SOLVED

$\boxed{1}$

Consider the function $f(x, y) = \sqrt{x + 2y}$ defined on the set

$$A = \{(x, y) : x^2 + y^2 \geq 1, 0 \leq y \leq x, x \leq 2\}.$$

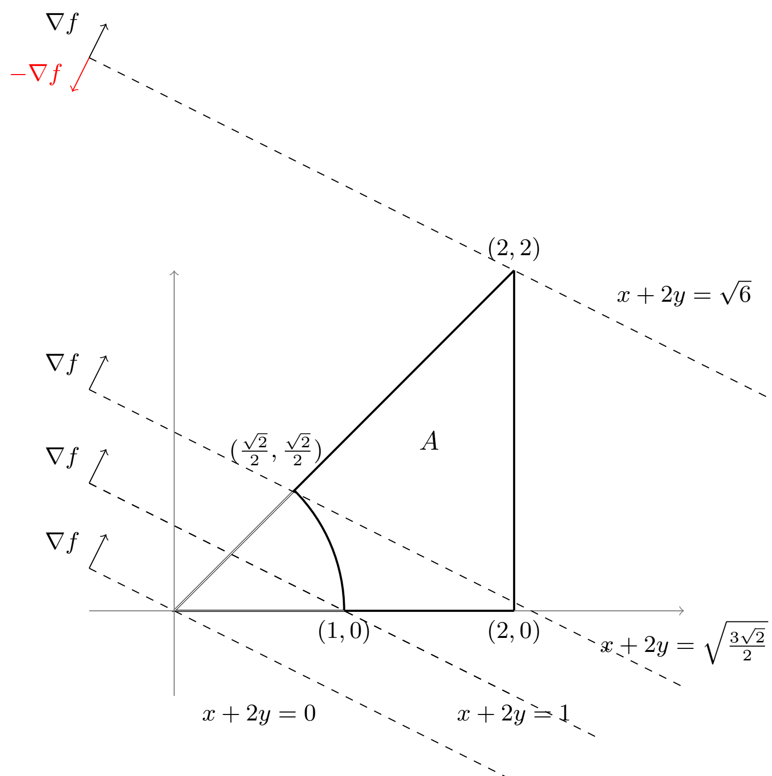
(a) (6 points) Draw the set A and discuss whether the function f and the set A satisfy the assumptions of the Weierstrass Theorem.

(b) (6 points) Draw the level curves of f on the set A, showing the directions in which f increases/decreases and determine (if they exist) the global extrema of f on A.

Solution:

(a) The function f is continuous in $\mathbb{R}^2_+$, thus continuous in A. The set A is closed, since it contains its boundary and it is bounded, since $\|(x, y)\| = \sqrt{x^2 + y^2} \leq \sqrt{x^2 + x^2} \leq \sqrt{2^2 + 2^2} = \sqrt{8}$, for all $(x, y) \in A$. Thus, the hypotheses of the Theorem of Weierstrass are fulfilled. The set A joint with some level curves of f and its gradient are shown in the figure below.

(b) The level curves of f are the lines $x + 2y = k$, where $k \geq 0$. They have slope $-\frac{1}{2}$. The gradient of f is $(1, 2)$ at every point, that shows the direction of quickest increase of f (and thus $(-1, -2)$ is the direction of quickest decrease of f). Then, $(2, 2)$, which is the intersection point of $x = 2$ with $y = x$, is the global maximum and $(1, 0)$, which is the intersection of $y = 0$ with $x^2 + y^2 = 1$, is the global minimum.



$\boxed{2}$

A firm sells two goods in amounts x and y, respectively.

(a) (6 points) Suppose that the profit function of the firm is

$$B(x,y) = -4x^2 - 24y^2 + 800x + 960y - 500.$$

Compute the critical points and study if they are global maximum.

(b) (6 points) Now suppose that the firm is unsure about its profit function. Precisely, the firm estimates that the profit function is

$$B(x,y) = -4x^2 - 24y^2 + 4axy + 800x + 960y - 500, \quad a \in \mathbb{R}.$$

Determine the interval(s) I of values of a, for which the profit function is strictly concave.

