February 2, 2022

1. INTRODUCTION TO MATHEMATICAL OPTIMIZATION

1. Mathematical Programs

Mathematical Programming is devoted to the study and resolution of optimization problems. An optimization problem consists in searching of extrema of functions in their domain or in subsets of their domain defined though equalities and/or inequalities.
Mathematical programs are expressed by

$$\text{opt}\, f(x) \quad \text{s.a: } x \in S, \tag{1.1}$$

where $f : D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}^n$ and $S \subseteq D$, and where "opt" denotes "to optimize" and more specifically, it can be substituted by "max" for "to maximize" or "min" for "to minimize".
All throughout this section, D denotes an **open subset** of $\mathbb{R}^n$.

Definition 1.1. *In a mathematical program we distinguish the following elements.*

- f *is the* objective function *and* S *is the* feasible set*.*
- *Program* (1.1) *is* impossible *if* $S = \emptyset$*.*
- *An element* $x_0 \in D$ *is* feasible *if* $x_0 \in S$*. Otherwise, it is non feasible.*
- *A feasible element* x_0 *is a local or global solution of* (1.1) *if* x_0 *is a local or global extremum of* f *restricted to* S*, respectively.*

Let us remember now the notion of local and global extrema of a function f on a set S.

Definition 1.2. *Let* $f : S \subseteq D \longrightarrow \mathbb{R}$*, with* $S \neq \emptyset$ *and let* $x_0 \in S$*. We say that* x_0 *is*

- *a local maximum (or relative) of* f *in* S *if and only if* $\exists r > 0$ *such that* $\forall x \in B(x_0, r) \cap S$, $f(x) \leq f(x_0)$.
- *a local minimum (or relative) of* f *in* S *if and only if* $\exists r > 0$ *such that* $\forall x \in B(x_0, r) \cap S$, $f(x_0) \leq f(x)$.
- *a global maximum (or absolute) of* f *in* S *if and only if* $\forall x \in S$, $f(x) \leq f(x_0)$.
- *a global minimum (or absolute) of* f *in* S *if and only if* $\forall x \in S$, $f(x_0) \leq f(x)$.

In all the cases above, if the inequality is strict (<) for $x \neq x_0$*, then* x_0 *is a strict local/global maximum/minimum, respectively*

Remark 1.3. *Note that a global extremum is local, and that the global extrema, if exist, are unique.*

Proposition 1.4. *Let* $f : S \subseteq D \longrightarrow \mathbb{R}$*, with* $S \neq \emptyset$ *and let* $x_0 \in S$*. The point* x_0 *is a local/global maximum of* f *in* S *iff* x_0 *is a local/global minimum of* $-f$ *in* S*.*

Remark 1.5. *By the Theorem of Weierstrass, if* f *is continuous in* S *and* S *is compact (closed and bounded), then Problem* (1.1) *admits global solutions.*

2. Classification of Mathematical Programs

Problem (1.1) is classified attending to the characteristics of the objective function and the feasible set.

Definition 2.1. *We say that Problem* (1.1) *is*

- *a problem without constraints if $S = D$.*
- *a problem with equality constraints if*

$$S = \{x \in D \,:\, g_1(x) = b_1, \dots, g_m(x) = b_m\},$$

where $g_i : D \longrightarrow \mathbb{R}$, $b_i \in \mathbb{R}$ $\forall i \in \{1, \dots, m\}$.

- *a problem with inequality constraints if the feasible set S is of the form*

$$S = \{x \in D \,:\, g_1(x) \leq b_1, \dots, g_m(x) \leq b_m\},$$

where $g_i : D \longrightarrow \mathbb{R}$ and $b_i \in \mathbb{R}$, for all $i = 1, \dots, m$.

Definition 2.2. *We say that the Problem* (1.1) *is a linear program if the objective function is a linear function, $f(x) = c_1x_1 + \dots + c_nx_n$, and the feasible set*

$$S = \{x \in D \,:\, g_1(x) \leq b_1, \dots, g_m(x) \leq b_m\}$$

is given by linear functions, $g_i(x) = a_{i1}x_1 + \dots + a_{in}x_n$ (in this case, S is called polytope).

