UC3M
Mathematical Optimization for Economics
Final Exam, 19 May 2023 **Solutions**

$\boxed{1}$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = xy$. Consider the order of Pareto defined on the set

$$A = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2, \quad x \geq 0\right\}.$$

(a) (10 points) Draw the set A. Justify whether the function f and the set A satisfy the hypotheses of the Theorem of Weiersstrass.

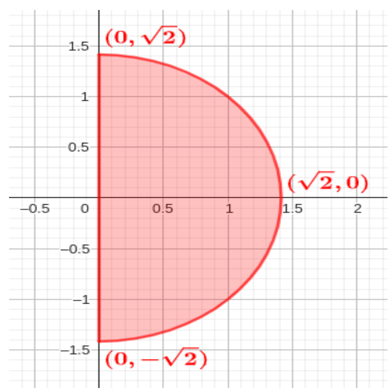
(b) (10 points) Calculate the maximal and minimal elements, and the maximum and the minimum of A, justifying your answers. If some of these elements do not exist, justify why.

(c) (10 points) Draw several level curves and several directions of fastest increase for the function f. Identify the global maximum and the global minimum of f on A on your drawing and calculate these points. Justify your answers.

Solution:

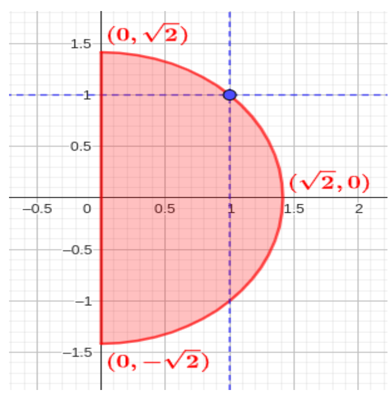
(a)

The set A is a semicircle, see the figure at the right. The set A is closed, since it contains its boundary points, and bounded, hence A is compact. The function f is continuous, thus the hypotheses of the Theorem of Weierstrass are fulfilled. Th function f attains global maximum and global minimum on A.

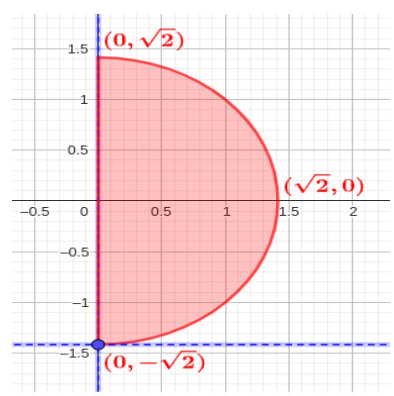


(b)

The figure represents the preference cone associated to a point with the order of Pareto. Clearly, the set of maximals is $\{(x, y) \in A : x^2 + y^2 = 2, y \geq 0\}$. A has no maximum, since the set of maximals is not a singleton.



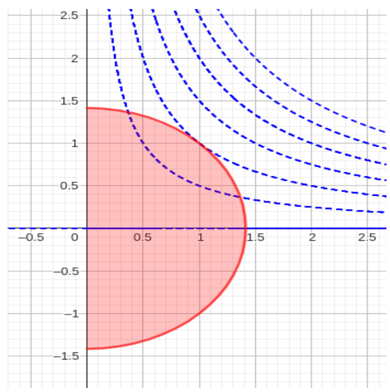
In this other graph, we see that the set A has only one minimal point, which is in fact the minimum of the set.The minimum of A is $\{(0, -\sqrt{2})\}$.



(c) The level curves are $C_k = \{(x, y) : xy = k\}$, $k \in .\mathbb{R}$. Note that C_0 is $OX \cup OY$ and for $k \neq 0$, C_k is the graph of the hyperbola $y = \frac{k}{x}$. On the set A for values $x > 0$, the hyperbolas with $k > 0$ move away from the origin in the NE direction with increasing k, whereas for $k < 0$ they move away from the origin in the SW direction with decreasing k. Set A, level curves and directions of maximum increase of f are represented in the graph above. Note that

$$\nabla f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y, x)$$

There is a unique maximizer of f on A, defined by the tangency point of the boundary of A with the level curve of highest level touching the set. See the graph below



To identify that point, we first compute the intersection points of a level curve $y = \frac{k}{x}$ with the curved boundary of A, $x^2 + y^2 = 2$.

$$\left.\begin{array}{r} y = \dfrac{k}{x} \\[2ex] x^2 + y^2 = 2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} y^2 = \dfrac{k^2}{x^2} \\[2ex] x^2 + \dfrac{k^2}{x^2} = 2 \Rightarrow x^4 + k^2 - 2x^2 = 0 \end{array}\right\} \xrightarrow{x^2 = t} t^2 - 2t + k^2 = 0 \Rightarrow$$

$$\Rightarrow t = \frac{2 \pm \sqrt{4 - 4k^2}}{2}$$

For the meeting point to be unique (the tangent point in this case), we impose that the discriminant of the second order equation is null, that is,

$$4 - 4k^2 = 0 \Rightarrow 4k^2 = 4 \Rightarrow k^2 = 1 \Rightarrow k = 1.$$

($k = -1$ is the minimum level of f on A). The tangent point is $(1, 1)$, which is the global maximum of f on A.

