
average			6.62	1.85	6.82	
Democrat	1982	1984	5.68	2.13	4.87	934
	1990	1992	5.51	1.81	6.05	740
	1995	1996	5.25	1.78	5.9	588
	1999	2000	5.42	1.96	5.34	555
	2006	2008	4.92	1.60	6	498
average			5.35	1.86	5.63	

Source: World Values Survey and the Manifesto Project (MRG/CMP/MARPOR). The original values from the Manifesto Project have been transformed to the 1-10 scale.

Table 3. Left-Right Position. Portugal

Party	WVS	Manifesto	WVS average	SD	Manifesto	Sample SizeWVS
Social democrats	1990	1991	7.24	1.83	5.11	285
	1999	1999	7.04	1.91	5.53	132
average			7.14	1.88	5.32	
Socialist	1990	1991	5.07	1.95	4.97	269
	1999	1999	5.01	2.10	4.7	299
average			5.04	2.05	4.835	

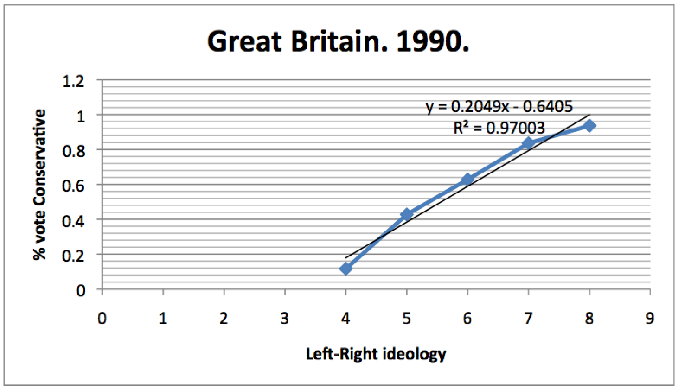
Source: World Values Survey and the Manifesto Project (MRG/CMP/MARPOR). The original values from the Manifesto Project have been transformed to the 1-10 scale.

7. Appendix 2

Here we provide some empirical examples of voting behavior and party strategies that are consistent with the assumptions in our model. We continue with the data provided in the World Values Survey Group et al. (1994) about the self-reported ideological position and which party respondents would vote for. We mainly focus on US and Great Britain. But, similar results can be obtained for other democracies with two major political parties.

Consider the case of Great Britain in year 1990. We take all the respondents that would vote either for the Conservative party or for the Labour party. Then, we compute for each left-right ideological position the percentage of those individuals who would vote for the Conservative party. Since the number of individuals reporting a given ideological positions outside the interval $[4, 8]$ is always very small (less than 50 people for each position), we only consider people in such interval (734

FIGURE 5



people representing 77% of all the individuals). Figure 1 plots such information and its best linear fit.

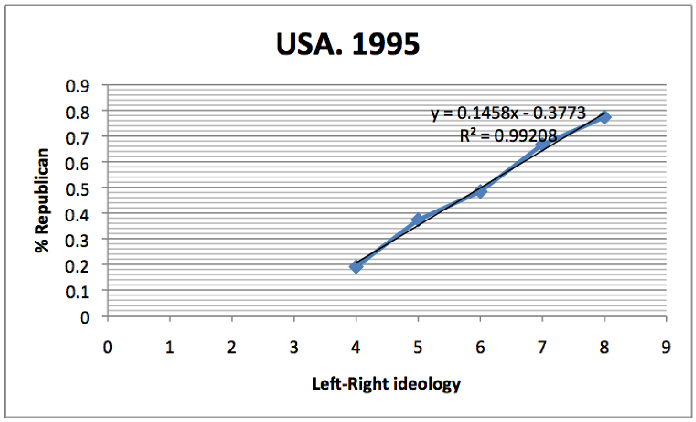
Only 11% of the individuals at the ideological position of 4 reported to be willing to vote Conservative. Such percentage increases in an apparently linear manner with the ideological position, reaching a value of 93% for individuals at position 8. That is, 93% of the individuals that reported an ideological position of eighth would vote for the Conservative party.