Definition 2.3. *We say that the Problem* (1.1) *is a convex program if S is a convex set and either*

- *f is concave and "opt=max', or*
- *f is convex and "opt=min".*

3. Geometric properties of mathematical programs

In this section we study some geometric properties of optimization problems that allow their graphical resolution when $n = 2$.

Definition 3.1. *The level k set of a function $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is the set*

$$C_k(f) = \{x \in D \,:\, f(x) = k\}.$$

When $n = 2$, they are called level curves.

Remark 3.2. *Level sets are disjoint, that is $C_k(f) \cap C_{k'}(f) = \emptyset$ if $k \neq k'$. Moreover, the set of all level sets of f fills D, that is*

$$\bigcup_k C_k(f) = D.$$

In which direction we face level sets with higher (lower) level k?

Theorem 3.3. *Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ with D open and let f be differentiable.*

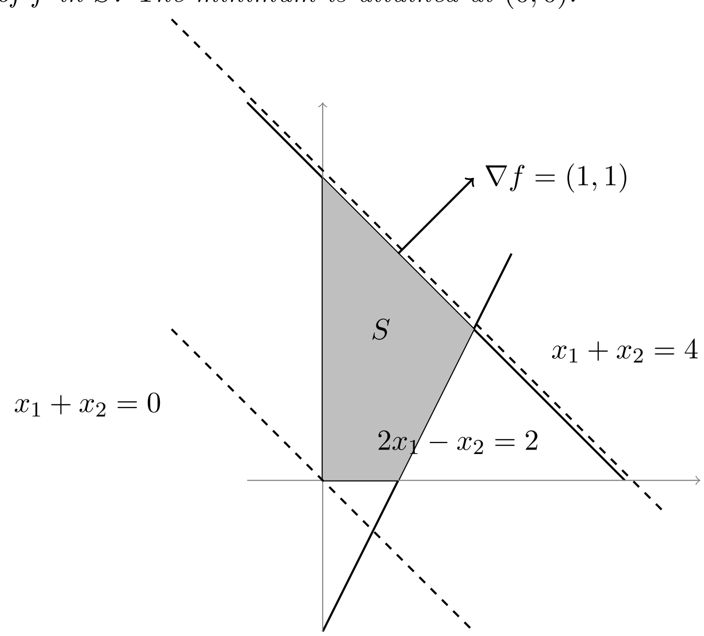
- *Given $x_0 \in D$, the directional derivative of f at x_0 takes the highest value in the direction of the gradient vector $\nabla f(x_0)$.*
- *Given $x_0 \in C_k(f)$, the gradient vector $\nabla f(x_0)$ is perpendicular to the level set $C_k(f)$ at x_0.*

Remark 3.4 (Tips for graphically solving (1.1) when $n = 2$.)**.** *Draw the set S and overlap on it some level curves of f, for different values of k. We are interested in the highest (lowest) level k for which $S \cap C_k(f) \neq \emptyset$. The points in this intersection are the global maximum (minimum) of f is S. The gradient vector gives the direction of maximum increase of f, and the opposite to the gradient vector gives the direction of maximum decrease of f.*

Example 3.5. *Solve diagrammatically, using contours, preference directions, and feasible sets, the following problems*

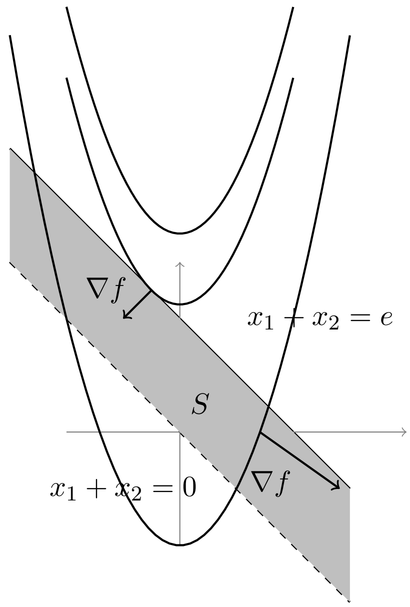
(1) opt $x_1 + x_2$ *subject to* $2x_1 - x_2 \leq 2$, $x_1 + x_2 \leq 4$, $x_1 \geq 0$, $x_2 \geq 0$.
(2) opt $x_1^2 - x_2$ *subject to* $\ln(x_1 + x_2) \leq 1$

(1) The level curves are lines $x_1 + x_2 = k$, $k \in \mathbb{R}$. The preference direction is $\nabla f(x_1, x_2) = (1, 1)$ for all (x_1, x_2), where $f(x_1, x_2) = x_1 + x_2$. The furthest contour line that contains points of the opportunity set is $x_1 + x_2 = 4$, thus the segment joining $(0, 4)$ and $(2, 2)$ is formed by global maximum of f in S. The minimum is attained at $(0, 0)$.



