UC3M
Mathematical Optimization for Economics
Final Exam July 2, 2024 **Soluciones**

1

Consider the plane set

$$A = \left\{ (x,y) \in \mathbb{R}^2 : x > 1,\ y \geq \frac{3}{4}(x-1),\ y \leq \frac{x}{x-1} \right\}.$$

(a) (10 puntos) Represent A graphically.

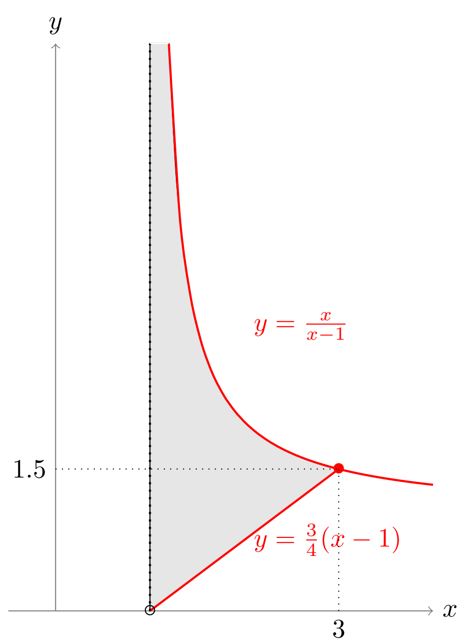
(b) (10 puntos) Considering the order of Pareto defined on A, find the maximals, minimals, maximum and minimum of A, justifying your answers. If you figure out that some of those elements do not exist, justify your answer.

(c) (10 puntos) Study graphically the existence of global extrema of $f(x,y) = y - 2x$ on A, and find them when they exist. Use the concepts of level curve and fastest increase/decrease directions, drawing some of these elements on the graph of A.

(d) (10 puntos) Study graphically the existence of global extrema of $f(x,y) = y - 0.5x$ on A, and find them when they exist. Use the concepts of level curve and fastest increase/decrease directions, drawing some of these elements on the graph of A.

Solución:

(a) The figure below represents A. Note that the mapping $x \mapsto \frac{x}{x-1}$ has a vertical asymptote at $x = 1$, and hence the set A is not bounded, nor closed.



(b) The upper boundary of A belongs to A and it is formed by points which are not comparable to each other. Also, there are no points in A which are preferred to the points in the upper boundary, thus this is the maximal set of A: $\{(x,y) \in \mathbb{R}^2 : 1 < x \leq 3, y = \frac{x}{x-1}\}$. There is no maximum. There are no minimals nor minimum. The element $(1,0)$ is a lower bound of A and clearly it is the greatest lower bound, but it is not in A, thus it is the infimum of A, but is not minimum.

(c) Let $f(x,y) = y - 2x$. The level curves given by $y - 2x = c$ are parallel lines of slope 2 and ordinate at the origin c. We draw the line with $c = 0$ and study whether parallel movements increase or decrease the ordinate at the origin over A.

The ordinate at the origin increases with NW movements. Since A is unbounded in the NW direction, there is no maximum of f in A. See the figure below, left.

The ordinate at the origin decreases with SE movements. $(3, 1.5)$ is the minimum. See the figure below, left.

Hence f has no maximum and $(3, 1.5)$ is the global minimum.

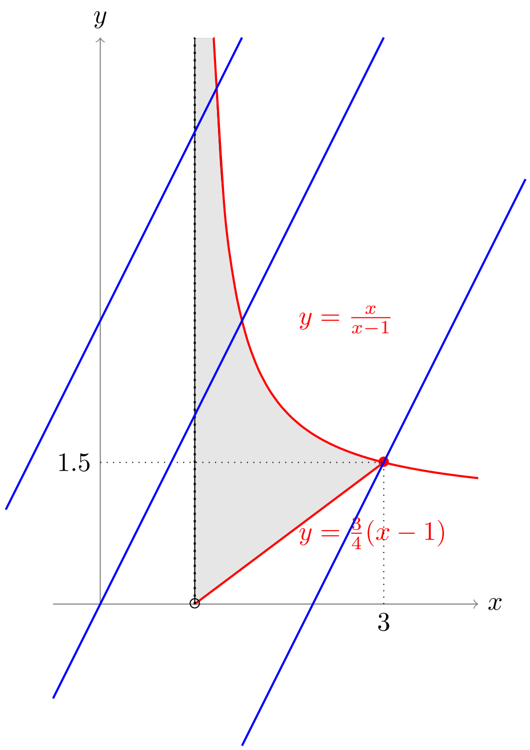
(d) Let $f(x, y) = y - 0.5x$. The level curves given by $y - 0.5x = c$ are parallel lines of slope 0.5 and ordinate at the origin c. We draw the line with $c = 0$ and study whether parallel movements increase or decrease the ordinate at the origin over A.

