*Surname:* .......................................... *Name:* .......................................

**MATHEMATICAL OPTIMIZATION. JUNE EXAM 2019-20**

**1.** 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the following function
$f(x,y) = y - x^3$and the sets $A_a = \{(x,y) : 0 \leq x \leq a, -e^{-x} \leq y \leq 0\}$.
$(a)$ *(0.4 points)* Draw the curves of level $c = -1, 0, 1$ of $f$ and the sets
$A_a$, where $a = 1, \infty$.
$(b)$ *(0.6 points)* Find, if they exist, the global maximizers and minimizers of $f$ on $A_a$.
Solution:
(a) the level curves are the graphs of $y = g_c(x)$, where $g_c(x) = x^3 + c$.
$A_a$ is limited below by the curve $y = -e^{-x}$, above by the horizontal axis, and at the left and the right by the vertical lines $x = 0$ and $x = a$ (if $a = 1$).
.
b) As the function $f$ increases when $x$ decreases and $y$ increases, it is obvious that the maximum is $0$, obtained at the point $(0,0)$, for any $a$.
Analogously, when $a = 1$, the minimum is obtained at the point $(1, -e^{-1})$ and takes the value $-e^{-1} - 1$, as $h(x) = f(x, -e^{-x}) = -e^{-x} - x^3$ , the value of $f$ along the curve $y = -e^{-x}$, is concave and obtains its minimum at $x = 0$ or $x = 1$.
But $h(1) = -e^{-1} - 1 < h(0) = -1$, so $-1 - e^{-1}$ is the minimum.
On the other hand, the minimum doesn't exist when $a = \infty$, because $f(x,0) = -x^3$ tends to $-\infty$ as $x$ tends to $\infty$.

**2.** Let $B(x,y) = 9x - 5x^2 + 6y - 20y^2 + 4axy$ be the profits of a firm when it sells $x$ units of the product E and $y$ units of the product F, where $a$ is an unknown parameter.
(a)*(0.5 points)* Find $a$ in order that $B(x,y)$ be a strictly concave function.
(b)*(0.5 points)* Let's suppose now that $a = 1$. Find the critical points of $B(x,y)$, and compute the production that maximizes the profits, if it exists.
Solution:
a) $HB(x,y) = \begin{pmatrix} -10 & 4a \\ 4a & -40 \end{pmatrix}$. So $\Delta_1 < 0, \Delta_2 = 400 - 16a^2 = 16(25 - a^2) > 0 \iff |a| < 5$.
b) $\nabla B(x,y) = (9 - 10x + 4y, 6 - 40y + 4x) = (0,0)$ if $(x,y)$ is a critical point.
Multiplying the first equation by $10$, and adding up the second, we obtain.
$90 - 100x + 6 + 4x = 0 \Rightarrow x = 1$, and susbstituting in the first equation: $y = 1/4$.

**3.** Consider the function $f(x,y) = 2\ln(1 + x) + \ln(1 + 2y)$ defined on
$A = \{(x,y) : x + y = 1, x > 0, y > 0\}$
(a)*(0.3 points)* Find the point $(x_0, y_0)$ that satisfies the Lagrange conditions of local extremum on $A$ for $f$.
(b)*(0.4 points)* Prove that $(x_0, y_0)$ is the global maximizer of $f$ on
$B = \{(x,y) : x + y \leq 1, x \geq 0, y \geq 0\}$.
Hint for b): observing that $f$ is both increasing on $x$ and $y$, and strictly concave, draw the set $C = \{(x,y) : f(x,y) \geq f(x_0, y_0)\}$.
(c)*(0.3 points)* Consider $f$ defined on $A^* = \{(x,y) : x + y = 1, x \geqslant 0, y \geqslant 0\}$. Find the point, if it exists, where $f$ obtains its minimum value.
Solution:
a) $\nabla f(x,y) = (2/(1 + x), 2/(1 + 2y)) = \lambda(1,1) \Rightarrow$ the point $(x,y)$ must satisfy:

$$\begin{cases} 1+x = 1+2y \\ x+y = 1 \end{cases} \Rightarrow (x_0,y_0) = (2/3,1/3).$$

b) The set $C$ must be convex, without segments in the frontier (as $f$ is strictly concave), and above the set $B$ (as $f$ is increasing in both variables), so $(x_0,y_0)$ is the global maximizer of $f$ on $B$.

c) As $A^*$ is compact and $f$ is continuous on that set, by the Weierstrass theorem obtains a minimum, which can only be obtained on $(1,0)$ or $(0,1)$.

As $f(0,1) = \ln 3 < \ln 4 = 2\ln 2 = f(1,0)$, the minimum value is obtained in $(0,1)$.

**4.** Consider a consumer that spends a rent $m$ of 36 euros on two products: $A$, which has a price of 9 euros, and $B$, which has a price of 1 euro.

Let's suppose that the utility function of the consumer is: $U(x,y) = 30\sqrt{xy}$,

where $x,y$ are the quantities of the products $A,B$ consumed, respectively.

(a)*(0.3 points)* If we suppose that the consumer obtains its maximum utility when he consumes $x = 2, y = 18$, what is the value of the Lagrange multiplier of this problem?

(b)*(0.3 points)* How much decreases, approximately, the utility of the consumer if his rent reduces to 35?

Hint for b): if you couldn't solve part a), consider then that the value of the Lagrange multiplier is 3.

(c)*(0.4 points)* Now, the consumer can take a small extra quantity of $A$ and $B$ without any cost, and we observe that the consumer has increased his consumption of product $B$ in $0.01$ units; i.e.: $x = 2.01$ now.

How much, approximately, has been the increase of his consumption of product $B$?

Solution:

a) Considering the Lagrange function $L(x,y;\lambda) = 30\sqrt{xy} + \lambda(36-9x-y)$,

if we take the partial derivative with respecto to $x$ we obtain:

$\partial L/\partial x = 15\sqrt{y/x} - 9\lambda = 0$; and now, taking into account that $(x,y) = (2,18)$,

we obtain that $45 = 9\lambda \Rightarrow \lambda = 5$.

b) As $\Delta U \approx \lambda\Delta m = -5$

c) As $\nabla U(2,9) = 30(3/2,1/6) = (45,5) = 5(9,1)$ when the consumer increases his consumption of $B$ in $0.01$ units, he must also have increased his consumption of $B$ in $0.09$ units, as the gradient of $U$ marks the direction of maximum increase of $U$.