March 5, 2025

MATHEMATICAL OPTIMIZATION FOR ECONOMICS

ECONOMICS, LAW-ECONOMICS, INTERNATIONAL STUDIES-ECONOMICS

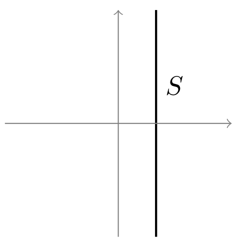
SHEET 1. INTRODUCTION TO MATHEMATICAL OPTIMIZATION

(1) *Show in a diagram the feasible set for an optimization problem of two variables, x_1 and x_2, where the constraint(s) is (are)*

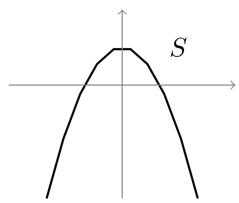
- *(a)* $x_1 = 10$
- *(b)* $x_1^2 + 3x_2 = 1$
- *(c)* $\ln(x_1 + x_2) \leq 1$
- *(d)* $e^{x_2} - x_1 \geq 0$, $x_1 \geq 1$, $x_2 < 0$
- *(e)* $x_2 \geq 4$, $(x_1 - 6)^2 + (x_2 - 4)^2 \leq 25$

Solution:

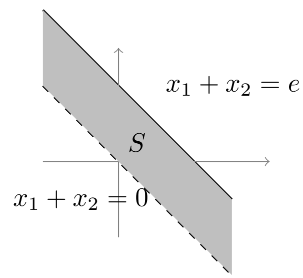
(a)



(b)



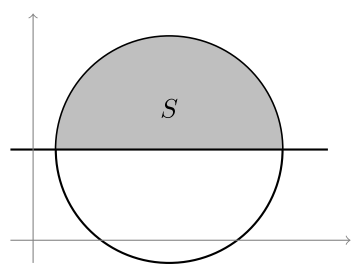
(c)



(d)

$S = \emptyset$

(e)

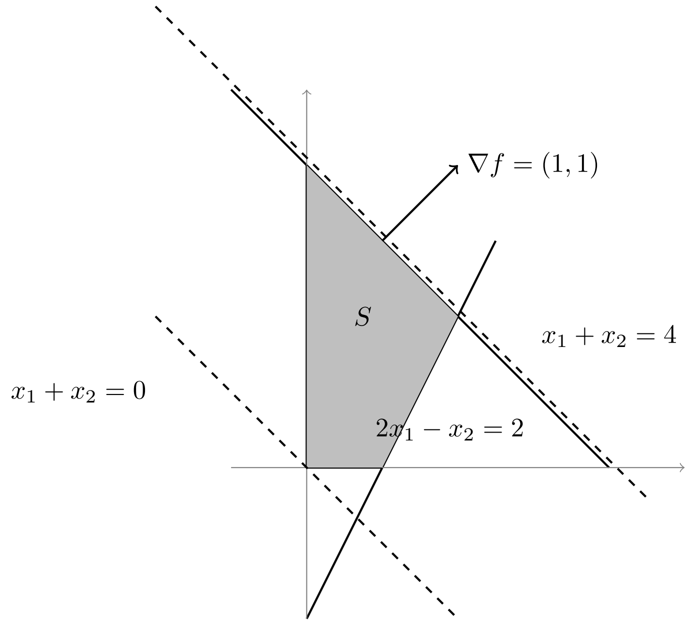


(2) *Solve diagrammatically, using contours, preference directions, and feasible sets, the following problems*

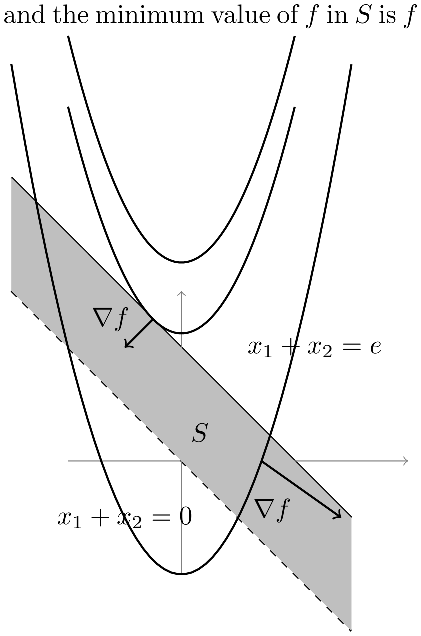
- *(a)* opt $x_1 + x_2$ *subject to* $2x_1 - x_2 \leq 2$, $x_1 + x_2 \leq 4$, $x_1 \geq 0$, $x_2 \geq 0$.
- *(b)* opt $x_1^2 - x_2$ *subject to* $\ln(x_1 + x_2) \leq 1$

Solution:

(a) The level curves are lines $x_1 + x_2 = k$, $k \in \mathbb{R}$. The preference direction is $\nabla f(x_1, x_2) = (1, 1)$ for all (x_1, x_2), where $f(x_1, x_2) = x_1 + x_2$. The furthest contour line that contains points of the opportunity set is $x_1 + x_2 = 4$, thus the segment joining $(0, 4)$ and $(2, 2)$ is formed by global maximum of f in S. The minimum is attained at $(0, 0)$.