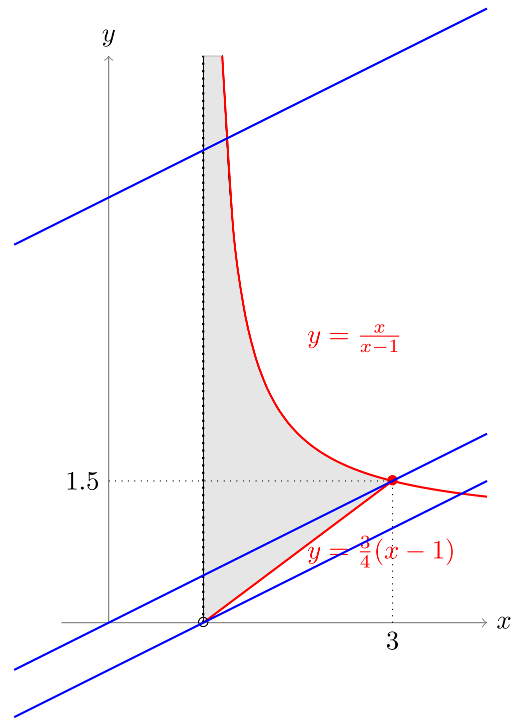
Again, the ordinate at the origin increases with NW movements. Since A is unbounded in the NW direction, there is no maximum of f in A. See the figure below, right.

The point $(1, 0)$ is the furthest point of the closure of A in the SE direction touching a level line, but it is not in A. Hence, f has no minimum in A. See the figure below, right.

Hence f has no maximum nor minimum in A.

The figure below left corresponds to $f(x, y) = y - 2x$, and right to $f(x, y) = y - 0.5x$





2

Consider the function $f(x,y) = x^3 + y^2 - 2xy - x + 6$.

(a) (10 puntos) Determine the critical points of f.

(b) (10 puntos) Classify the critical points obtained above. Clarify whether they are global extrema.

Solución:

(a) $\nabla f(x,y) = (3x^2 - 2y - 1, 2y - 2x) = (0,0)$. The critical points are the solutions of these two equations. We have $y = x$ and hence $3x^2 - 2x - 1 = 0$, obtaining solutions $x = 1$ and $x = -\frac{1}{3}$. Hence the critical points are $P = (1,1)$ and $Q = (-\frac{1}{3}, -\frac{1}{3})$.

(b) The Hessian matrix of f is

$$Hf(x,y) = \begin{pmatrix} 6x & -2 \\ -2 & 2 \end{pmatrix}.$$

Restricted to the critical points:

$$Hf(P) = \begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix}$$

is definite positive, thus P is a local minimum.

$$Hf(Q) = \begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}$$

is indefinite, thus Q is a saddle point.

P is not global, since $f(x,0) = x^3 - x$ tends to $-\infty$ as $x \to -\infty$.

3

Consider the following problem of Lagrange:

$$\begin{aligned} &\text{optimize} \quad && f(x,y,z) = 2x - y \\ &\text{subject to:} \quad && g(x,y,z) = (x-y)^2 + 3y^2 + z^2 = 39 \end{aligned}$$

(a) (5 puntos) Show without solving that the problem admits global solutions.

(b) (10 puntos) Construct the Lagrangian and find its critical points.

(c) (10 puntos) Determine the global maximum and minimum of f subject to the constraint and the optimal value of f in each case.

(d) (10 puntos) Assuming that the constraint becomes $g(x,y,z) = (x-y)^2 + 3y^2 + z^2 = 39 + h$, where h is a negligible quantity added to the independent term, give approximations to the optimal (maximum and minimum) values of f subject to the new constraint.

Solución:

(a) The feasible set S is compact, since it is closed since it is defined with a equality given by a continuous function, and it is bounded: $|z| \le \sqrt{39}$, $|y| \le \sqrt{13}$ and $|x| \le \sqrt{13} + \sqrt{39}$.

Since f is continuous, by the Theorem of Weierstrass, it attains global maximum and global minimum in A.

(b) $L(x,y,z,\lambda) = 2x - y - \lambda((x-y)^2 + 3y^2 + z^2 - 39)$.

$$\begin{cases} \frac{\partial L}{\partial x} &= 2 - 2(x-y)\lambda &= 0, \\ \frac{\partial L}{\partial y} &= -1 - (-2(x-y) + 6y)\lambda &= 0, \\ \frac{\partial L}{\partial z} &= -2z\lambda &= 0 \end{cases}$$

$$\lambda = \frac{1}{x-y} = \frac{1}{2x-8y} \Rightarrow y = \frac{x}{7}$$

Notice that $x \ne y$ and $2x \ne 8y$. Also, $\lambda \ne 0$. Substituting into g, we get

$$(x - \frac{x}{7})^2 + 3(\frac{x}{7})^2 + z^2 = 39 \Rightarrow \frac{36}{49}x^2 + \frac{3}{49}x^2 + z^2 = 39 \Rightarrow \frac{39}{49}x^2 + z^2 = 39.$$

From $2z\lambda = 0$ and $\lambda \ne 0$ we have $z = 0$. Hence $\frac{39}{49}x^2 = 39$, thus $x^2 = 49$ and $x = \pm 7$.

