UC3M
Optimización Matemática para la Economía
Examen Final, 19 de mayo de 2023 Soluciones

1

Sea $f : \mathbb{R}^2 \to \mathbb{R}$ la función $f(x,y) = xy$. Considere el orden de Pareto definido sobre el conjunto

$$A = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2, \quad x \geq 0\right\}.$$

(a) (10 puntos) Representar el conjunto A. Calcular, si existen, los elementos maximales y minimales, el máximo y el mínimo de A. Si alguno de los elementos anteriores no existen, justificar por qué.

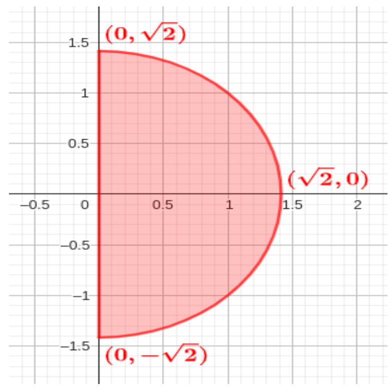
(b) (10 puntos) Justificar si la función f y el conjunto A verifican las hipótesis del Teorema de Weierstrass.

(c) (10 puntos) Representar algunas de las curvas de nivel y algunas de las direcciones de máximo crecimiento de la función f, superponiéndolas en el conjunto A. Identificar el máximo global y el mínimo global de f en A gráficamente y calcular ambos puntos. Justificar la respuesta.

Solución:

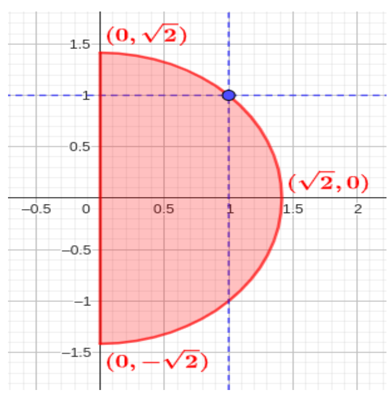
(a)

The set A is a semicircle, see the figure at the right. The set A is closed, since it contains its boundary points, and bounded, hence A is compact. The function f is continuous, thus the hypotheses of the Theorem of Weierstrass are fulfilled. Th function f attains global maximum and global minimum on A.

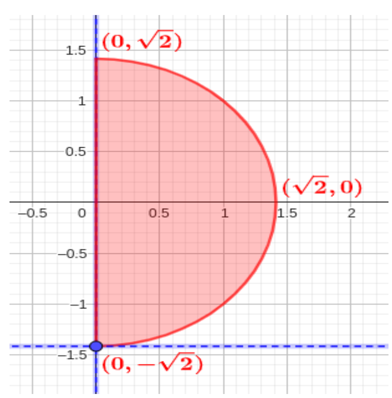


(b)

The figure represents the preference cone associated to a point with the order of Pareto. Clearly, the set of maximals is $\{(x,y) \in A : x^2 + y^2 = 2, y \geq 0\}$. A has no maximum, since the set of maximals is not a singleton.



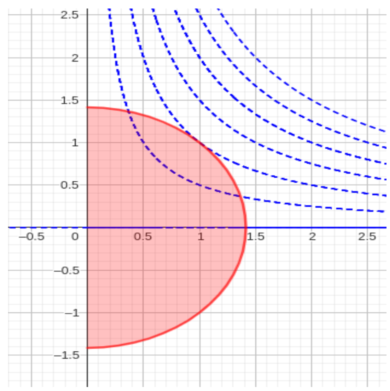
In this other graph, we see that the set A has only one minimal point, which is in fact the minimum of the set.The minimum of A is $\{(0, -\sqrt{2})\}$.



(c) The level curves are $C_k = \{(x,y) : xy = k\}$, $k \in .\mathbb{R}$. Note that C_0 is $OX \cup OY$ and for $k \neq 0$, C_k is the graph of the hyperbola $y = \frac{k}{x}$. On the set A for values $x > 0$, the hyperbolas with $k > 0$ move away from the origin in the NE direction with increasing k, whereas for $k < 0$ they move away from the origin in the SW direction with decreasing k. Set A, level curves and directions of maximum increase of f are represented in the graph above. Note that

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y, x)$$

There is a unique maximizer of f on A, defined by the tangency point of the boundary of A with the level curve of highest level touching the set. See the graph below



To identify that point, we first compute the intersection points of a level curve $y = \frac{k}{x}$ with the curved boundary of A, $x^2 + y^2 = 2$.

$$\left.\begin{array}{r} y = \dfrac{k}{x} \\ \\ x^2 + y^2 = 2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} y^2 = \dfrac{k^2}{x^2} \\ \\ x^2 + \dfrac{k^2}{x^2} = 2 \Rightarrow x^4 + k^2 - 2x^2 = 0 \end{array}\right\} \xrightarrow{x^2 = t} t^2 - 2t + k^2 = 0 \Rightarrow$$

$$\Rightarrow t = \frac{2 \pm \sqrt{4 - 4k^2}}{2}$$

For the meeting point to be unique (the tangent point in this case), we impose that the discriminant of the second order equation is null, that is,

$$4 - 4k^2 = 0 \Rightarrow 4k^2 = 4 \Rightarrow k^2 = 1 \Rightarrow k = 1.$$

($k = -1$ is the minimum level of f on A). The tangent point is $(1, 1)$, which is the global maximum of f on A.