(2) The level curves $\{x_1^2 - x_2 = k\}$, $k \in \mathbb{R}$, are convex parabolas. The gradient of $f(x_1, x_2) = x_1^2 - x_2$ is $\nabla f(x_1, x_2) = (2x_1, -1)$, which is the direction where the function is increasing fastest. There is no point of the opportunity set where a highest contour is attained, thus the problem has not a maximum. The point of the opportunity set where the lowest contour is attained is at the point of tangency of a parabola of level k and the line $x_1 + x_2 = e$. The slope of any of the parabolas at a point (x_1, x_2) is $2x_1$. The slope of the line $x_1 + x_2 = e$ is

-1. Thus, the tangency point satisfies $2x_1 = -1$, or $x_1 = -\frac{1}{2}$. Plugging this into the line, we obtain $x_2 = e + \frac{1}{2}$. Thus, the global minimum of f in S is the point $(-\frac{1}{2}, e + \frac{1}{2})$ and the minimum value of f in S is $f(-\frac{1}{2}, e + \frac{1}{2}) = \frac{1}{4} - e - \frac{1}{2} = -e - \frac{1}{4}$.



4. Ordered sets and monotone functions in the vectorial sense

In the real line there is a natural order defined as follows: given two real numbers x, y, we say that $x \leq y$ if $y - x \geq 0$ and say that $x < y$ if $y - x > 0$. This establishes a relation between x and y, which is an order relation. Real numbers can be related in many other ways. For instance, we can define the following relation: x is related with y if $x - y$ is an integer. We write xRy to indicate that x is related with y. The pair of numbers (x, y) which are related in this sense, xRy, form a subset of the product space $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. Hence a relation on the real line $\mathbb{R}$ is nothing but a subset of the plane $\mathbb{R}^2$. Obviously, relations can be defined in arbitrary sets.

Definition 4.1. *A binary relation on X is a subset $R \subseteq X \times X$. We say that $x \in X$ is related with $y \in X$, and write xRy, if $(x, y) \in R$.*

We are specially interested in relations that provide an order to the elements of the given set. This kind of relations R are denoted $\preceq$.

Definition 4.2. *Let X be a nonempty set and let $\preceq$ be a binary relation between elements in X. We say that $(X, \preceq)$ is an ordered set if the relation $\preceq$ is an order, that is, it is reflexive, antisymmetric and transitive:*

(i) Reflexive $x \preceq x$;
(ii) Antisymmetric $[x \preceq y$ and $y \preceq x] \Rightarrow x = y$;
(iii) Transitive $[x \preceq y$ and $y \preceq z] \Rightarrow x \preceq z$

for all $x, y, z \in X$.

Definition 4.3. *Given* $(X, \preceq)$, *the order* $\preceq$ *is called a total order if every pair of distinct elements is comparable:* $\forall x, y \in X$ *[*$x \preceq y$ *or* $y \preceq x$*].*
An order which is not total is called a partial order. In this case, there are two elements at least with no relation between them (x *and* y *such that, neither* $x \preceq y$ *nor* $y \preceq x$*); in this case they are incomparable.*

Remark 4.4. $(X, \preceq)$ *can be a totally or a partially ordered set.*

Example 4.5. *We illustrate the concepts above with two examples.*

(1) The real line with the usual order $(\mathbb{R}, \leq)$ *is a total order.*
(2) $(\mathbb{R}^2, \preceq_P)$ *is a partial order, where* $\preceq_P$ *is the Pareto order*

$$(x_1, y_1) \preceq_P (x_2, y_2) \Longleftrightarrow (x_1 \leq x_2, y_1 \leq y_2).$$

