1

Consider the function $f(x,y) = (x-2y)^2$ defined on the set

$$A = \{(x,y) : x+y \geq 0, y \leq 3, x \leq 3\}.$$

(a) (6 points) Draw the set A and discuss whether the function f and the set A satisfy the assumptions of the Weierstrass Theorem.

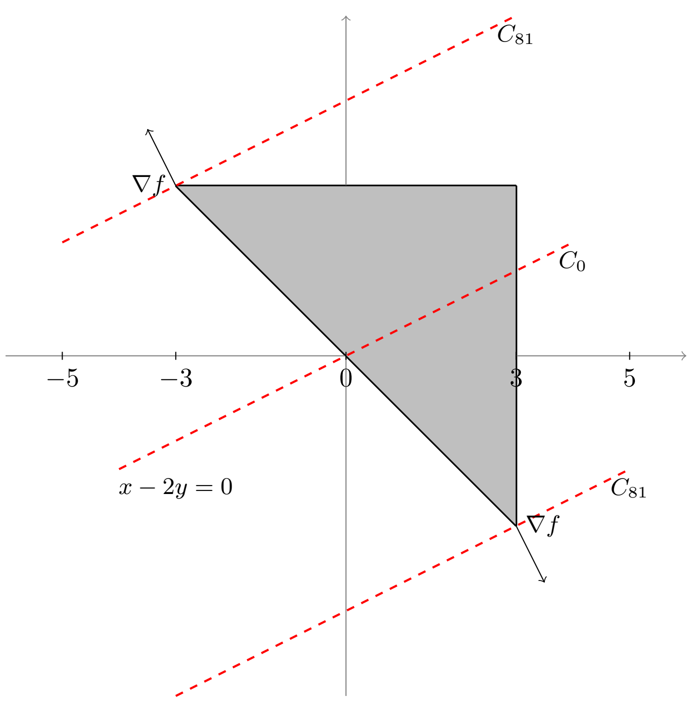
(b) (6 points) Draw the level curves of f on the set A, showing the directions in which f increases/decreases and determine (if they exist) the global extrema of f on A.

Solution:

(a) The function f is continuous in $\mathbb{R}^2_+$, thus continuous in A. The set A is closed, since it contains its boundary and it is bounded, since $x+y \geq 0$ implies $x \geq -y$, and $y \leq 3$ then implies $x \geq -3$. Thus, $|x| \leq 3$; the same bound is found for $|y|$. Hence, $\|(x,y)\| = \sqrt{x^2+y^2} \leq \sqrt{9+9} = 3\sqrt{2}$, for all $(x,y) \in A$. Thus, the hypotheses of the Theorem of Weierstrass are fulfilled. The set A joint with some level curves of f and its gradient are shown in the figure below.

(b) The level curves of f are given by $(x-2y)^2 = k^2$, where $k \in \mathbb{R}$. In fact, each level curve is formed by the pair of parallel lines $x - 2y = \pm k$, with slope $\frac{1}{2}$ (in the case $k = 0$, the level curve is the line $x - 2y = 0$). The gradient of f is $(2(x-2y), -4(x-2y))$ at every point, that shows the direction of quickest increase of f (and thus $-\nabla f(x,y)$ is the direction of quickest decrease of f). Notice that $\nabla f(x,y) = 2(x-2y)(1,-2)$, thus the direction of maximal increase of f is $(1,-2)$ at every point (x,y) for which $x > 2y$, and $-(1,-2) = (-1,2)$ if $x < 2y$.

Then, the points $(3,-3)$ and $(-3,3)$, are global maximum of f in A, with value $f(3,-3) = f(-3,3) = 81$. The global minima are attained along the points of the line $x - 2y = 0$ which belong to A, and the value is 0.



2

A firm sells two goods A and B in amounts x and y, respectively. The revenue of the firm is

$$R(x, y) = 800x + 960y - 2x^2 - 12y^2 + 4axy,$$

where a is an unknown parameter. The cost of producing x units of good A and y units of good B is

$$C(x, y) = 2x^2 + 12y^2.$$

(a) (6 points) Find the profit function $B(x, y)$ of the firm and find all values of a for which $B(x, y)$ is strictly concave.

(b) (6 points) The firm knows that $a = 2$. Find the critical points of the profit function $B(x, y)$ and calculate the output levels that maximize profits, if they exist.

Solution:

(a) The profit function is

$$B(x, y) = R(x, y) - C(x, y) = 800x + 960y - 4x^2 - 24y^2 + 4axy.$$

Noting that $\nabla B(x, y) = (800 - 8x + 4ay, 960 - 48y + 4ax)$, we have that the Hessian matrix is

$$\mathcal{H}B(x, y) = \begin{pmatrix} -8 & 4a \\ 4a & -48 \end{pmatrix}.$$

The principal minors are $D_1 = -8 < 0$ and $D_2 = 8 \cdot 48 - 16a^2 = 16(24 - a^2) > 0$ iff $a^2 < 24$, iff $|a| < \sqrt{24} = 2\sqrt{6}$, iff $a \in (-2\sqrt{6}, 2\sqrt{6})$.

(b) To find the critical points, we apply the necessary condition $\nabla B(x, y) = (0, 0)$, which is the system of equations (remember that $a = 2$)

$$\frac{\partial B}{\partial x} = 800 - 8x + 8y = 0,$$
$$\frac{\partial B}{\partial y} = 960 - 48y + 8x = 0.$$

Adding both equations we obtain $40y = 2760$, and thus $y = 44$. The only solution is $(x, y) = (144, 44)$, which is the only critical point of $B(x, y)$. By part (a), B is strictly concave, since $0 \le a = 2 < 2\sqrt{6}$, hence $(144, 44)$ is the unique global maximizer of B.