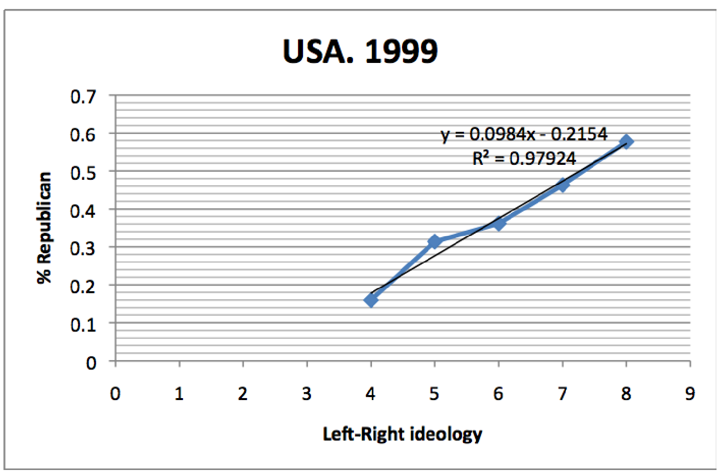
Figure 2 shows the same type of information for the case of the Republican party and the Democratic party in the US for the years 1995 and 1996[10]. We also obtain similar type of plots with such a good linear fit to the data for countries like, for example, Australia, Germany and New Zealand.

Take the Left-Right dimension as our policy space X. Then, these plots seem consistent with the existence of a non-policy variable y such that the conditional density function of ideal points, $f_2(\beta|\alpha)$, is independent of α. To see that, assume that voters have Euclidean preferences on the space $A \times B$ and all their ideal points, (α, β), are in the rectangle $[1, 10] \times [0, 1] \subseteq A \times B$. Figure 3 shows the case of Great Britain in 1990, as in Figure 1. But, **now the y-axis represents the non-policy variable**. We next assume that the policy proposal of the party, $P_j(\sigma_j(x)$, is the mean rule. Thus, these parties' proposals coincide with the average ideal point of the citizens supporting each party, and they are basically the same as the proposals reported in the Manifesto Project (see Appendix 1). In our case those policies are $x_1 = 4.13$ for the Labour party and $x_2 = 6.76$ for the Conservative party (see Table 1 in Appendix 1). The straight line $y = a + bx$ represents the best fit to the data showed in Figure 1, where $a = -0.6405$ and $b = 0.2049$. Given such 'separating

[10]These plots are very similar to the ones we have obtained using data from the National Election Study instead of the World Values Survey Group et al. (1994).

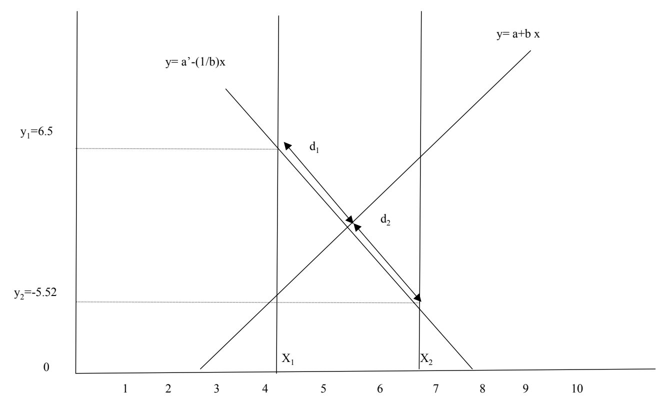
FIGURE 6





line' $y = a + bx$ we can find the non-policy positions y_1 and y_2. To do that we first need to find the line $y = a' - (1/b)x$ (perpendicular to $y = a + bx$), such that the intervals d_1 and d_2 shown in the figure have the same length. The values of the two non-policy variables are given by the intersections of such line $y = a' - (1/b)x$ and the vertical lines given by the policy positions x_1 and x_2. In our particular example we obtain $y_1 = 6.5$ and $y_2 = -5.52$. Thus, we can suppose that for the Labour party, $(x_1, y_1) = (4.13, 6.5)$ and for the Conservative party, $(x_2, y_2) = (6.76, -5, 52)$.

FIGURE 7



We can also assume that –given that preferences are Euclidean– all the agents with ideal positions on the Southeast side of $y = a + bx$ vote Conservative and all the agents with ideal positions on the other side of the line vote Labour. Thus, the two sides of the separating line $y = a + bx$ give the population partition $\sigma = (\sigma_1, \sigma_2)$.