(b) The level curves $\{x_1^2 - x_2 = k\}$, $k \in \mathbb{R}$, are convex parabolas. The gradient of $f(x_1, x_2) = x_1^2 - x_2$ is $\nabla f(x_1, x_2) = (2x_1, -1)$, which is the direction where the function is increasing fastest. There is no point of the opportunity set where a highest contour is attained, thus the problem has not a maximum. The point of the opportunity set where the lowest contour is attained is at the point of tangency of a parabola of level k and the line $x_1 + x_2 = e$. The slope of any of the parabolas at a point (x_1, x_2) is $2x_1$. The slope of the line $x_1 + x_2 = e$ is -1. Thus, the tangency point satisfies $2x_1 = -1$, or $x_1 = -\frac{1}{2}$. Plugging this into the line, we obtain $x_2 = e+\frac{1}{2}$. Thus, the global minimum of f in S is the point $(-\frac{1}{2}, e+\frac{1}{2})$ and the minimum value of f in S is $f(-\frac{1}{2}, e+\frac{1}{2}) = \frac{1}{4} - e - \frac{1}{2} = -e - \frac{1}{4}$.



(3) *Consider the function $f(x) = 1 - e^{-x^2}$. Does this function obtain a maximum or a minimum at $x = 0$?*

Solution: Note that $-x^2 \leq 0$ for all $x \in \mathbb{R}$. Since the exponential function is increasing, $e^{-x^2} \leq e^0 = 1$, hence $1 - e^{-x^2} = f(x) \geq 1 - 1 = 0$, for all $x \in \mathbb{R}$. On the other hand, $f(x) = 0$ iff $x = 0$, thus 0 is the unique global minimum of f.

(4) *Prove that*

(a) The two problems

$$\max f(x) \text{ subject to } x \in S$$

$$\min -f(x) \text{ subject to } x \in S$$

have the same solutions.

(b) The two problems

$$\max f(x) \text{ subject to } x \in S$$

$$\max F(f(x)) \text{ subject to } x \in S,$$

where F is a strictly increasing function $F : D \subseteq \mathbb{R} \to \mathbb{R}$, with $f(S) \subseteq D$, have the same solutions.

Solution:

(a) $x_0 \in S$ if a global maximum of f in S iff $f(x) \le f(x_0)$ for all $x \in S$ iff $-f(x) \ge f(x_0)$ for all $x \in S$ iff $(-f)(x) \ge (-f)(x_0)$ for all $x \in S$ iff x_0 is a global minimum of f in S.

(b) Let us prove that if $x_0 \in S$ is a global maximum of f in S, then x_0 is a global maximum of $F \circ f$ in S. By definition, $f(x) \le f(x_0$, for all $x \in S$. Since F is increasing, $F(f(x)) \le F(f(x_0))$, hence $(F \circ f)(x \le (F \circ f)(x_0)$, for all $x \in S$. Thus, x_0 is a global maximum of $F \circ f$ in S.
Now, let us prove that if $x_0 \in S$ is a global maximum of $F \circ f$ in S, then x_0 is a global maximum of f in S. We will use the contrapositive. Suppose that x_0 is not global maximum of f in S. Then there is $x \in S$ such that $f(x_0) < f(x)$. Since F is increasing, $(F \circ f)(x_0) = F(f(x_0)) < F(f(x)) = (F \circ f)(x)$, hence x_0 is not global maximum of $F \circ f$.

(5) *A firm produces two goods, denoted A and B. The cost per day is*

$$C(x, y) = 0.05x^2 + 0.05y^2 - 0.05xy + 2x + 6y + 100,$$

when x units of A and and y units of B are produced. The firm sells all it produces at prices 13 per unit of A and 10 per unit of B. Formulate the optimization problem of the firm if it wishes to maximize profits.

Now suppose that it is required that the units produced of A be at least twice the units produced of B. Formulate the new optimization problem of the firm if it wishes to maximize profits.

