$\boxed{1}$

Let the set

$$A = \left\{(x, y) \in \mathbb{R}^2 : \quad g(x) \le y \le h(x), \quad 2 \le x \le 4\right\}$$

with

$$g(x) = \ln(x-1), \quad h(x) = \frac{1}{x} + 2$$

and let the function

$$f(x, y) = \ln\left(y + \frac{x}{2} - 1\right).$$

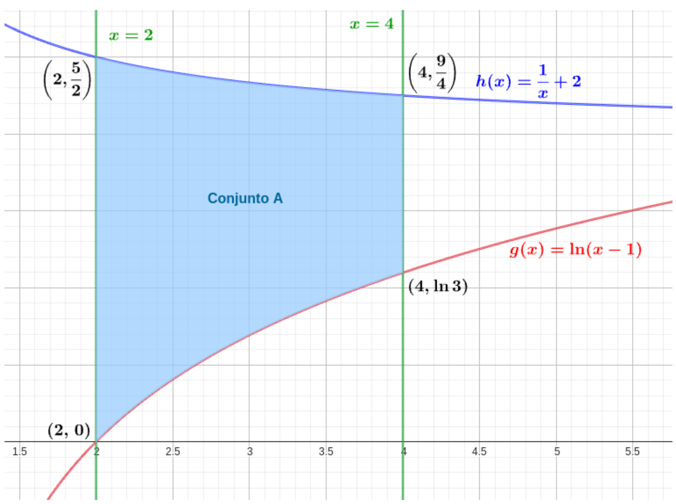
(a) (15 points) Draw the set A, justifying your drawing. Consider defined the order of Pareto on the set A; find maximals, minimals, maximum and minimum of A, justifying your answers. If you figure out that some of these elements do not exist, justify why.

(b) (5 points) Study if the function f and the set A fulfil the hypotheses of the Theorem of Weierstrass, and explain what does it mean.

(c) (10 points) Draw several level curves and several directions of fastest increase for the function f. Identify, the global maximum and the global minimum of f on A on your drawing and calculate these points if they exist. Justify your answers.

Solution:

(a) To obtain the figure below, note that h is decreasing and g is increasing, that $h(2) = \frac{5}{2}$, $g(2) = \ln 1 = 0$, $h(4) = 3$ and $g(4) = \ln 3 < 3$. Thus, $g(x) < 3 = h(4) \le h(x)$, for all $x \in [2, 4]$.



(b) The maximal points of A are those of the top boundary

$$\text{Maximal}(A) = \left\{(x, y) : \quad y = \frac{1}{x} + 2, \quad 2 \le x \le 4\right\}.$$

Then, there is no maximum.

The point $(2, 0) \in A$ is minimum and thus minimal.

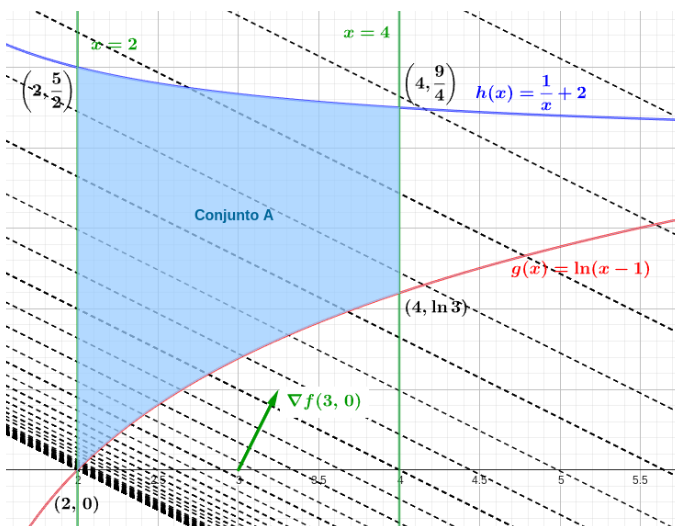
(c) A is closed, as it contains its boundary (since the set is defined by weak inequalities of continuous functions) and bounded, since it can be enclosed in a finite ball centered at $(0,0)$.

Function f is continuous on its domain. The domain of f is the set of points (x,y) where $y+\frac{x}{2}-1>0$. Note that the line $y+\frac{x}{2}-1$ equals 0 at the point $(2,0)\in A$ (remember that this is the furthest point of A in the southwest direction), and then it becomes negative at the right of that point. That is, $(2,0)$ is the only point of the set A which is not in the domain of f. But this suffices to get that f is not continuous on A. Thus, the hypotheses of the Theorem of Weierstrass are not fulfilled, and it could be the case that f does not attaint global maximum or global minimum on A.