Notice that the example described in Figure 3 represents a **multi–party equilibrium** since: i) $x_1 = 4.13$ is the average left-right position of agents in $\sigma_1(x)$, i.e. of the agents on the Northwest side of $y = a + bx$ and $x_2 = 6.76$ is the average of agents on $\sigma_2(x)$, the Southeast side of $y = a+bx$; ii) Given $(x_1, y_1) = (4.13, 6.5)$ and $(x_2, y_2) = (6.76, -5.52)$ the separating line that determines the population partition $\sigma = (\sigma_1(x), \sigma_2(x))$ is $y = a + bx$. Notice that this example violates our assumption that $Y = B$, since $y_1 = 6.5$ and $y_2 = -5.52$ do not belong to the set $B = [0, 1]$. However, this problem can be solved easily by changing the units of the policy variable x. For example if we consider the change of units $x' = 0.2x$ one can obtain a multi-party equilibrium with $(x'_1, y_1) = (0.826, 0.731)$ and $(x'_2, y_2) = (1.352, 0.218)$. In this case, agents with ideal positions on the Southeast side of $y = a + \frac{b}{0.2}x'$ vote Conservative and the vote shares of parties are the same as in the original multi-party equilibrium.

Since the ideal points on the non-policy variable y are all on the interval $[0, 1]$ and, by construction, for any given ideological position α those points coincide with the percentage of agents in favor of the Conservative party, the conditional distribution $f_2(\beta|\alpha)$ is independent of α.Thus, the data seems consistent with the existence of a non-policy variable y such that agents' preferences over it are independent of their preferences on the Left-Right ideological space.

Of course, we do not mean this example to be a conclusive proof that our model is correct. The only objective of this empirical exercise is to show that, in principle, our model could be consistent with some stylized facts. In particular, the example does not unquestionably prove the existence of a non-policy variable y. In principle, the variable y could be also chosen by parties (using some specific rule, perhaps different from the mean or median rule). However, the fact that in Figure 1 the true "separating line" $y = a + bx$ is a straight line strongly suggests the existence of a **non-policy** variable y. In the contrary case, i.e. in the event that this variable y was chosen by policy parties, it would be hard to believe that agents' preferences over the left-right variable x and their preferences over the variable y were independent, as the data suggest.

8. Appendix 3: Proofs

Proof Lemma 4. We do the proof for $i = 1$. Let $x = (x_1, x_2) \in X \times X$. Note that

$$\partial_1 g_1(\alpha; x) = f(\alpha, z(\alpha; x)) \frac{\alpha - x_1}{y_2 - y_1}$$

and

$$\partial_2 g_1(\alpha; x) = f(\alpha, z(\alpha; x)) \frac{x_2 - \alpha}{y_2 - y_1}$$

From the above expressions, the Fréchet derivative can be easily computed. We use the notation $e = (h, k)$ and $\|e\| = \sqrt{h^2 + k^2}$. Fix $x \in X \times X$. A simple computation shows that

$$\begin{aligned} z(\alpha; x + e) - z(\alpha; x) &= \frac{h(\alpha - x_1) + k(x_2 - \alpha) - (k^2 - h^2)/2}{y_2 - y_1} \\ &= \frac{(h - k)\alpha - x_1 h + x_2 k}{y_2 - y_1} + \frac{h^2 - k^2}{2(y_2 - y_1)} \end{aligned}$$

Note that there is $M > 0$ such that

$$\frac{|z(\alpha; x + e) - z(\alpha; x)|}{\|e\|} \leq M$$

for any $\alpha \in A$ and e such that $\|e\| \leq 1$. Let $\varepsilon > 0$. Note that

$$g_1(\alpha; x + e) - g_1(\alpha; (x)) = \int_{z(\alpha;x)}^{z(\alpha;x+e)} f(\alpha, \beta)\, d\beta$$

