1

Consider the function $f(x,y) = -\ln(x^2+y^2)$ defined on the set

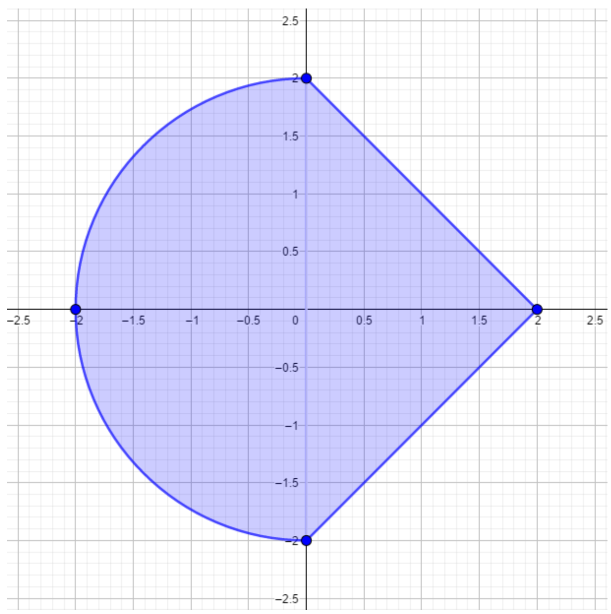
$$A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 4, x+y \leq 2, x-y \leq 2\}.$$

(a) (6 points) Draw the set A and discuss whether the function f and the set A satisfy the assumptions of the Theorem of Weierstrass. Can you ensure the existence of global extremes of f in A?

(b) (6 points) Draw the level curves of f of levels -1, 0 and 1 on the plane, showing the directions in which f increases/decreases and determine (if they exist) the global extrema of f on A. In case they do not exist, justify the reason.

Solution:

(a) Below is a representation of the set A.



A is closed (boundary is in A) and bounded, thus compact. The function f is not defined at $(0,0) \in A$. Thus, f is not continuous in A and cannot apply the Theorem of Weierstrass. In consequence the existence of global extrema cannot be assured (but they could exist, of course).

(b) Level curves of f are given by

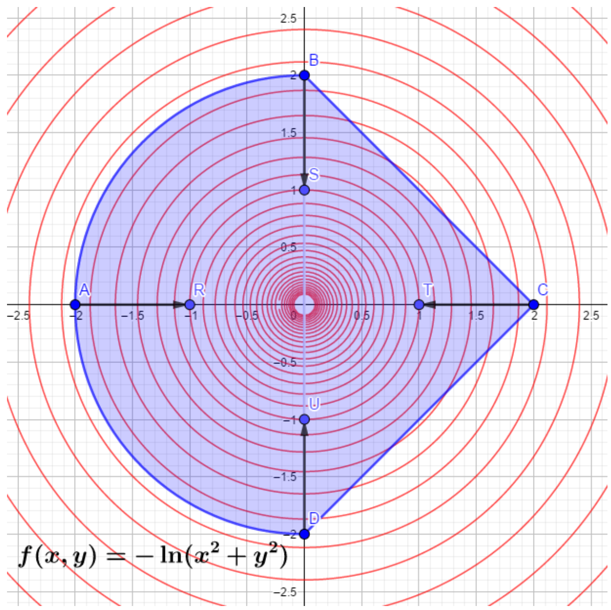
$$-\ln(x^2+y^2) = k \Rightarrow \ln(x^2+y^2) = -k \Rightarrow x^2+y^2 = e^{-k} = \frac{1}{e^k}$$

thus they are circumferences centred at $(0,0)$ an radius $\dfrac{1}{\sqrt{e^k}}$. Note that the radius decreases as k rises. This means that the function takes on greater and greater values as we move towards the origin of coordinates. This can be seen analytically with the gradient, that points in the direction of maximal growth of f.

$$\nabla f(x,y) = \left(\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}\right) = \left(-\frac{2x}{x^2+y^2}, -\frac{2y}{x^2+y^2}\right)$$

We illustrate the gradient at points $A = (-2,0)$, $B = (0,2)$, $C = (2,0)$ and $D = (0,-2)$ in the figure below.

- $\nabla f(-2,0) = \left(-\frac{2\,(-2)}{4+0}, -\frac{2\,0}{4+0}\right) = \left(\frac{4}{4}, 0\right) = (1,0).$
- $\nabla f(0,2) = \left(-\frac{2\,0}{0+4}, -\frac{2\,2}{0+4}\right) = \left(0, -\frac{4}{4}\right) = (0,-1).$
- $\nabla f(2,0) = \left(-\frac{2\,2}{4+0}, -\frac{2\,0}{4+0}\right) = \left(\frac{-4}{4}, 0\right) = (-1,0).$
- $\nabla f(0,-2) = \left(-\frac{2\,0}{0+4}, -\frac{2\,(-2)}{0+4}\right) = \left(0, \frac{4}{4}\right) = (0,1).$



Clearly, $\lim_{(x,y)\to(0,0)} f(x,y) = +\infty$, hence the f is not bounded above in the set A and thus has no global maximum. Regarding the points of minimum, they will be those points of A which are the the farthest from the $(0,0)$. It is clear that the global minima are attained in the arc $x^2+y^2=4$ with $x \le 0$. The minimum value of f in A is $-\ln(4)$.

