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SHEET 4. OPTIMIZATION WITH INEQUALITY CONSTRAINTS

(1) *Find the maximum of the function* $f(x,y,z) = xyz$ *on the set* $\{(x,y,z) \in \mathbb{R}^3 : x+y+z \leq 1, \quad x,y,z \geq 0\}$.

Solution: First we write the problem in the canonical form,

$$\begin{array}{ll} \max\limits_{x,y,z} & xyz \\ \text{s.t.} & x+y+z \leq 1 \\ & -x \leq 0 \\ & -y \leq 0 \\ & -z \leq 0 \end{array}$$

The Lagrangian is

$$L = xyz + \lambda_1(1-x-y-z) + \lambda_2 x + \lambda_3 y + \lambda_4 z$$

and the Kuhn–Tucker equations are

$$\begin{aligned}
(1) && \frac{\partial L}{\partial x} = yz - \lambda_1 + \lambda_2 &= 0 \Leftrightarrow \lambda_1 = yz + \lambda_2 \\
(2) && \frac{\partial L}{\partial y} = xz - \lambda_1 + \lambda_3 &= 0 \Leftrightarrow \lambda_1 = xz + \lambda_3 \\
(3) && \frac{\partial L}{\partial z} = xy - \lambda_1 + \lambda_4 &= 0 \Leftrightarrow \lambda_1 = xy + \lambda_4 \\
(4) && \lambda_1(1-x-y-z) &= 0 \\
(5) && \lambda_2 x &= 0 \\
(6) && \lambda_3 y &= 0 \\
(7) && \lambda_4 z &= 0 \\
(8) && \lambda_1, \lambda_2, \lambda_3 &\geq 0 \\
(9) && x+y+z &\leq 1 \\
(10) && x,y,z &\geq 0
\end{aligned}$$

Case 1 $\lambda_1 = 0$. In this case, from equations (1), (2) y (3) we obtain that

$$yz + \lambda_2 = xz + \lambda_3 = xy + \lambda_4 = 0$$

But all the variables are positive, so

$$yz = xz = xy = \lambda_2 = \lambda_3 = \lambda_4 = 0 \qquad (11)$$

The equations (11) yield an infinite number of solutions in which at least two of the variables x, y, z vanish. The value of the objective function at these solutions is 0.

Case 2 $\lambda_1 > 0$. Then, $x + y + z = 1$. If, for example $x = 0$, we obtain from equations (2) y (3) that

$$\lambda_3 = \lambda_4 = \lambda_1 > 0$$

and from the equations (6) y (7) we see that $y = z = 0$. But this contradicts that $x+y+z = 1$. We conclude that $x > 0$. Analogously, one can show that $y, z > 0$. By equations (5), (6) y (7) we see that

$$\lambda_2 = \lambda_3 = \lambda_4 = 0$$

and equations (1), (2) y (3) imply that

$$yz = xz = xy$$

that is

$$x = y = z$$

And since $x + y + z = 1$, we conclude that

$$x = y = z = \frac{1}{3}; \quad \lambda_1 = \frac{1}{9}$$

The value of the objective function is

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}$$

So the point

$$x = y = z = \frac{1}{3}$$

corresponds to the maximum.

(2) *Find the minimum of the function $f(x, y) = 2y - x^2$ on the set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, \quad x, y \geq 0\}$.*

Solution: First we write the problem in the canonical form,

$$\begin{array}{rl} \max\limits_{x,y,z} & x^2 - 2y \\ \text{s.t.} & x^2 + y^2 \leq 1 \\ & -x \leq 0 \\ & -y \leq 0 \end{array}$$

The Lagrangian is

$$L = x^2 - 2y + \lambda_1(1 - x^2 - y^2) + \lambda_2 x + \lambda_3 y$$

and the Kuhn–Tucker equations are

$$
\begin{aligned}
\frac{\partial L}{\partial x} = 2x - 2\lambda_1 x + \lambda_2 &= 0 \Leftrightarrow 2\lambda_1 x = 2x + \lambda_2 && (12)\\
\frac{\partial L}{\partial y} = -2 - 2\lambda_1 y + \lambda_3 &= 0 \Leftrightarrow \lambda_3 = 2 + 2\lambda_1 y && (13)\\
\lambda_1(1 - x^2 - y^2) &= 0 && (14)\\
\lambda_2 x &= 0 && (15)\\
\lambda_3 y &= 0 && (16)\\
\lambda_1, \lambda_2, \lambda_3 &\geq 0 && (17)\\
x^2 + y^2 &\leq 1 && (18)\\
x, y &\geq 0 && (19)
\end{aligned}
$$

Since all the variables that appear in equation (13) are positive, we have that

$$\lambda_3 \geq 2 > 0$$

so

$$y = 0$$

and from equation (13) we see that

$$\lambda_3 = 2$$

Case 1: $x = 0$**.** By equation (14) we see that $\lambda_1 = 0$ y by equation (12) we see that $\lambda_2 = 0$. That is, of solution of the Kuhn–Tucker equations is

$$x^* y^* = 0; \quad \lambda_1 = \lambda_2 = 0, \quad \lambda_3 = 2$$

The value of the objective function at these solutions is $f(0,0) = 0$.

Case 2: $x > 0$**.** Then, $\lambda_2 = 0$ and by equation (12), $\lambda_1 = 1$. Now we obtain from equation (14) that $x^2 + y^2 = 1$. And since $y = 0$, $x \geq 0$ we have that

$$x = 1$$

Another solution of the Kuhn–Tucker equations is

$$x^* = 1, \quad y^* = 0; \quad \lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = 2$$

The value of the objective function at these solutions is $f(1,0) = -1$. Therefore, the minimum is attained at the point $(1,0)$.