We are ready now to compute $D\sigma(e; x) : \mathbb{R}^2 \to C(A)$, the derivative of σ at the point x. We continue to use the notation $e = (h, k)$. We will show that $D\sigma(e; x)(\alpha)$ takes the following value

$$D\sigma(e; x)(\alpha) = \frac{f(\alpha, z(\alpha; x)}{y_2 - y_1} (h(\alpha - x_1) + k(x_2 - \alpha)) = \int_{z(\alpha;x)}^{z(\alpha;x+e)} f(\alpha, z(\alpha; x))\, d\beta - \frac{k^2 - h^2}{2|y_2 - y_1|}$$

Note that $D\sigma(e; x)(\alpha)$, as defined in the above equation, is linear in e and continuous in all the variables. Note that,

$$g_1(\alpha; x+e) - g_1(\alpha; (x)) - D\sigma(e; x)(\alpha) = \int_{z(\alpha;x)}^{z(\alpha;x+e)} (f(\alpha, \beta) - f(\alpha, z(\alpha; x)))\, d\beta + \frac{k^2 - h^2}{2|y_2 - y_1|}$$

Since z is continuous and $f(\alpha, \beta)$ is uniformly continuous in $A \times [z(\alpha; x), z(\alpha; x+e)]$, given $\varepsilon > 0$, there is a $0 < \delta < 1$ such that if $\|e\| \leq \delta$, then

$$|f(\alpha, \beta) - f(\alpha, z(\alpha; x))| \leq \frac{\varepsilon}{M}$$

Thus, as long as $\|e\| \leq \delta$ we have that

$$|g_1(\alpha; x+e) - g_1(\alpha;(x)) - D\sigma(e;x)(\alpha)| \leq \frac{\varepsilon}{M}\,|z(\alpha;x+e) - z(\alpha;x)| + \frac{k^2+h^2}{2|y_2-y_1|}$$

So, for any $\alpha \in A$ the following holds

$$\frac{|g_1(\alpha; x+e) - g_1(\alpha;(x)) - D\sigma(e;x)(\alpha)|}{\|e\|} \leq \varepsilon + \frac{k^2+h^2}{2\|e\|} = \varepsilon + \frac{\|e\|}{2|y_2-y_1|}$$

Hence, have that

$$\sup_{\alpha\in A} \frac{|g_1(\alpha; x+e) - g_1(\alpha;(x)) - D\sigma(e;x)(\alpha)|}{\|e\|} \leq \varepsilon + \frac{\|e\|}{2|y_2-y_1|}$$

Therefore,

$$\frac{\|g_1(\alpha; x+e) - g_1(\cdot;x) - D\sigma(e;x)\|}{\|e\|} \leq \varepsilon + \frac{\|e\|}{2|y_2-y_1|}$$

Since $\varepsilon > 0$ is arbitrary, we see that

$$\lim_{\|e\|\to 0} \frac{\|\sigma_1(x+e) - \sigma_1(x) - D\sigma(e;x)\|}{\|e\|} = 0$$

and the Lemma is proved. □

Proof Lemma 15. For each $i = 1, 2$, the maps $\nu_i \mapsto \nu_i(A)$ and $\nu_i \mapsto \int_A \alpha d\,\nu_i(\alpha)$ are linear. It is easy to see that they are also continuous. For example, given $\varepsilon > 0$, let

$$\delta = \frac{\varepsilon}{\int_A \alpha\, d\alpha}$$

If $\sup\{|f(\alpha) - g(\alpha)| : \alpha \in A\} \leq \delta$ then

$$\left|\int_A \alpha f(\alpha)\, d\alpha - \int_A \alpha g(\alpha)\, d\alpha\right| \leq \int_A \alpha|f(\alpha) - g(\alpha)|\, d\alpha \leq \delta \int_A \alpha\, d\alpha = \varepsilon$$

So $\nu_i \mapsto \int_A \alpha d\,\nu_i(\alpha)$ is continuous. The proof that $\nu_i \mapsto \nu_i(A)$ is continuous is similar. Hence, then maps $\nu_i \mapsto \nu_i(A)$ and $\nu_i \mapsto \int_A \alpha d\,\nu_i(\alpha)$ are Fréchet differentiable. Since for every $i = 1, 2$ we have that $\nu_i(A) \neq 0$, the mapping $\nu_i \mapsto \frac{\int_A \alpha d\,\nu_i(\alpha)}{\nu_i(A)}$ is also differentiable. Therefore, so Q.