Solution:

(a) To find the critical points, we apply the necessary condition $\nabla B(x,y) = (0,0)$, which is the system of equations

$$\frac{\partial B}{\partial x} = -8x + 800 = 0,$$
$$\frac{\partial B}{\partial y} = -48y + 960 = 0.$$

The only solution is $(x,y) = (100, 20)$. The Hessian of f is

$$\mathcal{H}B(x,y) = \begin{pmatrix} -8 & 0 \\ 0 & -48 \end{pmatrix},$$

which is negative definite for all x, y, thus B is strictly concave. Then (100,20) is a global maximum.

(b) Noting that $\nabla B(x,y) = (-8x + 4ay + 800, -48y + 4ay + 960)$, we have that the Hessian matrix is

$$\mathcal{H}B(x,y) = \begin{pmatrix} -8 & 4a \\ 4a & -48 \end{pmatrix}.$$

The principal minors are $D_1 = -8 < 0$ and $D_2 = 8 \cdot 48 - 16a^2 = 16(24 - a^2) < 0$ iff $a^2 < 24$, iff $|a| < \sqrt{24} = 2\sqrt{6}$, iff $a \in (-2\sqrt{6}, 2\sqrt{6}) = I$. (Note that part (a) corresponds with $a = 0$).

$\boxed{3}$

Consider the function $f(x,y) = 12x\sqrt{y}$ on the set $\{(x,y) : 3x + 4y = 12, y > 0\}$.

(a) (6 points) Obtain the Lagrange equations and find the critical points.

(b) (6 points) Classify the critical points found in part (a).

(c) (6 points) Suppose that the equality constraint changes to $3x + 4y = 13$. Without solving the problem, give an estimate of the new optimal value of $f(x,y)$.

Solution:

(a) The Lagrangian is $L(x,y,\lambda) = 12x\sqrt{y} + \lambda(12 - 3x - 4y)$. The Lagrange equations are $\nabla L = (0,0,0)$, that is

$$\frac{\partial L}{\partial x} = 12\sqrt{y} - 3\lambda = 0,$$
$$\frac{\partial L}{\partial y} = \frac{12x}{2\sqrt{y}} - 4\lambda = 0,$$
$$\frac{\partial L}{\partial z} = 12 - 3x - 4y = 0.$$

Equating $\lambda = \frac{12}{3}\sqrt{y} = \frac{12x}{8\sqrt{y}}$, we find $y = \frac{3x}{8}$. Plugging this into the constraint we get $12 - 3x - \frac{4\cdot 3x}{8} = 12 - \frac{36x}{8} = 0$, and solving, $x = \frac{8}{3}$, $y = 1$. The multiplier is $\lambda = \frac{12}{3}\sqrt{1} = 4$.

(b) The Hessian matrix of L with respect to (x,y) is the matrix

$$\mathcal{H}_{(x,y)}L(x,y,\lambda) = \begin{pmatrix} 0 & \frac{6}{\sqrt{y}} \\ \frac{6}{\sqrt{y}} & -\frac{3x}{\sqrt{y^3}} \end{pmatrix}.$$

Evaluating at $(\frac{8}{3}, 1, 4)$, it becomes the matrix

$$\mathcal{H}_{(x,y)}L(8/3,1,4) = \begin{pmatrix} 0 & 6 \\ 6 & -8 \end{pmatrix},$$

which defines an indefinite quadratic form, thus we have to restrict to the tangent subspace, $\{(v_1,v_2) : \nabla g(\frac{8}{3},1)\cdot(v_1,v_2) = 0\}$. Since $\nabla g(x,y) = (3,4)$ for all x,y, the tangent subspace is determined by the identity $3v_1 + 4v_2 = 0$, from which we obtain $v_2 = -\frac{3}{4}v_1$. The quadratic form with matrix $\mathcal{H}_{(x,y)}L(8/3,1,4)$ when restricted to the tangent subspace becomes

$$(v_1, (-3/4)v_1)\begin{pmatrix} 0 & 6 \\ 6 & -8 \end{pmatrix}\begin{pmatrix} v_1 \\ -\frac{3}{4}v_1 \end{pmatrix} = -\frac{27}{2v_1^2} < 0,$$

which is negative definite, thus $(\frac{8}{3}, 1)$ is a local maximum.