Solution: The profit function is

$$\pi(x, y) = 13x + 10y - (0.05x^2 + 0.05y^2 - 0.05xy + 2x + 6y + 100).$$

The optimization problems is

$$\max \pi(x, y) \text{ subject to } x \ge 0,\ y \ge 0.$$

With the additional restriction, the problem is

$$\max \pi(x, y) \text{ subject to } x \ge 0,\ y \ge 0,\ x \ge 2y.$$

(6) (June 18) Consider the order of Pareto defined on the set $A \subseteq \mathbb{R}^2$ limited by the graph of the function $g(x) = 10 - \dfrac{6}{3 - x}$ and the segment that links the points $(4, 16)$ and $(6, 12)$.

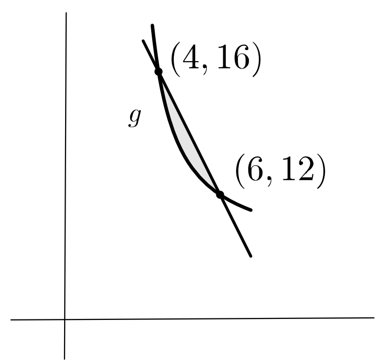
(a) Represent the set A and find the maximal and minimal points of A.

(b) Consider the function $F : A \longrightarrow \mathbb{R}$ defined by $F(x, y) = ax + y$, where $a > 0$. Discuss the points where F attains the maximum value.

Suggestion: for part (a), study the monotonicity of the function g, as well as its concavity or convexity; for part (b), consider the cases $a > 2$, $a = 2$ and $a < 2$.

Solution:

(a) The function $g(x) = 10 - \dfrac{6}{3-x}$ is decreasing and convex on the interval $[4, 6]$ because $g'(x) = -6(3-x)^{-2} < 0$, $g"(x) = -12(3-x)^{-3} > 0$ on that interval. Since the function $g(x)$ is convex, the segment joining the points $(4, g(4))$ and $(6, g(6))$ will be nowhere underneath the graph of $g(x)$. Therefore, the set A will have an approximately graph as shown in the figure below.



Knowing that the segment intercepts the point $(4, 16)$ and has slope equal to -2 the segment lies on the line $y = 16 - 2(x-4)$. From the graph we can deduce that:

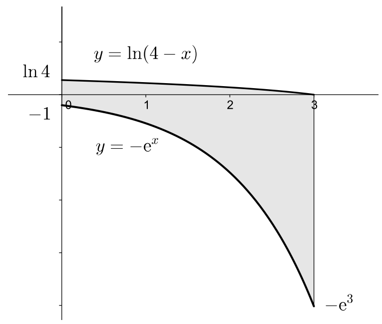
$$\text{maximals}(A) = \{(x, y) : 4 \leq x \leq 6, y = 16 - 2(x-4)\},$$
$$\text{minimals}(A) = \{(x, y) : 4 \leq x \leq 6, y = g(x)\}.$$

(b) F is monotone in the order of Pareto, thus F attains maximum within its maximals.
On the other hand, the set of maximals of F is a segment and the level curves of F are lines, thus
  (i) when $a = 2$, all maximal points of A are maximum of F.
  (ii) when $a < 2$, the maximum of F is $(4, 16)$.
  (iii) when $a > 2$, the maximum of F is $(6, 12)$.

(7) *(Jan 19) Let $f, g : [0, 3] \longrightarrow \mathbb{R}$ be the functions defined by $f(x) = -e^x$ and $g(x) = \ln(4-x)$. Consider the order of Pareto defined on the set $A = \{(x, y) : 0 \leq x \leq 3, f(x) \leq y \leq g(x)\}$.*
  (a) Draw approximately the set A and find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.
  (b) Consider the function $F : A \longrightarrow \mathbb{R}$ defined by $F(x, y) = ax + y$, where $a > 0$. Discuss the points where f attains the maximum value.

Solution:

(a) Both f and g are decreasing on their domains, as they have negative derived functions. So, the drawing of A will be, approximately, as shown in the figure below.



With this graph, there is no maximum nor minimum of A and

$$\text{maximals}(A) = \{(x, g(x)) : 0 \leq x \leq 3\},$$
$$\text{minimals}(A) = \{(x, f(x)) : 0 \leq x \leq 3\}.$$

(b) Case 1. Suppose that the level line of $F(x, y)$ is tangent to $g(x) = \ln(4 - x)$ at the point (x, y). Then, the slopes of the graph of $y = g(x)$ and the level line of $F(x, y)$ are equal, so $g'(x) = -1/(4 - x) = -a$. Hence, when $x = 0$, there is tangency when $a = 1/4$. When $x = 3$, there is tangency when $a = 1$. From that, we can deduce that when $1/4 \leq a \leq 1$, the maximum of $f(x, y)$ is obtained at $(x = 4 - 1/a, y = \ln((4 - x)) = \ln(1/a) = -\ln a$.