Geometrically, this means that (x_1, y_1) *is on the left and below with respect to* (x_2, y_2)*. Therefore, (0,1) and (1,0) are not comparable.*

Definition 4.6. *if* $(X, \preceq)$ *is a totally ordered set, we define, for any subset* $A \subset X, \quad A \neq \emptyset$*:*

maximum of (A): $\max(A) = M$ *is the greatest element in* $A \Leftrightarrow$ $[\forall a \in A \Rightarrow a \preceq M$ *and* $M \in A]$.

minimum of (A): $\min(A) = m$ *is the least element in* $A \Leftrightarrow$ $[\forall a \in A \Rightarrow m \preceq a$ *and* $m \in A]$.

Example 4.7. *Let* $(\mathbb{R}, \leq)$.
Any finite subset A *of* $\mathbb{R}$ *has both maximum and minimum.*
$A = [0, 1]$ *has both maximum and minimum*
$A = [0, 1)$ *has minimum but not maximum.*
$A = (-\infty, 1]$ *has maximum but not minimum.*

Remark 4.8. *Similarly, it is possible to define maximum and minimum in a partially ordered set* $(X, \preceq)$*, but it is not a very useful concept.*

Example 4.9. *Let* $A = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$ *has no maximum.*

To solve the lack shown in this example, we introduce the following definitions.

Definition 4.10. *If* $(X, \preceq)$ *is a partially ordered set, for any* $A \subset X$, $A \neq \emptyset$ *, we define:*

maximal elements of (A): *A maximal element of* A *is an element that is not smaller than any other element in* A
$\text{maximal}(A) = \{a \in A : \nexists a' \in A,\ a' \neq a$ *and* $a \preceq a'\}$,

minimal elements of (A): *A minimal element of* A *is an element that is not greater than any other element in* A
$\text{minimal}(A) = \{a \in A : \nexists a' \in A,\ a' \neq a$ *and* $a' \preceq a\}$

Example 4.11. *Let* $(\mathbb{R}^2, \preceq_P)$. *The set* $A = \{(x, y) : x \geq 0, y \geq 0, x+y \leq 1\}$ *has maximal elements:* $maximal(A) = \{(x, y) \in A : x + y = 1, \quad 0 \leq x \leq 1\}$.

Remark 4.12. *Observe that, if a maximum* M *exists, then it is the unique maximal element. Similarly, if a minimum exists, is the unique minimal element.*
But the opposite is not true: the set $A = \{(x, y) : x = 0, 0 \leq y \leq 1\} \cup \{0 < x < 1, y = 0\}$ *has an unique maximal element, the point* $(0, 1)$*, but it has not a maximum.*

Remark 4.13. *Maximal and minimal elements are also known as Pareto optima. It is a basic concept in economics since the beginning of the 20th century.*

Definition 4.14. *We say that a function* $f : (X, \preceq) \longrightarrow \mathbb{R}$ *is a monotone increasing function when it satisfies that:* $z_1 \preceq z_2 \Longrightarrow f(z_1) \leq f(z_2)$. *For instance, if* $(X, \preceq)$ *is a subset of the plane with the Pareto order, it means that:*

$$z_1 = (x_1, y_1) \preceq (x_2, y_2) = z_2 \Longrightarrow f(z_1) = f(x_1, y_1) \leq f(x_2, y_2) = f(z_2)$$

Example 4.15.
1. *(1)* $f(x, y) = Ax + By$, *is monotone increasing if and only if* $\min(A, B) \geq 0$.
2. *(2)* $f(x, y) = x^\alpha \cdot y^\beta$, *defined on* $[0, \infty) \times [0, \infty)$, *it is also monotone increasing when* $\alpha, \beta > 0$.

With respect to strictly increasing functions, there are several possible definitions.
The most usual one, for the case of the Pareto order, is:

Definition 4.16. *We say that a function* $f : (A, \preceq_P) \longrightarrow \mathbb{R}$ *is strictly increasing on* $A \subset \mathbb{R}^2$, *when it satisfies that:*

$$z_1 = (x_1, y_1) \preceq_P (x_2, y_2) = z_2, \; z_1 \neq z_2 \Longrightarrow f(z_1) = f(x_1, y_1) < f(x_2, y_2) = f(z_2)$$

Example 4.17.
1. *(1)* $f(x, y) = Ax + By$, *is strictly increasing if* $\min(A, B) > 0$.
2. *(2) on the other hand,* $f(x, y) = x^\alpha \cdot y^\beta$, *defined on* $(0, \infty) \times (0, \infty)$, *it is also a strictly increasing function when* $\alpha, \beta > 0$.