We have found 2 critical points

$P = (7, 1, 0)$, with $\lambda_P = \frac{1}{7-1} = \frac{1}{6}$ and

$Q = (-7, -1, 0)$, with $\lambda_Q = \frac{1}{(-7)-(-1)} = -\frac{1}{6}$.

There are no more critical points, since all points are regular:

$\nabla g(x,y,z) = (2(x-y), -2(x-y)+6y, 2z) = (0,0,0) \Leftrightarrow x = y, z = 0,$ and $-2(x-y)+6y = 0 \Leftrightarrow x = y = z = 0$,

but $(0,0,0)$ does not satisfy the constraint.

(c) By part (a) above, we evaluate f at P and Q, obtaining $f(P) = 13$ and $f(Q) = -13$. P is maximum and Q is minimum.

(d) The change of the maximum is approximately $h\lambda_P = \frac{h}{6}$ and the minimum $h\lambda_Q = -\frac{h}{6}$. Hence, if we denote by P_h and by Q_h the maximum and the minimum of f on the new feasible set, the optimal values will be

$$f(P_h) \approx f(P) + \frac{h}{6} = 13 + \frac{h}{6}$$

and

$$f(Q_h) \approx f(Q) - \frac{h}{6} = -13 - \frac{h}{6},$$

respectively.

4

Consider the Kuhn–Tucker problem

$$\begin{aligned} \text{maximize} \quad & x + y \\ \text{subject to:} \quad & x^2 + y^2 \le 4, \\ & y \le 1. \end{aligned}$$

(a) (10 puntos) Discuss whether the optimization problem is a convex one, and if the necessary Kuhn–Tucker conditions are also sufficient.

(b) (15 puntos) Solve the Kuhn–Tucker necessary conditions and solve the optimization problem.

Solución:

(a) Yes, it is a convex program, since the objective function is concave (linear) and the feasible set is convex. The feasible set $A = \{(x, y) : x^2 + y^2 \le 4, y \le 1\}$ is the intersection of a circle with a closed semiplane, thus it is convex. The K–T conditions are necessary and sufficient and thus any point satisfying them is a global maximum.

(b) $L(x, y, \lambda, \mu) = x + y + \lambda(4 - x^2 - y^2) + \mu(1 - y)$. Note that $L_x = 1 - 2\lambda x$ and $L_y = 1 - 2\lambda y - \mu$. The K–T conditions are

$$1 - 2\lambda x = 0, \tag{1}$$
$$1 - 2\lambda y - \mu = 0, \tag{2}$$
$$\lambda(4 - x^2 - y^2) = 0, \tag{3}$$
$$\mu(1 - y) = 0, \tag{4}$$
$$\lambda \ge 0, \tag{5}$$
$$\mu \ge 0, \tag{6}$$
$$4 - x^2 - y^2 \ge 0, \tag{7}$$
$$1 - y \ge 0 \tag{8}$$

(I) Case $\lambda = 0$, any μ

From (1) we obtain the contradiction 1=0.

(II) Case $\lambda > 0$ and $\mu = 0$.

From (1) and (2), $x = y$. Plugging this into (3), $4 - 2x^2 = 0$, that is, $x = \pm\sqrt{2}$. We obtain two points, $(x_1, y_1) = (\sqrt{2}, \sqrt{2})$ and $(x_2, y_2) = (-\sqrt{2}, -\sqrt{2})$.

From (1) we obtain $\lambda = \pm\frac{1}{2\sqrt{2}}$. We eliminate the point that corresponds to negative values of λ, $(x_2, y_2) = (-\sqrt{2}, -\sqrt{2})$.

However, (x_1, y_1) does not satisfy (8), since $1 - \sqrt{2} < 0$.

(III) Case $\lambda > 0$ and $\mu > 0$. These are points intersection of $x^2+y^2 = 4$ with the line $y = 1$: $(x_3, y_3) = (\sqrt{3}, 1)$, $(x_4, y_4) = (-\sqrt{3}, 1)$.

From (1), we obtain $\lambda = \pm\frac{1}{2\sqrt{3}}$; we eliminate the point corresponding to the negative multiplier, that is the point (x_4, y_4). The value of μ corresponding to (x_3, y_3) can be found from (2), obtaining $\mu = 1 - \frac{1}{\sqrt{3}} > 0$. All conditions (1)–(8) are fulfilled by $(x_3, y_3) = (\sqrt{3}, 1)$.

By (a), this is the global maximum of f in A.