Case 2.
i) when $0 < a < 1/4$, the level line of $F(x, y)$ that passes through the point $(0, \ln 4)$ is even more horizontal that when $a = 1/4$. So the maximum of $f(x, y)$ is obtained at $(x = 0, y = \ln 4)$.
ii) when $1 < a$, the level line of $F(x, y)$ that passes through the point $(3, 0)$ is even steeper that when $a = 1$. So the maximum of $f(x, y)$ is obtained at $(x = 3, y = 0)$.

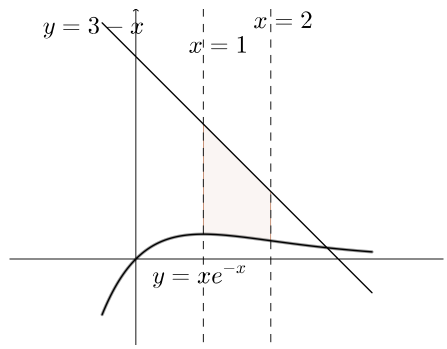
(8) *(June 19) Let the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $h(x) = xe^{-x}$. Consider the order of Pareto defined on the set $A = \{(x, y) : 1 \leq x \leq 2,\ h(x) \leq y \leq 3 - x)\}$.*
  (a) *Draw approximately the set A and find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.*
  (b) *Consider the function $F : A \longrightarrow \mathbb{R}$ defined by $F(x, y) = ax + y$, where $a > 0$. Discuss the points where f attains the maximum value.*

Solution:

(a) Because the function is continuous and $h'(x) = e^{-x}(1-x) < 0$ if $1 < x < 2$, then it is decreasing on the closed interval $[1, 2]$. Moreover, the straight line $3 - x$ is also decreasing. Since both functions are decreasing and $\max h(x) = h(1) = \frac{1}{e} < 1 = 3-2 = \min(3-x)$, we can deduce that $h(x) < 3-x$, for every $x \epsilon [1, 2]$. So, the drawing of A will be, approximately, as shown in the figure below.

Hence the Pareto order describes the set in the following way: there are no maximum nor minimum and

$$\text{maximals}(A) = \{(x, 3 - x)) : 1 \leq x \leq 2\},$$
$$\text{minimals}(A) = \{(x, xe^{-x}) : 1 \leq x \leq 2\}.$$



(b) Case 1. When $a = 1$, the slope of any level line of $F(x, y)$ is the same as the segment that unites the points $(1, 2)$ and $(2, 1)$; i.e.: -1. So, int this case, the maximizers are all the points of the segment that unites the points $(1, 2)$ and $(2, 1)$.

Case 2. When $a < 1$, the slope of any level line of $F(x, y)$ is less than the segment that unites the points $(1, 2)$ and $(2, 1)$. So, int this case, the maximizer is the point $(1, 2)$.

Case 3. When $1 < a$, the slope of any level line of $F(x, y)$ is more than the segment that unites the points $(1, 2)$ and $(2, 1)$. So, int this case, the maximizer is the point $(2, 1)$.

(9) *Consider defined the order of Pareto on the following sets.*

$$A = \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 1\}$$
$$B = \{(x, y) \in \mathbb{R}^2 | \; |x| \leq 1; \; |y| \leq 1\}$$
$$C = \{(x, y) \in \mathbb{R}^2 | \; 0 \leq y \leq 4 - x^2\}$$
$$D = \{(x, y) \in \mathbb{R}^2 | \; x^2 - 9 \leq y \leq 0\}$$
$$E = \left\{(x, y) \in \mathbb{R}^2 : |x| \leq y \leq 6 - x^2\right\}.$$

Obtain, if they exist, the maximun and the minimun, the maximals and the minimals of the sets above.

Solution:

(a) maximals$(A) = \{(x, y) : x + y = 1\}$.
(b) maximals$(B) = \max B = \{(1, 1)\}$; minimals$(B) = \min B = \{(-1, -1)\}$.
(c) maximals$(C) = \{(x, y) : y = 4 - x^2, 0 \leq x \leq 2\}$; minimals$(C) = \min C = \{(-2, 0)\}$.
(d) maximals$(D) = \max D = \{(3, 0)\}$; minimals$(D) = \{(x, y) : y = x^2 - 9, -3 \leq x \leq 0\}$.
(e) maximals$(E) = \left\{(x, y) \in \mathbb{R}^2 : y = 6 - x^2, 0 \leq x \leq 2\right\}$; minimals$(E) = \left\{(x, y) \in \mathbb{R}^2 : y = -x, -2 \leq x \leq 0\right\}$.