Theorem 4.18. *Let* f *be defined on* $A \subset \mathbb{R}^2$, *with the Pareto order.*
- *a) If* f *is a monotone increasing function and* A *is a set bounded above by its maximal elements, the maximum of* f *will be reached at some of the maximal points of* A *and, perhaps, in some other points of* A *which are not maximals.*
- *b) If* f *is strictly increasing and reaches a global maximum, this maximum will be only obtained at a point in the set of maximal elements of* A.

Remark 4.19. *A set* A *is bounded above by its maximal elements when, for any* $a \in A$, *there exists a maximal element* M, *depending on* a, *such that* $a \preceq M$.

Example 4.20.
1. *(1) if* f *is a constant function, the maximum will be obtained on all the points of* A. *Thus, over all the maximal elements of* A *supposing that any exists.*

(2) *If f is a continuous function, strictly increasing and defined on* $A = \{(x, y) : x \geq 0,\ y \geq 0,\ x + y \leq 1\}$, *then f will get its global maximum on the set* $B = \{(x, y) : x \geq 0,\ y \geq 0, x + y = 1\}$, *i.e., on the set of maximal elements of A.*

On the same way, it can be defined a monotone decreasing and a strictly decreasing function over the Pareto order.

Definition 4.21. *We say that a function $f : (X, \preceq) \longrightarrow \mathbb{R}$ is a monotone decreasing function when it satisfies that: $z_1 \preceq z_2 \Longrightarrow f(z_1) \geq f(z_2)$. For instance, if $(X, \preceq)$ is a subset of the plane with the Pareto order, it means that:*

$$z_1 = (x_1, y_1) \preceq (x_2, y_2) = z_2 \Longrightarrow f(z_1) = f(x_1, y_1) \geq f(x_2, y_2) = f(z_2)$$

Example 4.22. (1) *$f(x, y) = Ax + By$, is monotone decreasing if and only if* $\max(A, B) \leq 0$.
(2) *$f(x, y) = -x^\alpha \cdot y^\beta$, defined on $[0, \infty) \times [0, \infty)$ it is also monotone decreasing when $\alpha, \beta > 0$.*

With respect to strictly decreasing functions, there are several possible definitions.
The most usual one, for the case of the Pareto order, is:

Definition 4.23. *We say that a function $f : (A, \preceq_P) \longrightarrow \mathbb{R}$ is a strictly decreasing function, with $A \subset \mathbb{R}^2$, when it satisfies that:*

$$z_1 = (x_1, y_1) \preceq_P (x_2, y_2) = z_2, z_1 \neq z_2 \Longrightarrow f(x_1, y_1) > f(x_2, y_2)$$

Example 4.24. (1) *$f(x, y) = Ax + By$, is strictly decreasing if $max(A, B) < 0$.*
(2) *On the other hand, $f(x, y) = -x^\alpha \cdot y^\beta$, defined on $(0, \infty) \times (0, \infty)$, it is also a strictly decreasing function when $\alpha, \beta > 0$.*

Theorem 4.25. *Let f defined on $A \subset \mathbb{R}^2$, with the Pareto order.*

a) *If f is a monotone decreasing function and A is a set bounded above by its maximal elements, the minimum of f will be reached at some of the maximal points of A and, perhaps, in some other points of A which are not maximals.*
b) *If f is strictly decreasing and reaches a global minimum, this minimum will be only obtained at a point in the set of maximal elements of A.*

Example 4.26.

(1) *if f is a constant function, the minimum will be obtained on all the points of A. Thus, over all maximal elements of A supposing that any exists.*
(2) *If f is a continuous function, strictly decreasing, defined on*

$$A = \{(x, y) : x \geq 0,\ y \geq 0,\ x + y \leq 1\},$$

then f will get its global minimum on the set $B = \{(x, y) : x \geq 0,\ y \geq 0, x + y = 1\}$, i.e., on the set of maximal elements of A.