(d) The level curves of f are lines given by

$$f(x)=\ln\left(y+\frac{x}{2}-1\right)=c\Rightarrow y+\frac{x}{2}-1=e^{c}\;\Rightarrow\; y=-\frac{x}{2}+1+e^{c}.$$

The lines of this family have all slope $-\frac{1}{2}$ and y–intercept $1+e^{c}$.



As $1+e^{c}$ increases with c, the value of f grows when the y–intercept of the level lines grows, that is, when the level line moves parallel in a northeast direction. Given the slope of the lines and the shape of set A, clearly $(4,\frac{9}{4})$ is the global maximum of f on A, with $f(4,\frac{9}{4})=\ln\frac{13}{4}$. Similarly, the value of f decreases when moving the level lines in a southwest direction. Note that when $c\to-\infty$, $1+e^{c}$ decreases to 1, but this limiting value is never attained. Thus, the level lines $y=-\dfrac{x}{2}+1+e^{c}$ converge in this sense to the line $y=-\dfrac{x}{2}+1$, which is the line that passes through the point $(2,0)$, where the function f is not defined. Thus f decreases along the set A in the southwest direction to the value $-\infty$, meaning that it is not bounded from below, and thus that there is no global minimum of f on A.

$\boxed{2}$

Consider the function $f(x,y) = 3xy - x^2y - xy^2$.

(a) (10 points) Find the critical points of f.

(b) (10 points) Classify the critical points found above (local maximum, local minimum or saddle point).

(c) (5 points) Determine whether the local extrema of f found above are global extrema.

Solution:

(a) $\nabla f(x,y) = (y(3-2x-y), x(3-x-2y)) = (0,0)$. The critical points are the solutions to this system of equations. It is easy to find that the solutions are $P = (0,0)$, $Q = (3,0)$, $R = (0,3)$ and $S = (1,1)$.

(b) The Hessian matrix of f is

$$Hf(x,y) = \begin{pmatrix} -2y & 3-2x-2y \\ 3-2x-2y & -2x \end{pmatrix}.$$

At the critical points, the sign of the Hessian is as follows.

$$Hf(P) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

is indefinite, thus P is a saddle point.

$$Hf(Q) = \begin{pmatrix} 0 & -3 \\ -3 & -6 \end{pmatrix}$$

is indefinite, thus Q is a saddle point.

$$Hf(R) = \begin{pmatrix} -6 & -3 \\ -3 & 0 \end{pmatrix}$$

is indefinite, thus R is a saddle point.

$$Hf(S) = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$$

is negative definite, thus S is a local maximum.

(c) $f(x,x) = 3x^2 - 2x^3$ is a cubic polynomial, which it is known that takes every value of the real line, thus f is not bounded above nor below.

$\boxed{3}$

Consider the following problem:

$$\max \qquad f(x,y) = x^3 + y^3$$

$$\text{subject to:} \quad g(x,y) = x + 4y = 63$$

(a) (10 points) Construct the Lagrangian of the problem and find its critical points.

(b) (10 points) Classify the critical points found above (local maximum, local minimum, or saddle point).

(c) (5 points) Determine whether the local extrema found above are global extrema of f subject to the constraint.

(d) (10 points) Now suppose that the constraint becomes $g(x,y) = 63$ and we add the nonnegative conditions $x \geq 0$, $y \geq 0$. Is it possible that any of the local extrema of f found above are global extrema of f subject to the new constraints?

Solution:

(a) The Lagrangian is

$L(x,y;\lambda) = x^3 + y^3 + \lambda(63 - x - 4y)$.

The necessary conditons of Lagrange are

i) $\partial L/\partial x(x,y;\lambda) = 3x^2 - \lambda = 0$

ii) $\partial L/\partial y(x,y;\lambda) = 3y^2 - 4\lambda = 0$

iii) $\partial L/\partial \lambda(x,y;\lambda) = 63 - x - 4y = 0$

Multiplying the first equation by 4 and equating to the second equation we find

$12x^2 = 4\lambda = 3y^2 \Longrightarrow y = 2x$ o $y = -2x$.

Substituting x, y into de third equation:

i) $9x = 63 \Longrightarrow (x^*, y^*) = (7, 14)$.

ii) $-7x = 63 \Longrightarrow (x^*, y^*) = (-9, 18)$.

(b) To classify the critical points obtained above we use second order conditions, calculating the Hessian matrix of L with respect to variables x, y. This Hessian matrix coincides with the Hessian matrix of f since the constraint $g(x,y)$ is linear. Then

$Hf(x,y) = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix}$; we have two cases

i) $Hf(7,14) = \begin{pmatrix} 42 & 0 \\ 0 & 84 \end{pmatrix}$ positive definite.

