March 22, 2024

SHEET 3. OPTIMIZATION WITH EQUALITY CONSTRAINTS

(1) *Find and classify the extreme points of the following functions under the given restrictions.*
 (a) $f(x,y,z) = x + y + z$ *in* $x^2 + y^2 + z^2 = 2$.
 (b) $f(x,y) = \cos(x^2 - y^2)$ *in* $x^2 + y^2 = 1$.

Answer: (a) $(\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}})$ global maximum and $(-\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}})$ global minimum; (b) the four points $(\pm 1, 0)$, $(0, \pm 1)$ are global minima and the four points $(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$ are global maxima.

(2) *Minimize* $x^4 + y^4 + z^4$ *on the plane* $x + y + z = 1$.

Answer: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a local minimum.

(3) *A company makes two products,* P_1 *and* P_2. *If the company sells* x_1 *units of* P_1 *and* x_2 *units of* P_2 *it receives a net profit of* $R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$. *Find* x_1 *and* x_2 *that maximize net profit.*

Answer: $(3, 6)$ is the global maximum.

(4) *The prices for two goods produced by a monopolist are*

$$p_1 = 256 - 3q_1 - q_2$$
$$p_2 = 222 + q_1 - 5q_2$$

where p_1, p_2 *are the prices and* q_1, q_2 *are the quantities produced. The cost function is* $C(q_1, q_2) = q_1^2 + q_1q_2 + q_2^2$. *Find the quantities that maximize profit.*

Answer: $(q_1, q_2) = (28, 16)$ is the global maximum.

(5) *The production function for a firm is* $4x + xy + 2y$, *where* x *is labor and* y *is capital. The total budget that the company can spend is 2000\$. Each unit of labor costs 20\$, whereas each unit of capital costs 4\$. Find the optimal level of production for the firm.*

Answer: $(49.4, 253)$ is the global maximum.

(6) *An editor has been assigned a budget of* 60.000 *to be spent on advertising and production of a new book. She estimates that spending* x *thousand euro in production and* y *thousand euro in advertising she can sell* $f(x,y) = 20x^{3/2}y$ *books. If she wants to maximize sales,*

how much should she allocate to advertising and how much should she allocate to production?

Answer: $(36000, 24000)$ is the global maximum.

(7) *A store sells two products which are close substitutes. The manager has found that if he sells the products at the prices P_1 and P_2, the net profit is $R = 500P_1 + 800P_2 + 1,5P_1P_2 - 1,5P_1^2 - P_2^2$. Find the optimal prices for the manager.*

Answer: $(\frac{2800}{3}, 600)$ is the global maximum.

(8) *The utility function of a consumer is $u(x, y) = \frac{1}{3}\ln x + \frac{2}{3}\ln y$, where x and y are the consumption goods, with prices, respectively, p_1 and p_2. The rent of the agent is M. Find the demand of the agent for each good.*

Answer: $x = \frac{M}{3p_1}$, $y = \frac{2M}{3p_2}$.

(9) *Find and classify the extreme points of the function f on the given set.*
   (a) $f(x, y, z) = x^2 + y^2 + z^2$ *on the set* $\{(x, y, z) \in \mathbb{R}^3 : x + 2y + z = 1, 2x - 3y - z = 4\}$.
   (b) $f(x, y, z) = (y + z - 3)^2$ *on the set* $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y + z = 2, x + y^2 + 2z = 2\}$.
   (c) $f(x, y, z) = x + y + z$ *on the set* $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, x - y - z = 1\}$.
   (d) $f(x, y, z) = x^2 + y^2 + z^2$ *on the set* $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z, x + y + z = 4\}$.

Answer: (a) $(\frac{92}{59}, -\frac{19}{59}, \frac{5}{59})$ is the (unique) global minimum, since the optimization program is convex and Lagrange conditions are both necessary and sufficient; (b) $(-\frac{1}{2}, 1, \frac{3}{4})$ and $(1, 1, 0)$ local minimum. It can be proved that the former is global; (c) $(1, 0, 0)$ global maximum and $(-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3})$ global minimum.

(10) *[Final Exam May 2022].* Consider the problem of Lagrange:

$$\text{Optimize } f(x, y) = xy - 3x - 6y$$
$$\text{subject to: } g(x, y) = 2x + 4y = 40.$$

   (a) Find all critical points of the problem.
   (b) Find all local extrema of $f(x, y)$ subject to the constraint. Justify whether the local extrema are global extrema.
   (c) Suppose that $f(x, y)$ is the profit function of a firm and that $2x + 4y = 40$ is the budget constraint, both in thousands of euros.
   Approximately, what would be the added benefit of increasing the company's funds by 1,000 euros?

Answer: (a) $(x^*, y^*) = (10, 5)$ with multiplier $\lambda = 1$; (b) tangent space is $\{(2v, -v) : v \in \mathbb{R}\}$ and the restricted Hessian at the critical point becomes $-4v^2$, thus it is a local maximum. In fact, it is global as can be seen by substituting the constraint into the objective function; (3) The multiplier is the derivative of the value function, $\lambda = V'(b)|_{b=40}$. Since $\lambda = 1$, then $\triangle V \approx \lambda \triangle b$ and hence $\triangle V \approx \triangle b = 1000$.

(11) *[Final Exam June 2022].* A state-owned company managed by the Government of country G produces two goods A and B, of which it sells x and y units per day, respectively. The cost function is given by

$$C(x, y) = xy + 2x^2 + y^2.$$

The unitary price of good A is $p_A(x, y) = 3 - 2x - \alpha y$, where $\alpha > 0$ is a positive parameter. The unitary price of good B is constant and equal to 1, that is, $p_B(x, y) = 1$.

(a) Find the range of values of α for which the profit function of the firm is a concave function.

(b) Let $\alpha = 1$. Find the values of x and y which maximize the firm's profits.

(c) Let $\alpha = 1$. The Government requires the firm managers to produce goods A and B in quantities such that the unitary price of good A is 2, that is, $p_A(x, y) = 3 - 2x - y = 2$. Find the values of x and y which maximize the firm's profits under this constraint.

(d) Let $\alpha = 1$. Suppose now that the Government requires the firm managers to produce goods A and B in quantities such that the unitary price of good A is $2 + \frac{1}{6}$, that is, $p_A(x, y) = 3 - 2x - y = 2 + \frac{1}{6}$. Without solving the new Lagrange problem, give an estimate of the increment (positive or negative) of the optimal profits of the firm with respect to the case solved in part (c) above when the unitary price of A was fixed to 2. Base your answer in the value of the multiplier found in part (c).

Answer: (a) The profits function Π is concave iff $0 < \alpha \leq 3$; (b) $(x^*, y^*) = (1/3, 1/6)$ is the global maximum; (c) $(x^{**}, y^{**}) = (3/8, 1/4)$ is the global maximum, with multiplier $\lambda = -\frac{1}{4}$; (d) the new constraint is $2x + y = 1 - \frac{1}{6}$, thus $\triangle b = -\frac{1}{6}$. Since $-\frac{1}{4} = \lambda = V'(1)$, and $V'(b) \approx \triangle V / \triangle b$, we get $\triangle V \approx (-\frac{1}{4})(-\frac{1}{6}) = \frac{1}{24} > 0$.