$\boxed{2}$

Consider the function

$$f(x, y) = e^{1+ax^2+by^2},$$

where a and b are unknown parameters, both different from 0.

(a) (6 points) Find the critical points of f.

(b) (6 points) For each of the critical points found in the item above, determine the range of values of the parameters a and b, for which the critical point considered is

- A local maximum.
- A local minimum.
- A saddle point.

Solution:

(a) Critical points are those where either the function is not differentiable os the gradient is the null vector.

$$\nabla f(x, y) = \left(\frac{\partial f(x, y)}{\partial x}, \frac{\partial f(x, y)}{\partial y} \right) = (e^{1+ax^2+by^2}\, 2ax, e^{1+ax^2+by^2}\, 2by)$$

$$\nabla f(x, y) = 0 \Rightarrow \left\{ \begin{array}{l} 2ax\, e^{1+ax^2+by^2} = 0 \\ 2by\, e^{1+ax^2+by^2} = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x = 0 \\ y = 0 \end{array} \right\},$$

since that both a and b are different from 0. The only critical point is thus $(0, 0)$.

(b) To classify the critical point, we use the second order sufficient conditions, which depends on the sign of quadratic form associated to the Hessian matrix $Hf(x, y)$:

$$\frac{\partial^2 f(x, y)}{\partial x^2} = e^{1+ax^2+by^2}\,(2ax)^2 + e^{1+ax^2+by^2}\, 2a = 2\, a\, e^{1+ax^2+by^2}(2ax^2 + 1)$$

$$\frac{\partial^2 f(x, y)}{\partial y \partial x} = 2ax\, e^{1+ax^2+by^2}\, 2by = 4\, a\, b\, x\, y\, e^{1+ax^2+by^2} = \frac{\partial^2 f(x, y)}{\partial x \partial y}$$

$$\frac{\partial^2 f(x, y)}{\partial y^2} = e^{1+ax^2+by^2}\,(2by)^2 + e^{1+ax^2+by^2}\, 2b = 2\, b\, e^{1+ax^2+by^2}(2by^2 + 1)$$

$$\mathrm{H}f(x, y) = \begin{pmatrix} 2\, a\, e^{1+ax^2+by^2}(2ax^2 + 1) & 4\, a\, b\, x\, y\, e^{1+ax^2+by^2} \\ 4\, a\, b\, x\, y\, e^{1+ax^2+by^2} & 2\, b\, e^{1+ax^2+by^2}(2by^2 + 1) \end{pmatrix}$$

At $(0, 0)$ we get

$$\mathrm{H}f(0, 0) = \begin{pmatrix} 2a\, e & 0 \\ 0 & 2b\, e \end{pmatrix},$$

a diagonal matrix.

- $\mathrm{H}f(0, 0)$ is negative definite iff $a < 0$ and $b < 0$, thus $(0, 0)$ is a local maximum (it is global indeed).
- $\mathrm{H}f(0, 0)$ es positive definite iff $a > 0$ and $b > 0$, thus $(0, 0)$ is a local minimum (it is global indeed).
- $\mathrm{H}f(0, 0)$ is indefinite iff $ab < 0$, thus $(0, 0)$ is a saddle point.

$\boxed{3}$

Consider the problem of Lagrange:

$$\text{Opt. } f(x,y) = 2x^3 - y^3 \quad \text{s.t.: } x^2 + y^2 = 5$$

(a) (3 points) Obtain the Lagrange equations.

(b) (6 points) Find the critical points of the Lagrangian.

(c) (3 points) Find, if they exist, the global maximum and the global minimum.

(d) (6 points) Suppose that the constraint changes to $x^2 + y^2 = 5.1$, that is, the problem becomes now

$$\text{Opt. } f(x,y) = 2x^3 - y^3 \quad \text{s.t.: } x^2 + y^2 = 5.1.$$

Without solving the problem again, calculate approximately the maximum value of $f(x,y)$ after this change.

Solution:

(a) The Lagrangian is $L(x,y,\lambda) = 2x^3 - y^3 + \lambda(5 - x^2 - y^2)$ and the Lagrange equations are

(i) $6x^2 - 2\lambda x = 0 \to 2x(3x - \lambda) = 0$

(ii) $-3y^2 - 2\lambda y = 0 \to -y(3y + 2\lambda) = 0$

(iii) $x^2 + y^2 = 5$

(b) Let us solve the Lagrange equations to find the critical points.

If $x = 0$ in (i), then from (iii) $y = \pm\sqrt{5}$.

If $y = 0$ in (ii), then from (iii) $x = \pm\sqrt{5}$.