Hence $(7,14)$ is a local minimizer of $f(x,y)$, subject to the constraint.

ii) $Hf(-9,18) = \begin{pmatrix} -54 & 0 \\ 0 & 108 \end{pmatrix}$ indefinite, but restricted to the tangent plane $\{(v,w) : v + 4w = 0\} = \{(-4w, w)\}$, it becomes

$(-4w, w) \begin{pmatrix} -54 & 0 \\ 0 & 108 \end{pmatrix} \begin{pmatrix} -4w \\ w \end{pmatrix} = -756w^2 < 0$, if $(v,w) \neq (0,0)$. Thus, $Hf(-9,18)$ restricted to the tangent plane is negative definite and so $(-9,18)$ is a local maximizer.

(c) Clearly, $\lim_{x \longrightarrow \infty} f(x,y) = \infty$, and $\lim_{x \longrightarrow -\infty} f(x,y) = -\infty$, for suitable selection of points (x, y) satisfying the constraint, thus there is no global extrema of f on the constraint set.

(d) The point $(7, 14)$ satisfies the new constraints and since y since the feasible set now is compact, the function $f(x, y)$ must have a global maximum and a global minimum. To check that $(7, 14)$ is the global minimum, we must compare with the points $(63, 0)$ and $(0, 63/4)$.

$f(63, 0) = 63^3 = 9^3 7^3 > 9.7^3 = 7^3 + 14^3 = f(7, 14)$,

$f(0, 63/4) = 63^3/4^3 = 9^3 7^3/64 > 9 \cdot 7^3 = f(7, 14)$, since simplifying, $81 > 64$.

Hence $(7, 14)$ is the global minimizer.

$\boxed{4}$

Consider the program

$$\begin{array}{ll} \text{minimize} & x^2 + 2y^2 + 2x - 4y \\ \text{s.t.} & y \leq x \\ & x \geq -y \end{array}$$

(a) (20 points) Obtain the solutions of the Kuhn-Tucker equations for the program.

(b) (10 points) Justify that the program has a global solution.

Solution:

1. Writing the program in standard form

$$\begin{array}{ll} \text{maximize} & -(x^2 + 2y^2 + 2x - 4y) \\ \text{s.t.} & -x + y \leq 0 \\ & -x - y \leq 0 \end{array}$$

The lagrangian is

$L(x, y, \lambda_1, \mu) = -x^2 - 2y^2 - 2x + 4y + \lambda(x - y) + \mu(x + y)$

The K-T equations correspond to the following equalities and inequalities:

$$-2x - 2 + \lambda + \mu = 0 \quad (1)$$
$$-4y + 4 - \lambda + \mu = 0 \quad (2)$$
$$\lambda(x - y) = 0 \quad (3)$$
$$\mu(x + y) = 0 \quad (4)$$
$$-x + y \leq 0 \quad (5)$$
$$-x - y \leq 0 \quad (6)$$
$$\lambda, \mu \geq 0 \quad (7)$$

Solving the first four equations:

- in equation (3) option $x - y = 0$ gives

$$-2x - 2 + \lambda + \mu = 0$$
$$-4x + 4 - \lambda + \mu = 0$$
$$\mu(x + y) = 0$$

If $x + y = 0$ we have that $x = 0$ e $y = 0$ y $\mu = -1 < 0$ which does not satisfy (7).
If $\mu = 0$ solving for the first two equations $x = \frac{1}{3}$ $y = \frac{1}{3}$ $\lambda = \frac{8}{3}$ y $\mu = 0$

- in equation (3) the option $\lambda = 0$ gives

$$-2x - 2 + \mu = 0$$
$$-4y + 4 + \mu = 0$$
$$\mu(x + y) = 0$$

If $\mu = 0$ we get $x = -1$ $y = 1$ which does not satisfy (5).
And if $x + y = 0$ we have

$$\begin{aligned} -2x - 2 + \mu &= 0 \\ 4x + 4 + \mu &= 0 \end{aligned}$$

with solution $x = -1$ $y = 1$ which does not satisfy (5).

2. The original program is convex since the goal function is convex

$$Hf(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix},$$

where $f(x,y) = x^2 + 2y^2 + 2x - 4y$, and the inequalities, being linear, define a convex feasible region. Therefore the solution of the K-T equations, $x = \frac{1}{3}$ $y = \frac{1}{3}$ $\lambda = \frac{8}{3}$ y $\mu = 0$ corresponds to a global maximum of the standard program and to a global minimum of the original one.

Notice that the region is not bounded so Weierstrass theorem cannot be applied.