3

Let $M = \{(x, y, z) : x + y + z = 1\}$ and $f(x, y, z) = 3x^2 + 3y^2 + z^2 + 2yz$.

(a) (6 points) Obtain the Lagrange equations and find the critical points of the problem of minimizing f on M.

(b) (6 points) Show that there is a unique constrained global minimum and find the minimum value of f on M.

(c) (6 points) Let $N = \{(x, y, z) : x+y+z = 0.5\}$. Without solving the new problem, give an estimate of the minimum value of f on N.

Solution:

(a) The Lagrangian is $L(x, y, z, \lambda) = 3x^2 + 3y^2 + z^2 + 2yz + \lambda(1 - x - y - z)$. The Lagrange equations are $\nabla L = (0, 0, 0)$, that is

$$\begin{aligned} \frac{\partial L}{\partial x} &= 6x - \lambda = 0, \\ \frac{\partial L}{\partial y} &= 6y + 2z - \lambda = 0, \\ \frac{\partial L}{\partial z} &= 2z + 2y - \lambda = 0, \\ x + y + z &= 1. \end{aligned}$$

We obtain $\lambda = 6x = 6y + 2z = 2z + 2y$, and hence $y = 0$ and $x = \frac{\lambda}{6}$, $z = \frac{\lambda}{2}$. Substituting into the constraint we get $\frac{\lambda}{6} + \frac{\lambda}{2} = 1$, or $\lambda = \frac{3}{2}$. Thus, $x = \frac{1}{4}$, $y = 0$ and $z = \frac{3}{4}$ is the only critical point (with $\lambda = \frac{3}{2}$).

(b) It can be checked that f is strictly convex, since the Hessian matrix of f is given by

$$Hf(x, y, z) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 2 \end{pmatrix},$$

which is positive definite, and M is a convex set, thus the critical point of the Lagrangian is a global minimum of f on M. The minimum value of f on M is thus $V = f(\frac{1}{4}, 0, \frac{3}{4}) = \frac{3}{4}$.

(c) The Lagrange multiplier coincides with the derivative of the value function. Thus, by the definition of the derivative of a function, $V'(b) = \lim_{\Delta b \to 0} \frac{\Delta V}{\Delta b} = \lambda$, we have

$$\Delta V \approx \lambda \cdot \Delta b,$$

where ΔV denotes the increment in the optimal value and Δb denotes the increment in the independent term of the constraint, and Δb is small. Noting that $\Delta b = -0.5$ and that from parts (a) and (b) $\lambda = 1.5$, we have that the increment in value is $\Delta V \approx -0.75$. As the optimal value of f on M was 0.75, the optimal value on N will be approximately $0.75 - 0.75 = 0$. (The true optimal value is 0.1875; what we get with the Lagrange multiplier is only an approximation).

4

Consider the function $f(x, y) = 10xy$ defined on the set

$$A = \{(x, y) \,:\, x^2 + 2y^2 \leq 1\}.$$

(a) (6 points) Find the Kuhn–Tucker necessary optimality conditions to the problem

$$\max f(x, y) \quad \text{subject to } (x, y) \in A.$$

(b) (6 points) Find all the solutions of the Kuhn–Tucker conditions established in part (a).

(c) (6 points) Find the maximum of f on A. Is $(0, 0)$ the minimizer of f on A?

Solution:

(a) The Lagrangian is $L(x, y, \lambda) = 10xy + \lambda(1 - x^2 - 2y^2)$. The K–T necessary conditions of optimality are

$$\frac{\partial L}{\partial x} = 10y - 2\lambda x = 0,$$
$$\frac{\partial L}{\partial y} = 10x - 4\lambda y = 0,$$
$$\lambda \frac{\partial L}{\partial \lambda} = \lambda(1 - x^2 - 2y^2) = 0,$$
$$\lambda \geq 0,$$
$$1 - x^2 - 2y^2 \geq 0.$$

(b) First explore the solutions of the system formed by the equalities, and then check the rest of conditions.

Case 1. Suppose that $x = 0$ or $y = 0$; then the other variable is 0 too, and the KT conditions are fulfilled with $\lambda = 0$. Reciprocally, if $\lambda = 0$, then $(x, y) = (0, 0)$ fulfills KT conditions.

Case 2. Suppose that $x \neq 0$, $y \neq 0$ and $\lambda > 0$. Then, from the two first equations in KT conditions, we can solve $\lambda = 5y/x = 5x/2y$, hence $x^2 = 2y^2$. Since $\lambda > 0$, the third equation implies that $4y^2 = x^2 + 2y^2 = 1$, thus $y^2 = \frac{1}{4}$ and then $y = \pm\frac{1}{2}$. Also, $x^2 = 2(\frac{1}{4}) = \frac{1}{2}$, thus $x = \pm\frac{1}{\sqrt{2}}$. We have found 4 points:

$$\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2}\right),$$

but notice that $\lambda = -\frac{5}{2}\sqrt{2} < 0$ for the points that have coordinates of different sign, hence these points do not fulfill KT conditions. For the points $(\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{2})$, $\lambda = \frac{5}{2}\sqrt{2} > 0$, and the KT conditions hold.

(c) To see which points are global maxima, we realize that the Weierstrass Theorem applies, since f is continuous and the feasible set is the interior and frontier of an ellipse, thus a compact set. It suffices then to evaluate f on the candidates, and to pick the point(s) with the highest value.

$$f(0, 0) = 0,$$
$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \frac{10}{2\sqrt{2}} = \frac{5\sqrt{2}}{2} = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right).$$

So, the maximizers are $(\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{2})$. Obviously, $(0, 0)$ is not the minimizer, since $f(-\frac{1}{2}, \frac{1}{2}) < 0 = f(0, 0)$.