In a similar fashion, it can be shown that the minimum of f on A is the point $(1, -1)$.

$\boxed{2}$

Consider the function $f(x,y) = (1-xy)^2$.

(a) (10 points) Find the critical points of f.

(b) (10 points) Classify the critical points found above.

(c) (10 points) Say whether the function f has global maximum and/or global minimum and find them when they exist.

Solution:

(a) The gradient of f is $(-2y(1-xy), -2x(1-xy))$. Hence, the critical points satisfy the system

$$\begin{cases} y(1-xy) = 0 \\ x(1-xy) = 0. \end{cases}$$

This system admits the solutions $(0,0)$ and (x,y) such that $xy = 1$, or $y = \frac{1}{x}$, where $x \neq 0$. If we denote $x = t$, these points are $(t, \frac{1}{t})$, with $t \neq 0$.

(b) The Hessian matrix of f is

$$Hf(x,y) = \begin{pmatrix} 2y^2 & -2+4xy \\ -2+4xy & 2x^2 \end{pmatrix}.$$

At the point $(0,0)$, $Hf(0,0) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$, which is indefinite, thus $(0,0)$ is a saddle point of f. At points of the form $(t, \frac{1}{t})$

$$Hf(t, \frac{1}{t}) = \begin{pmatrix} \frac{2}{t^2} & 2 \\ 2 & 2t^2 \end{pmatrix}.$$

The principal minors are $\frac{1}{t^2} > 0$ and $2\cancel{t^2}\frac{2}{\cancel{t^2}} - 4 = 0$, thus $Hf(t, \frac{1}{t})$ is positive semidefinite, and the second order conditions do not allow to classify these points. The only information that we get is that the points $(t, \frac{1}{t})$ cannot be maximum.

(c) Note that $f(t, \frac{1}{t}) = 0$ and that $f(x,y) = (1-xy)^2 \geq 0$ for all (x,y), thus all points $(t, \frac{1}{t})$ attain the global minimum of f. However, f has no global maximum, since $f(1,t) = (1-t)^2$ goes to $+\infty$ as $t \to \infty$.

$\boxed{3}$

The production function for a firm is $5x + xy + 3y$, where x is labor and y is capital. Each unit of labor costs 15 monetary units (m.u.), whereas each unit of capital costs 3 m.u. The total budget that the company spends is 3000 m.u.

(a) (15 points) By solving the Lagrange equations associated to the optimization problem, find the optimal level of production for the firm. (Note that the answer which is not based on the Lagrange equations will have 0 points).

(b) (15 points) How does respond the maximum production to changes in the total budget? How does it impact a budget increase of 1 m.u.? If we want our production to increase 1%, what increase of budget should we made approximately.

Solution:

(a) This optimization problem is

$$\begin{array}{ll} \text{Maximize} & f(x,y) = 5x + xy + 3y \\ \text{subject to} & 15x + 3y = 3000 \end{array}$$

Lagrangian is $L(x, y, \lambda) = 5x + xy + 3y + \lambda(3000 - 15x - 3y)$. Note that

$$\nabla(15x + 3y) = (15, 3) \neq (0, 0),$$

thus the Theorem of Lagrange can be applied.

Lagrange's equations

$$\begin{cases} \frac{\partial L}{\partial x}(x, y, \lambda) &=& 5 + y - 15\lambda = 0 \\ \frac{\partial L}{\partial y}(x, y, \lambda) &=& x + 3 - 3\lambda = 0 \\ \frac{\partial L}{\partial \lambda}(x, y, \lambda) &=& 3000 - 15x - 3y = 0 \end{cases}$$

The first two equations allow to equate $\lambda = \frac{5+y}{15} = \frac{x+3}{3}$. Then we obtain a relation between x and y, for instance $y = 5x + 10$, which can be substituted into the third equation to obtain $30x + 30 = 3000$, that is, $x + 1 = 100$, and thus $x = 99$. This value gives $y = 5 \times 99 + 10 = 505$, and $\lambda = \frac{99}{3} + 1 = 33 + 1 = 34$.