In a similar fashion, it can be shown that the minimum of f on A is the point $(1, -1)$.

2

Se considera la función $f(x,y) = (1-xy)^2$.

(a) (10 puntos) Determinar los puntos críticos de f.

(b) (10 puntos) Clasificar los puntos críticos hallados en el apartado anterior.

(c) (10 puntos) Hallar, si existen, lo máximos y los mínimos globales de f.

Solución:

(a) The gradient of f is $(-2y(1-xy), -2x(1-xy))$. Hence, the critical points satisfy the system

$$\begin{cases} y(1-xy) = 0 \\ x(1-xy) = 0. \end{cases}$$

This system admits the solutions $(0,0)$ and (x,y) such that $xy = 1$, or $y = \frac{1}{x}$, where $x \neq 0$. If we denote $x = t$, these points are $(t, \frac{1}{t})$, with $t \neq 0$.

(b) The Hessian matrix of f is

$$Hf(x,y) = \begin{pmatrix} 2y^2 & -2+4xy \\ -2+4xy & 2x^2 \end{pmatrix}.$$

At the point $(0,0)$, $Hf(0,0) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$, which is indefinite, thus $(0,0)$ is a saddle point of f. At points of the form $(t, \frac{1}{t})$

$$Hf(t, \frac{1}{t}) = \begin{pmatrix} \frac{2}{t^2} & 2 \\ 2 & 2t^2 \end{pmatrix}.$$

The principal minors are $\frac{1}{t^2} > 0$ and $2\cancel{t^2}\frac{2}{\cancel{t^2}} - 4 = 0$, thus $Hf(t, \frac{1}{t})$ is positive semidefinite, and the second order conditions do not allow to classify these points. The only information that we get is that the points $(t, \frac{1}{t})$ cannot be maximum.

(c) Note that $f(t, \frac{1}{t}) = 0$ and that $f(x,y) = (1-xy)^2 \geq 0$ for all (x,y), thus all points $(t, \frac{1}{t})$ attain the global minimum of f. However, f has no global maximum, since $f(1,t) = (1-t)^2$ goes to $+\infty$ as $t \to \infty$.

3

La función de producción de una empresa es $5x + xy + 3y$, donde x denota las unidades de mano de obra y y las de capital. Cada unidad de mano de obra cuesta 15 unidades monetarias (u.m.), mientras que cada unidad de capital cuesta 3 u.m. El presupuesto total que gasta la empresa es de 3000 u.m.

(a) (15 puntos) Utilizando las ecuaciones de Lagrange del problema, encontrar el nivel de producción óptimo de la empresa. (Nota: la respuesta que no esté basada en las ecuaciones de Lagrange será calificada con 0 puntos).

(b) (15 puntos) ¿Cómo responde la producción óptima a cambios en el presupuesto total de la empresa? ¿Cuál es el impacto de un aumento presupuestario de 1 u.m.? ¿Qué aumento aproximado de presupuesto haría incrementar la producción un 1%?

Solución:

(a) This optimization problem is

$$\begin{array}{ll} \text{Maximize} & f(x,y) = 5x + xy + 3y \\ \text{subject to} & 15x + 3y = 3000 \end{array}$$

Lagrangian is $L(x, y, \lambda) = 5x + xy + 3y + \lambda(3000 - 15x - 3y)$. Note that

$$\nabla(15x + 3y) = (15, 3) \neq (0, 0),$$

thus the Theorem of Lagrange can be applied.

Lagrange's equations

$$\begin{cases} \frac{\partial L}{\partial x}(x,y,\lambda) &= 5 + y - 15\lambda = 0 \\ \frac{\partial L}{\partial y}(x,y,\lambda) &= x + 3 - 3\lambda = 0 \\ \frac{\partial L}{\partial \lambda}(x,y,\lambda) &= 3000 - 15x - 3y = 0 \end{cases}$$

The first two equations allow to equate $\lambda = \frac{5+y}{15} = \frac{x+3}{3}$. Then we obtain a relation between x and y, for instance $y = 5x + 10$, which can be substituted into the third equation to obtain $30x + 30 = 3000$, that is, $x + 1 = 100$, and thus $x = 99$. This value gives $y = 5 \times 99 + 10 = 505$, and $\lambda = \frac{99}{3} + 1 = 33 + 1 = 34$.