(3) *Solve the optimization problem*

$$
\begin{cases}
\min & x^2 + y^2 - 20x \\
s.t. & 25x^2 + 4y^2 \leq 100
\end{cases}
$$

Solution: We write the problem as

$$\begin{cases} \max & -x^2 - y^2 + 20x \\ \text{s.t.} & 25x^2 + 4y^2 \leq 100 \end{cases}$$

The associated Lagrangian is

$$L = -x^2 - y^2 + 20x + \lambda(100 - 25x^2 - 4y^2)$$

and the Kuhn-Tucker equations are

$$-2x + 20 - 50\lambda x = 0 \tag{20}$$

$$-2y - 8\lambda y = 0 \tag{21}$$

$$25x^2 + 4y^2 \leq 100 \tag{22}$$

$$\lambda(100 - 25x^2 - 4y^2) = 0 \tag{23}$$

$$\lambda \geq 0 \tag{24}$$

From (20) we see that $y(1 + 4\lambda) = 0$, so if $y \neq 0$, then $\lambda = -1/4$ which contradicts (24). We conclude that $y = 0$ and the system reduces to

$$-2x + 20 - 50\lambda x = 0 \tag{25}$$

$$x^2 \leq 4 \tag{26}$$

$$\lambda(4 - x^2) = 0 \tag{27}$$

$$\lambda \geq 0 \tag{28}$$

If $\lambda = 0$ from (25) we obtain that $x = 10$. But this does not satisfy (26). Therefore, $\lambda \neq 0$ and from (27) we see that $x^2 = 4$. There are two possibilities $x = \pm 2$. Solving in (25) we obtain

$$\lambda = \frac{10 - x}{25x}$$

so that if $x = -2$ then $\lambda = -12/50$ does not satisfy (28). Therefore, the solution is

$$x = 2, \quad y = 0, \qquad \lambda = \frac{8}{50}$$

(4) *Solve the optimization problem*

$$\begin{cases} \max & x + y - 2z \\ s.t. & z \geq x^2 + y^2 \\ & x, y, z \geq 0 \end{cases}$$

Solution: First we write the problem in the canonical form,

$$\begin{array}{ll} \max\limits_{x,y,z} & x+y-2z \\ \text{s.t.} & x^2+y^2-z \leq 0 \\ & -x \leq 0 \\ & -y \leq 0 \\ & -z \leq 0 \end{array}$$

The Lagrangian is

$$L = x + y - 2z + \lambda_1(z - x^2 - y^2) + \lambda_2 x + \lambda_3 y + \lambda_4 z$$

and the Kuhn–Tucker equations are

$$\frac{\partial L}{\partial x} = 1 - 2\lambda_1 x + \lambda_2 = 0 \Leftrightarrow 2\lambda_1 x = 1 + \lambda_2 \tag{29}$$

$$\frac{\partial L}{\partial y} = 1 - 2\lambda_1 y + \lambda_3 = 0 \Leftrightarrow 2\lambda_1 y = 1 + \lambda_3 \tag{30}$$

$$\frac{\partial L}{\partial z} = -2 + \lambda_1 + \lambda_4 = 0 \Leftrightarrow \lambda_1 + \lambda_4 = 2 \tag{31}$$

$$\lambda_1(x^2 + y^2 - z) = 0 \tag{32}$$

$$\lambda_2 x = 0 \tag{33}$$

$$\lambda_3 y = 0 \tag{34}$$

$$\lambda_4 z = 0 \tag{35}$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \tag{36}$$

$$x, y, z \geq 0 \tag{37}$$

From equation (29), we see that if $\lambda_1 = 0$ o $x = 0$, then $1 + \lambda_2 = 0$ so $\lambda_2 < 0$. Therefore, we must have that $\lambda_1 > 0$ y $x > 0$. Similarly, from equation (30) we obtain that $y > 0$. From equations (33) and (34) we see now that $\lambda_2 = \lambda_3 = 0$. The Kuhn–Tucker equations may now be written as follows

$$2\lambda_1 x = 1 \tag{38}$$

$$2\lambda_1 y = 1 \tag{39}$$

$$\lambda_1 + \lambda_4 = 2 \tag{40}$$

$$x^2 + y^2 = z \tag{41}$$

$$\lambda_4 z = 0 \tag{42}$$

$$\lambda_1, \lambda_4 \geq 0 \tag{43}$$

$$x, y, z \geq 0 \tag{44}$$

Since $\lambda_1 > 0$, from equations (38) and (39) we obtain that

$$x = y = \frac{1}{2\lambda_1}$$

so

$$z = 2x^2 > 0$$

And we deduce from equation (42) that $\lambda_4 = 0$ and from equation (40) that $\lambda_1 = 2$. Therefore, the solution is

$$x = y = \frac{1}{4}, \quad z = \frac{1}{8}; \qquad \lambda_1 = 2, \quad \lambda_2 = \lambda_3 = \lambda_4 = 0$$

(5) *Solve the optimization problem*

$$\begin{cases} \max & x^2 - 2xy + 4y^2 \\ s.t. & x + y \leq 4 \\ & y \geq 2x \\ & x, y \geq 0 \end{cases}$$

Solution: The function $f(x, y) = x^2 - 2xy + 4y^2$ is convex and the opportunity set is convex and compact, thus f attains the maximum at some of the vertex of S, which are $(0, 0)$, $(0, 4)$ and $(2, 4)$. Evaluating we have

$$f(0, 0) = 0, \quad f(0, 4) = 64, \quad f(2, 4) = 52.$$

Hence, $(0, 4)$ is the global maximum of f in S.