□

Lemma 24.

$$\lim_{x\to(\mu,\mu)} \partial_i\phi_j(x) = \frac{1}{\lambda_j}\int_A (\alpha-\mu)(\partial_i g_j(\alpha;(\mu,\mu))d\alpha$$

o

Proof. Note that

$$\partial_i\phi_j(x) = \frac{1}{(\int_A g_j(\alpha;x)\,d\alpha)^2}\left(\int_A \alpha(\partial_i g_j(\alpha;x)\,d\alpha \int_A g_j(\alpha;x)\,d\alpha - \int_A \alpha g_j(\alpha;x)\,d\alpha \int_A (\partial_i g_j(\alpha;x)\,d\alpha\right)$$

$$= \frac{\int_A (\alpha - \phi_j(x))\,\partial_i g_j(\alpha;x)\,d\alpha}{(\int_A g_j\,(\alpha;x)d\alpha)}$$

Thus,

$$\lim_{x\to(\mu,\mu)} \partial_i\phi_j(x) \quad = \quad \frac{1}{\lambda_j}\int_A (\alpha-\mu)\partial_i g_j(\alpha;(\mu,\mu))d\alpha$$

and the Lemma follows. □

Proof of Proposition 17. Recall that, given the proposals $x = (x_1, x_2)$ of the parties, we use the notation

$$g_j(\alpha; x) = \int_{\{(\alpha,\beta):||(x_j,y_j)-(\alpha,\beta)||\geq||(x_i,y_i)-(\alpha,\beta)||, i\neq j\}} f(\alpha,\beta)d\beta$$

when we want to make explicit the dependence of the density functions that describe the induced population partitions on the policies proposed by the parties. We have seen in the proof of Lemma 4 that

$$\partial_1 g_1(\alpha; x) = f(\alpha, z(\alpha; x))\partial_1 z(t; x)|_{t=\alpha} = f(\alpha, z(\alpha; x))\frac{\alpha - x_1}{y_2 - y_1}$$

Furthermore,

$$\partial_2 g_1(\alpha; x) = f(\alpha, z(\alpha; x))\partial_2 z(t; x)|_{t=\alpha} = f(\alpha, z(\alpha; x))\frac{x_2 - \alpha}{y_2 - y_1} \tag{7}$$

which implies that

$$\partial_1 g_1(\alpha; (\mu,\mu)) = -\partial_2 g_1(\alpha; (\mu,\mu)) = f\left(\alpha, \frac{y_1+y_2}{2}\right)\frac{\alpha-\mu}{y_2-y_1}$$

Since for party 2 the relevant population density is

$$g_2(\alpha; x) = \int_{z(\alpha;x)}^{\infty} f(\alpha,\beta)\, d\beta$$

we get that

$$\begin{aligned} \partial_2 g_2(\alpha; (\mu,\mu)) &= -\partial_1 g_2(\alpha; (\mu,\mu)) = \partial_1 g_1(\alpha; x) \\ &= f\left(\alpha, \frac{y_1+y_2}{2}\right)\frac{\alpha-\mu}{y_2-y_1} \end{aligned} \tag{8}$$

To ease the notation we will write $g_i = g_i(\alpha; (\mu, \mu))$. Applying Lemma 24 and using the above formulae for $\partial_i g_j$ we have to establish conditions under which

$$\begin{aligned} \lim_{x\to(\mu,\mu)} |B(x)| &= \begin{vmatrix} \frac{1}{\lambda_1}\int_A(\alpha-\mu)\partial_1 g_1 d\alpha - 1 & \frac{1}{\lambda_1}\int_A(\alpha-\mu)\partial_2 g_1 d\alpha \\ \frac{1}{\lambda_2}\int_A(\alpha-\mu)\partial_1 g_2 d\alpha & \frac{1}{\lambda_2}\int_A(\alpha-\mu)\partial_2 g_2 d\alpha - 1 \end{vmatrix} \\ &= \begin{vmatrix} \frac{1}{\lambda_1}\int_A(\alpha-\mu)\partial_1 g_1 d\alpha - 1 & -\frac{1}{\lambda_1}\int_A(\alpha-\mu)\partial_1 g_1 d\alpha \\ -\frac{1}{\lambda_2}\int_A(\alpha-\mu)\partial_1 g_1 d\alpha & \frac{1}{\lambda_2}\int_A(\alpha-\mu)\partial_1 g_1 d\alpha - 1 \end{vmatrix} \\ &= 1 - \frac{1}{\lambda_1\lambda_2}\int_A(\alpha-\mu)\partial_1 g_1 d\alpha \\ &= 1 - \frac{1}{\lambda_1\lambda_2\,(y_2-y_1)}\int_A(\alpha-\mu)^2 f\left(\alpha, \frac{y_1+y_2}{2}\right)d\alpha < 0 \end{aligned}$$

