February 21, 2020

# MATHEMATICAL OPTIMIZATION FOR ECONOMICS (2019-20)

*ECONOMICS, LAW-ECONOMICS, INTERNATIONAL STUDIES-ECONOMICS*

## SHEET 2. UNCONSTRAINED OPTIMIZATION

(1) *Find and classify the critical points of the following functions.*
- (a) $f(x, y) = x^2 - y^2 + xy$.
- (b) $f(x, y) = x^2 + y^2 + 2xy$.
- (c) $f(x, y) = e^{x \cos y}$.
- (d) $f(x, y) = e^{1+x^2-y^2}$.
- (e) $f(x, y) = x \sin y$.
- (f) $f(x, y) = xe^{-x}(y^2 - 4y)$

**Answers:** (a) $(0, 0)$ saddle point; (b) $(x, -x)$, $x \in \mathbb{R}$ are global min; (c) $(0, \frac{\pi}{2} + k\pi)$, $k = 0, \pm 1, \pm 2, \ldots$; (d) $(0, 0)$ saddle point; (e) $(0, k\pi)$, $k = 0, \pm 1, \pm 2, \ldots$ saddle points; (f) $(0, 0)$ and $(0, 4)$ are saddle points and $(1, 2)$ is a local min.

(2) *Find the critical points of the following functions. For which points the second derivative criterion does not give any information?*
- (a) $f(x, y) = x^3 + y^3$.
- (b) $f(x, y) = \left((x - 1)^2 + (y + 2)^2\right)^{1/2}$.
- (c) $f(x, y) = x^3 + y^3 - 3x^2 + 6y^2 + 3x + 12y + 7$.
- (d) $f(x, y) = x^{2/3} + y^{2/3}$

**Answers:** (a) $(0, 0)$ saddle point, no information; (b) $(1, -2)$ global min, no information; (c) $(1, -2)$ saddle point, no information; (d) $(0, 0)$ global min, no information.

(3) *Let* $f(x, y) = (3 - x)(3 - y)(x + y - 3)$.
- (a) *Find and classify the critical points.*
- (b) *Does* $f$ *have absolute extrema? (hint: consider the line* $y = x$*)*

**Answers:** (a) $(3, 0)$, $(0, 3)$ and $(3, 3)$ are saddles; $(2, 2)$ is a local max; (b) $f$ does not attain global extrema.

(4) *Find the values of the constants* $a$, $b$ *and* $c$ *so that the function* $f(x, y) = ax^2y + bxy + 2xy^2 + c$ *has a local minimum at the point* $(2/3, 1/3)$ *and the minimum value at that point is* $-1/9$.

**Answers:** $a = 1$, $b = -2$ and $c = \frac{1}{27}$.

(5) *The income function is* $R(x,y) = x(100-6x) + y(192-4y)$ *where* $x$ *and* $y$ *are the number of articles sold. If the cost function is* $C(x,y) = 2x^2+2y^2+4xy-8x+20$ *determine the maximum profit.*

**Answers:** $x = 3$, $y = 15$ is the unique global max. The maximum profit is $B(3,15)$=3(100-18)+15(192-60)-(36+450+180-24+20)=246+1980-662=1810.

(6) *A milk store produces* $x$ *units of whole milk and* $y$ *units of skim milk. The price for whole milk is* $p(x) = 100-x$ *and the price for skim milk is* $q(y) = 100-y$*. The cost of production is* $C(x,y) = x^2+xy+y^2$*. How should the company choose* $x$ *and* $y$ *to maximize profits?*

**Answers:** $(20,20)$ is the global maximum.

(7) *A monopolist produces a good which is bought by two types of consumers. The consumers of type 1 are willing to pay* $50-5q_1$ *euros in order to purchase* $q_1$ *units of the good. The consumers of type 2 are willing to pay* $100-10q_2$ *euros in order to purchase* $q_2$ *units of the good. The cost function of the monopolist is* $c(q) = 90+20q$ *euros. How much should the monopolist produce in each market?*

**Answers:** $q_1 = 3$, $q_2 = 4$.

(8) *In the method of least squares of regression theory, the line* $y = a+bx$ *is fit to the data*

$$(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)$$

*by minimizing the quadratic errors*

$$E(a,b) = \sum_{i=1}^{n}(y_i-(a+bx_i))^2$$

*by choice of the two parameters* $a$ *(the intercept) and* $b$ *(the slope).*

(a) *Determine the necessary conditions for minimizing* $E(a,b)$ *by choice of* $a$ *and* $b$*. Show that the sufficient conditions are met.*

(b) *Find* $a$ *and* $b$*.*

(c) *Apply the results to the data*

$$(0,0),(1,1),(4,2),(16,4)$$

*and draw a graph.*

**Answers:** (a) (1) $\frac{\partial E}{\partial a} = -2\sum_{i=1}^{n}(y_i-(a+bx_i)) = 0$, (2) $\frac{\partial E}{\partial b} = -2\sum_{i=1}^{n}x_i(y_i-(a+bx_i)) = 0$. Consider the $i$th summand in the definition of the function $E$, $(y_i-(a+bx_i))^2$. It is a convex function of variables $(a,b)$, since its Hessian matrix

$$\begin{pmatrix} 2 & 2x_i \\ 2x_i & 2x_i^2 \end{pmatrix}$$

positive semidefinite. Thus, $E$ is the sum of $n$ convex functions, so it is convex. Critical points of convex functions are global minimum; (b) Denoting $\overline{x}$, $\overline{y}$ the mean values of $x$ and $y$, respectively, the equations (1) and (2) above are $a + b\overline{x} = \overline{y}$ and $a\overline{x} + b\sum_{i=1}^{n} \frac{x_i^2}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i$. Solving

$$b = \frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y}}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2}$$

and $a = -b\overline{x} + \overline{y}$.