If $x \neq 0$ and $y \neq 0$, then (i) $3x - \lambda = 0 \to \lambda = 3x$ and (ii) $3y + 2\lambda = 0 \to \lambda = -\frac{3y}{2}$. Equating the two values of λ, we obtain $y = -2x$. Plugging this equality into (iii) $x^2 + (-2x))^2 = 5 \to 5x^2 = 5 \to x = \pm 1$.

Hence, the six critical points, with the corresponding value of the multipliers are

Critical point 1 $(0, \sqrt{5})$, $\lambda = -\frac{3\sqrt{5}}{2}$.

Critical point 2 $(0, -\sqrt{5})$, $\lambda = \frac{3\sqrt{5}}{2}$.

Critical point 3 $(\sqrt{5}, 0)$, $\lambda = 3\sqrt{5}$.

Critical point 4 $(-\sqrt{5}, 0)$, $\lambda = -3\sqrt{5}$.

Critical point 5 $(1, -2)$, $\lambda = 3$.

Critical point 6 $(-1, 2)$, $\lambda = -3$.

(c) The set $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 5\}$ is compact and the function $f(x,y) = 2x^3 - y^3$ is continuous, thus we evaluate the function on the critical points and select the global maximum and the global minimum.

The global maximum is attained at the critical point 3, with $f(\sqrt{5}, 0) = 10\sqrt{5}$.

The global minimum is attained at the critical point 4, with $f(-\sqrt{5}, 0) = -10\sqrt{5}$.

(d) The global maximum is obtained at $(\sqrt{5}, 0)$, where $f(\sqrt{5}, 0) = 10\sqrt{5}$. When the independent term b changes from 5 to 5.1, we get

$$\Delta \text{ optimal value } \approx \lambda \Delta b = 3\sqrt{5}(5.1 - 5) = (0.3)\sqrt{5}.$$

In consequence, the new optimal value is approximately $10\sqrt{5} + 0.3\sqrt{5} = (10.3)\sqrt{5}$.

$\boxed{4}$

Consider the function $f(x,y) = \dfrac{x^4}{2} - y^4 + 2y^2$ defined on the set

$$A = \{(x,y) \,:\, x^2 + y^2 \leq 4\}.$$

(a) (6 points) Establish the Kuhn–Tucker necessary optimality conditions to the problem

$$\max f(x,y) \quad \text{subject to } (x,y) \in A.$$

(b) (9 points) Find all the solutions of the Kuhn–Tucker conditions established in part (a).

(c) (3 points) Find the global maximum of f on A.

Solution:

(a) The Lagrangian is $L(x,y,\lambda) = \dfrac{x^4}{2} - y^4 + 2y^2 + \lambda(4 - x^2 - y^2)$. The K–T necessary conditions of optimality are

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 2x^3 - 2\lambda x = 2x(x^2 - \lambda) = 0, \\
\frac{\partial L}{\partial y} &= -4y^3 + 4y - 2\lambda y = 2y(-2y^2 + 2 - \lambda) = 0, \\
\lambda\frac{\partial L}{\partial \lambda} &= \lambda(4 - x^2 - y^2) = 0, \\
\lambda &\geq 0, \\
4 - x^2 - y^2 &\geq 0.
\end{aligned}$$

(b) We work first with the system of equalities

(i) $2x(x^2 - \lambda) = 0$

(ii) $2y(-2y^2 + 2 - \lambda) = 0$

(iii) $\lambda(4 - x^2 - y^2) = 0$.

We find the following solutions (x,y) and λ (not all of them satisfy K–T):

(1) $(0,0)$ with $\lambda = 0$.

(2) $(0,\pm 1)$ with $\lambda = 0$, since both (i) and (iii) are fulfilled, and (ii) is $-2y^2 + 2 - \lambda = -2 + 2 = 0$.

(3) $(0,\pm 2)$ with $\lambda = -6$. For, supposing that $x = 0$, we get from (iii) $y = \pm 2$ and then from (ii) $\lambda = -2y^2 + 2 = -8 + 2 = -6$

(4) $(\pm 2, 0)$ with $\lambda = 4$, since supposing that $y = 0$, we get from (iii) $x = \pm 2$ and then from (i) $\lambda = x^2 = 4$.

There are no solutions with both $x \neq 0$ and $y \neq 0$. For, if this were the case, then we would have from (i) $\lambda = x^2 > 0$ and from (ii) $\lambda = 2 - 2y^2$, hence $x^2 = 2 - 2y^2$. Plugging this equality into the feasibility condition, which holds with an equality because $\lambda > 0$, we would obtain $x^2 + y^2 = 2 - 2y^2 + y^2 = 4$, and thus $-y^2 = 2$, which is impossible. Of course, we can eliminate case (3) since $\lambda < 0$.

In summary, these are the points satisfying K–T conditions: $(0,0)$, $(0,\pm 1)$, and $(\pm 2, 0)$.

(c) Since that f is continuous and A is compact, we can apply the Theorem of Weierstrass and conclude that f attains global maximum on A. Evaluating, we get $f(0,0) = 0$, $f(0,\pm 1) = 1$ and $f(\pm 2, 0) = 8$, thus $(\pm 2, 0)$ are the points where f attains the maximum value on A.