(c) The Lagrange multiplier coincides with the derivative of the value function. Thus, by the definition of the derivative of a function, we have

$$\Delta V \approx \lambda \cdot \Delta b,$$

where ΔV denotes the increment in the optimal value and Δb denotes the increment in the "constraint". Noting that $\Delta b = 13 - 12$, we have that the increment in value is $\Delta V \approx 4$. As the old optimal value was $f(\frac{8}{3}, 1) = 32$, the new optimal value will be 36.

$\boxed{4}$

Consider the function $f(x,y) = x^2 + 2y^2 + y - 1$ defined on the set

$$A = \{(x,y) \,:\, x^2 + y^2 \leq 1\}.$$

(a) (5 points) Find the Kuhn–Tucker necessary optimality conditions to the problem

$$\max f(x,y) \quad \text{subject to } (x,y) \in A.$$

(b) (13 points) Find all the solutions of the Kuhn–Tucker conditions established in part (a) and find the maximum of f on A.

Solution:

(a) The Lagrangian is $L(x,y,\lambda) = x^2 + 2y^2 + y - 1 + \lambda(1 - x^2 - y^2)$. The K–T necessary conditions of optimality are

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 2x - 2\lambda x = 0,\\
\frac{\partial L}{\partial y} &= 4y + 1 - 2\lambda y = 0,\\
\lambda\frac{\partial L}{\partial z} &= \lambda(1 - x^2 - y^2) = 0,\\
\lambda &\geq 0,\\
1 - x^2 &- y^2 \geq 0.
\end{aligned}$$

(b) Let us explore the solutions of the system formed by the equalities. First suppose that $\lambda > 0$. Then, $x^2 + y^2 = 1$. From the first equation in part (a) we get either $x = 0$ or $\lambda = 1$. If $x = 0$, then either $y = 1$ or $y = -1$. Note that $x = 0$ and $y = 1$ leads to the value $\lambda = \frac{5}{2} > 0$, when plugged into the second equation of part (b). The point $(0, 1, \frac{5}{2})$ fulfills all the K–T conditions. Now consider $x = 0$ and $y = -1$. In this case, the multiplier is $\lambda = \frac{3}{2} > 0$. Thus, also the point $(0, -1, \frac{3}{2})$ fulfills all the K–T conditions. Suppose now that $x \neq 0$ and thus $\lambda = 1$. We get from the first equation in part (a) that x is arbitrary, and from the second one that $y = -\frac{1}{2}$, hence using the constraint, $x^2 = 1 - y^2 = 1 - \frac{1}{4} = \frac{3}{4}$, and thus we get two possibilities, $x = \frac{\sqrt{3}}{2}$, and $x = -\frac{\sqrt{3}}{2}$. We collect two candidates: $(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1)$.

It remains the case with $\lambda = 0$. This gives $x = 0$ and $y = -\frac{1}{4}$. An additional solution is thus $(0, -\frac{1}{4}, 0)$.

To see which points are global maximum, we realize that the Weierstrass Theorem applies. It suffices then to evaluate f on the candidates, and to pick the maximum value.

$$\begin{aligned}
f(0,1) &= 2,\\
f(0,-1) &= 0,\\
f\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) &= \frac{9}{4} + 2\cdot\frac{1}{4} - \frac{1}{2} - 1 = \frac{5}{4},\\
f\left(0, -\frac{1}{4}\right) &= 2\frac{1}{16} - \frac{1}{4} - 1 = -\frac{9}{8}.
\end{aligned}$$