which clearly holds if

$$|y_2 - y_1| < \frac{1}{\lambda_1\lambda_2}\int_A(\alpha-\mu)^2 f\left(\alpha, \frac{y_1+y_2}{2}\right)d\alpha$$

And the Proposition is proved. □

Proof of Lemma 19. Let $j = 1, 2$ and consider the function $F_j : X \times X \times X \to \mathbb{R}$ defined by

$$F(x_1, x_2, y) = \int_{-\infty}^{y} g_j(\alpha; x) d\alpha$$

Since $f(\alpha, \beta)$ is equivalent to Lebesgue measure, for each $x = (x_1, x_2) \in X \times X$, the equation

$$F(x_1, x_2, y) = \frac{\lambda_j}{2}$$

has a unique solution $y = \phi_j(x)$ in the interior of X. And since

$$\frac{\partial F}{\partial y} = g_j(y; x) > 0$$

me may apply the Implicit Function Theorem to conclude that there is an open neighborhood U of x in $X \times X$, an open neighborhood V of y in X and a continuously differentiable function $\phi_j : U \to V$ such that

$$F(x_1, x_2, \phi_j(x)) = \frac{\lambda_j}{2}$$

Hence, the function $\phi = (\phi_1, \phi_2) : V \times V \to X$ is continuously differentiable. □

Proof of Proposition 20. Differentiating ϕ implicitly with respect to x_i in Equation(6), we obtain

$$g_j(\phi_j(x); x)\partial_i\phi_j(x) + \int_{-\infty}^{\phi_j(x_j)} \partial_i g_j(\alpha; x) d\alpha = \frac{1}{2}\int_{-\infty}^{\infty} \partial_i g_j(\alpha; x) d\alpha$$

Taking the limit of the expression as $x \to (m, m)$ we obtain

$$f_1(m)\partial_i\phi_j(m) + \int_{-\infty}^{m} \partial_i g_j(\alpha; m) d\alpha = \frac{1}{2}\int_{-\infty}^{\infty} \partial_i g_j(\alpha; m) d\alpha \tag{9}$$

with

$$f_1(m) = \int_{-\infty}^{\frac{y_1+y_2}{2}} f(m, \beta)\, d\beta$$

Now, taking into account equations (7) and (8) the formulas for $\partial_i g_j(\alpha; x)$, we see that

$$\partial_1\phi_1(m) = \partial_2\phi_2(m) = -\partial_1\phi_2(m) = -\partial_2\phi_1(m)$$

Thus the determinant in (4) becomes,

$$B(x) = \begin{vmatrix} \partial_1\phi_1(m) - 1 & \partial_2\phi_1(m) \\ \partial_1\phi_2(m) & \partial_2\phi_2(m) - 1 \end{vmatrix} = \begin{vmatrix} \partial_1\phi_1(m) - 1 & -\partial_1\phi_1(m) \\ -\partial_1\phi_1(m) & \partial_1\phi_1(m) - 1 \end{vmatrix} = 1 - 2\partial_1\phi_1(m)$$

From equations (9), (7) and (8) we get that

$$\partial_1\phi_1(x) = \frac{1}{(y_2 - y_1)f_1(m)} \left(\frac{1}{2}\int_{-\infty}^{\infty} (\alpha - m) f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha - \int_{-\infty}^{m} (\alpha - m) f\left(\alpha, \frac{y_1 + y_2}{2}\right) d\alpha \right)$$

And the proposition follows. □

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