To see if the critical point found is the solution, note that we have to check the second order conditions. The Hessian matrix of the Lagrangian with respect to variables (x, y),

$$H_{(x,y)}L(x, y, \lambda) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

is indefinite at the point $(99, 505, 34)$ (and at every point), thus we make a more local study. We know that $\nabla(15x+3y) = (15, 3)$, thus the tangent space of the constraint $15x+3y = 3000$ at the point $(99, 505)$ (and in fact at every point of the feasible set, since the constraint is linear) is given by the set of points (h, k) such that $(15, 3) \cdot (h, k) = 15h + 3k = 0$. Thus, the tangent space is the set $\{(h, -5h) : h \in \mathbb{R}\}$. The quadratic form defined by $H_{(x,y)}L(99, 505, 34)$ above is $2hk$, which restricted to the tangent space becomes $-10h^2$, which is definite negative, thus $(99, 505)$ is a (local) maximum.

The optimal production is thus $f(99, 505) = 52005$.

(b) Change in production due to an infinitesimal change in budget is given by the Lagrange multiplier. In this case $\lambda = 34 > 0$, thus a budget increase of 1 m.u. produces an increase of approximately 34 in production (approximately since a rise of 1 m.u. is not an infinitesimal change).

Now, 1% of optimal production is $\frac{52005}{100} = 520.05$. Since 1 unit of budget increment changes production in approximately 34 units, $\frac{520.05}{34} \approx 15,3$ m.u. of budget increase will produce the 1% change in production.

$\boxed{4}$

Consider the Kuhn–Tucker problem

$$\max\ -x^2 + 2y^2 - 4y + 10$$
$$\text{subject to: } x^2 + (y-2)^2 \leq 1.$$

(a) (10 points) Justify if the function f and the set A satisfy the conditions of the Theorem of Weierstrass. Is the problem a convex problem?

(b) (10 points) Find all the points that satisfy the necessary Kuhn–Tucker conditions (do not forget the value of the multiplier).

(c) (10 points) Find the solution(s) of the Kuhn–Tucker problem.

Solution:

(a) Let $f(x,y) = -x^2 + 2y^2 - 4y + 10$, which is obviously a continuous function, because it is a polynomial, and let $g(x,y) = x^2 + (y-2)^2$. The set A denotes the feasible set given by $g(x,y) \leq 1$. This is a disk with boundary centered at $(0,2)$ and radius 1, it is clearly compact. Thus the conditions of Weierstrass' Theorem are fulfilled.

Clearly A is a convex set; however, the function f is not concave: its Hessian matrix is $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$, which is indefinite. Thus, the problem is not a convex problem.

(b) The Lagrangian is $L(x,y,\lambda) = -x^2 + 2y^2 - 4y + 10 + \lambda(1 - x^2 - (y-2)^2)$.

Note that $\nabla g(x,y) = (2x, 2(y-2)) = (0,0)$ only if $(x,y) = (0,2)$, which is interior to A. Thus every point of A is regular. The Khun–Tucker conditions are

$$\text{KT} \quad \begin{cases} \frac{\partial L}{\partial x}(x,y,\lambda) & = & -2x - 2x\lambda = 0 \\ \frac{\partial L}{\partial y}(x,y,\lambda) & = & 4y - 4 - 2(y-2)\lambda = 0 \\ \lambda \frac{\partial L}{\partial \lambda}(x,y,\lambda) & = & \lambda(1 - x^2 - (y-2)^2) = 0 \end{cases}$$

join with the inequalities $\lambda \geq 0$ and $x^2 + (y-2)^2 \leq 1$.

<u>Let us see if there are solutions with $\lambda = 0$</u>. Plugging this into the equations above we obtain

$$\begin{cases} -2x = 0 \\ 4y - 4 = 0 \\ 0 = 0 \end{cases}$$

with solution $x = 0$, $y = 1$. Note that $0^2 + (1-2)^2 = 1 \leq 1$, thus the point $(x,y,\lambda) = (0,1,0)$ satisfies K–T conditions.

<u>Let us see if there are solutions with $\lambda > 0$</u>.

From the first equation in KT, $-2x(1+\lambda) = 0$, we obtain again $x = 0$, and from the third one, the solutions saturate the constraint, $x^2 + (y-2)^2 = 1$, thus $(y-2)^2 = 1$, obtaining the solutions $y = 1$ and

$y = 3$. $(0, 1)$ was already found with $\lambda = 0$. Plugging the value $y = 3$ into the second eq. in KT we obtain $12 - 4 - 2\lambda = 0$, that is, $\lambda = 4 > 0$. Hence the point $(x, y, \lambda) = (0, 3, 4)$ satisfies K–T conditions.

There are no more possibilities.

(c) By part (a), the problem admits solution. By K–T Theorem, the solutions must satisfy the necessary K–T conditions. We have found candidates $(0, 1)$ and $(0, 3)$ in part (b). Thus, we simply evaluate f to pick the greatest value. Since $f(0, 1) = 8$ and $f(0, 3) = 16$, the solution is $(0, 3)$.