February 2, 2022

MATHEMATICAL OPTIMIZATION FOR ECONOMICS

ECONOMICS, LAW-ECONOMICS, INTERNATIONAL STUDIES-ECONOMICS

SHEET 1. INTRODUCTION TO MATHEMATICAL OPTIMIZATION

(1) Show in a diagram the feasible set for an optimization problem of two variables, x_1 and x_2, where the constraint(s) is (are)

(a) $x_1 = 10$

(b) $x_1^2 + 3x_2 = 1$

(c) $\ln(x_1 + x_2) \leq 1$

(d) $e^{x_2} - x_1 \geq 0$, $x_1 \geq 1$, $x_2 < 0$

(e) $x_2 \geq 4$, $(x_1 - 6)^2 + (x_2 - 4)^2 \leq 25$

(2) Solve diagrammatically, using contours, preference directions, and feasible sets, the following problems

(a) opt $x_1 + x_2$ subject to $2x_1 - x_2 \leq 2$, $x_1 + x_2 \leq 4$, $x_1 \geq 0$, $x_2 \geq 0$.

(b) opt $x_1^2 - x_2$ subject to $\ln(x_1 + x_2) \leq 1$

(3) Consider the function $f(x) = 1 - e^{-x^2}$. Does this function obtain a maximum or a minimum at $x = 0$?

(4) Prove that

(a) The two problems

$$\max f(x) \text{ subject to } x \in S$$

$$\min -f(x) \text{ subject to } x \in S$$

have the same solutions.

(b) The two problems

$$\max f(x) \text{ subject to } x \in S$$

$$\max F(f(x)) \text{ subject to } x \in S,$$

where F is a strictly increasing function $F : D \subseteq \mathbb{R} \to \mathbb{R}$, with $f(S) \subseteq D$, have the same solutions.

(5) A firm produces two goods, denoted A and B. The cost per day is

$$C(x, y) = 0.05x^2 + 0.05y^2 - 0.05xy + 2x + 6y + 100,$$

when x units of A and and y units of B are produced. The firm sells all it produces at prices 13 per unit of A and 10 per unit of B. Formulate the optimization problem of the firm if it wishes to maximize profits.

Now suppose that it is required that the units produced of A be at least twice the units produced of B. Formulate the new optimization problem of the firm if it wishes to maximize profits.

(6) (June 18) Consider the order of Pareto defined on the set $A \subseteq \mathbb{R}^2$ limited by the graph of the function $g(x) = 10 - \dfrac{6}{3 - x}$ and the segment that links the points $(4, 16)$ and $(6, 12)$.

(a) Represent the set A and find the maximal and minimal points of A.

(b) Consider the function $F : A \longrightarrow \mathbb{R}$ defined by $F(x, y) = ax + y$, where $a > 0$. Discuss the points where f attains the maximum value.

Suggestion: for part (a), study the monotonicity of the function g, as well as its concavity or convexity; for part (b), consider the cases $a > 2$, $a = 2$ and $a < 2$.

(7) (Jan 19) Let $f, g : [0, 3] \longrightarrow \mathbb{R}$ be the functions defined by $f(x) = -e^x$ and $g(x) = \ln(4 - x)$. Consider the order of Pareto defined on the set $A = \{(x, y) : 0 \leq x \leq 3, f(x) \leq y \leq g(x)\}$.

(a) Draw approximately the set A and find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.

(b) Consider the function $F : A \longrightarrow \mathbb{R}$ defined by $F(x, y) = ax + y$, where $a > 0$. Discuss the points where f attains the maximum value.

(8) (June 19) Let the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $h(x) = xe^{-x}$. Consider the order of Pareto defined on the set $A = \{(x, y) : 1 \leq x \leq 2,\ h(x) \leq y \leq 3 - x)\}$.

(a) Draw approximately the set A and find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.

(b) Consider the function $F : A \longrightarrow \mathbb{R}$ defined by $F(x, y) = ax + y$, where $a > 0$. Discuss the points where f attains the maximum value.

(9) Consider defined the order of Pareto on the following sets.

$$A = \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 1\}$$
$$B = \{(x, y) \in \mathbb{R}^2 |\ |x| \leq 1;\ |y| \leq 1\}$$
$$C = \{(x, y) \in \mathbb{R}^2 |\ 0 \leq y \leq 4 - x^2\}$$
$$D = \{(x, y) \in \mathbb{R}^2 |\ x^2 - 9 \leq y \leq 0\}$$
$$E = \left\{(x, y) \in \mathbb{R}^2 : |x| \leq y \leq 6 - x^2\right\}.$$

Obtain, if they exist, the maximun and the minimun, the maximals and the minimals of the sets above.