To see if the critical point found is the solution, note that we have to check the second order conditions. The Hessian matrix of the Lagrangian with respect to variables (x, y),

$$H_{(x,y)}L(x, y, \lambda) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

is indefinite at the point $(99, 505, 34)$ (and at every point), thus we make a more local study. We know that $\nabla(15x + 3y) = (15, 3)$, thus the tangent space of the constraint $15x + 3y = 3000$ at the point $(99, 505)$ (and in fact at every point of the feasible set, since the constraint is linear) is given by the set of points (h, k) such that $(15, 3) \cdot (h, k) = 15h + 3k = 0$. Thus, the tangent space is the set $\{(h, -5h) : h \in \mathbb{R}\}$. The quadratic form defined by $H_{(x,y)}L(99, 505, 34)$ above is $2hk$, which restricted to the tangent space becomes $-10h^2$, which is definite negative, thus $(99, 505)$ is a (local) maximum.

The optimal production is thus $f(99, 505) = 52005$.

(b) Change in production due to an infinitesimal change in budget is given by the Lagrange multiplier. In this case $\lambda = 34 > 0$, thus a budget increase of 1 m.u. produces an increase of approximately 34 in production (approximately since a rise of 1 m.u. is not an infinitesimal change).

Now, 1% of optimal production is $\frac{52005}{100} = 520.05$. Since 1 unit of budget increment changes production in approximately 34 units, $\frac{520.05}{34} \approx 15,3$ m.u. of budget increase will produce the 1% change in production.

4

Considerar el problema de Kuhn–Tucker siguiente

$$\max \ -x^2 + 2y^2 - 4y + 10$$

$$\text{sujeto a: } x^2 + (y-2)^2 \leq 1.$$

(a) (10 puntos) Justificar si el conjunto A y la función f satisfacen las condiciones del Teorem de Weierstrass. ¿Se trata de un problema convexo?

(b) (10 puntos) Encontrar todos los puntos que sumplen las condiciones necesarias de Kuhn–Tucker, incluyendo el valor del multiplicador, en cada caso.

(c) (10 puntos) Encontrar la(s) soluciones del problema de Kuhn–Tucker.

Solución:

(a) Let $f(x,y) = -x^2 + 2y^2 - 4y + 10$, which is obviously a continuous function, because it is a polynomial, and let $g(x,y) = x^2 + (y-2)^2$. The set A denotes the feasible set given by $g(x,y) \leq 1$. This is a disk with boundary centered at $(0,2)$ and radius 1, it is clearly compact. Thus the conditions of Weierstrass' Theorem are fulfilled.

Clearly A is a convex set; however, the function f is not concave: its Hessian matrix is $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$, which is indefinite. Thus, the problem is not a convex problem.

(b) The Lagrangian is $L(x,y,\lambda) = -x^2 + 2y^2 - 4y + 10 + \lambda(1 - x^2 - (y-2)^2)$.

Note that $\nabla g(x,y) = (2x, 2(y-2)) = (0,0)$ only if $(x,y) = (0,2)$, which is interior to A. Thus every point of A is regular. The Khun–Tucker conditions are

$$\text{KT} \quad \begin{cases} \frac{\partial L}{\partial x}(x,y,\lambda) &= -2x - 2x\lambda = 0 \\ \frac{\partial L}{\partial y}(x,y,\lambda) &= 4y - 4 - 2(y-2)\lambda = 0 \\ \lambda \frac{\partial L}{\partial \lambda}(x,y,\lambda) &= \lambda(1 - x^2 - (y-2)^2) = 0 \end{cases}$$

join with the inequalities $\lambda \geq 0$ and $x^2 + (y-2)^2 \leq 1$.

<u>Let us see if there are solutions with $\lambda = 0$</u>. Plugging this into the equations above we obtain

$$\begin{cases} -2x = 0 \\ 4y - 4 = 0 \\ 0 = 0 \end{cases}$$

with solution $x = 0$, $y = 1$. Note that $0^2 + (1-2)^2 = 1 \leq 1$, thus the point $(x,y,\lambda) = (0,1,0)$ satisfies K–T conditions.

<u>Let us see if there are solutions with $\lambda > 0$</u>.

From the first equation in KT, $-2x(1+\lambda) = 0$, we obtain again $x = 0$, and from the third one, the solutions saturate the constraint, $x^2 + (y-2)^2 = 1$, thus $(y-2)^2 = 1$, obtaining the solutions $y = 1$ and $y = 3$. $(0,1)$ was already found with $\lambda = 0$. Plugging the value $y = 3$ into the second eq. in KT we obtain $12 - 4 - 2\lambda = 0$, that is, $\lambda = 4 > 0$. Hence the point $(x,y,\lambda) = (0,3,4)$ satisfies K–T conditions.

There are no more possibilities.

(c) By part (a), the problem admits solution. By K–T Theorem, the solutions must satisfy the necessary K–T conditions. We have found candidates $(0,1)$ and $(0,3)$ in part (b). Thus, we simply evaluate f to pick the greatest value. Since $f(0,1) = 8$ and $f(0,3) = 16$, the solution is $(0,3)$.