UC3M
Mathematical Optimization for Economics
Final Exam, 20 June, 2023 **Solutions**

$\boxed{1}$

Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{1}{2 - (x + y)}.$$

Consider the order of Pareto defined on the set

$$A = \left\{(x, y) \in \mathbb{R}^2 : -1 \le x + y \le 1, \quad -(1 - x)^2 \le y \le (1 + x)^2\right\}.$$

(a) (10 points) Draw the set A, reasoning your answer. Calculate, if they exist, the maximal and minimal elements, and the maximum and the minimum of A, justifying your answers. If some of these elements do not exist, give reasons.

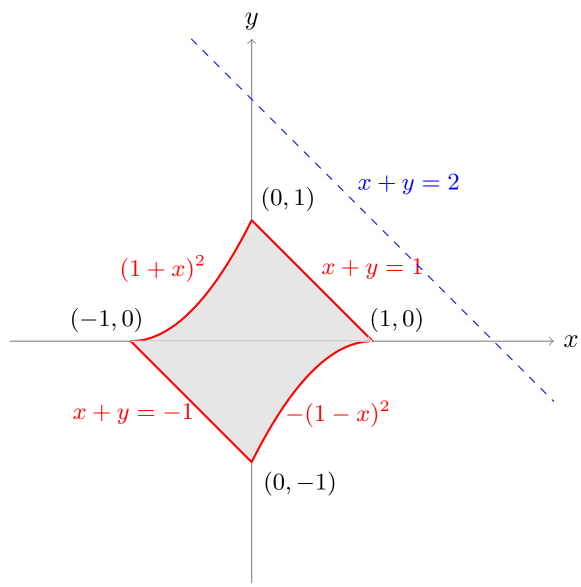
(b) (10 points) Justify whether the function f and the set A satisfy the hypotheses of the Theorem of Weierstrass.

(c) (10 points) Draw several level curves of f, superimposing them on the set A. Identify the points of A, if they exist, where f attains global maximum and/or minimum, and calculate the value of f on that points.

Solution:

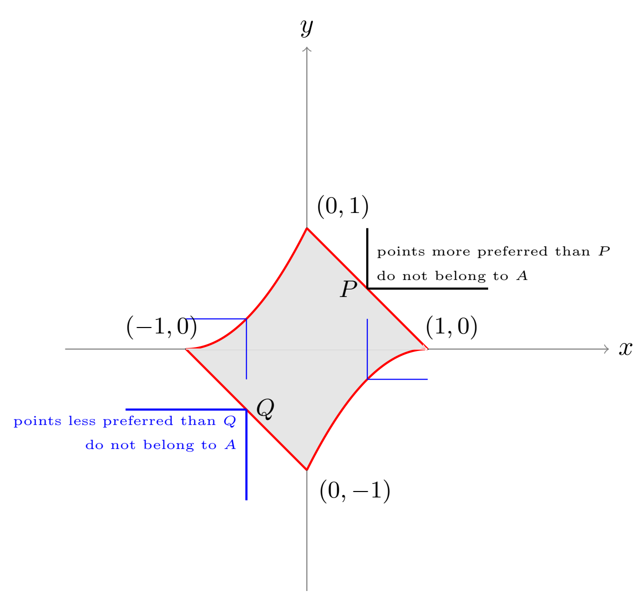
(a)

The set A is bounded by two branches of parabola and two lines with negative slope, see the figure at the right. The set A is closed, since it contains its boundary points, and bounded, hence A is compact. The function f is continuous on A since it is the quotient of continuous functions, where the denominator, $2-(x+y)$, is not null on A, since for all $(x, y) \in A$, $x + y \le 1$. Thus, the hypotheses of the Theorem of Weierstrass are fulfilled. The function f attains global maximum and global minimum on A.



(b)

The figure at the right represents four preference cones in the Pareto order, attached to different boundary points of A. Clearly, the set of maximals is the segment that joins $(1, 0)$ with $(0, 1)$, and the set of minimals is the segment joining $(-1, 0)$ and $(0, -1)$. A has no maximum nor minimum since the set of maximals (minimals) is not a singleton.



(c) The level curves are

$$C_k = \left\{ (x,y) \, : \, \frac{1}{2-(x+y)} = k \right\}, \quad \text{for } k \neq 0.$$

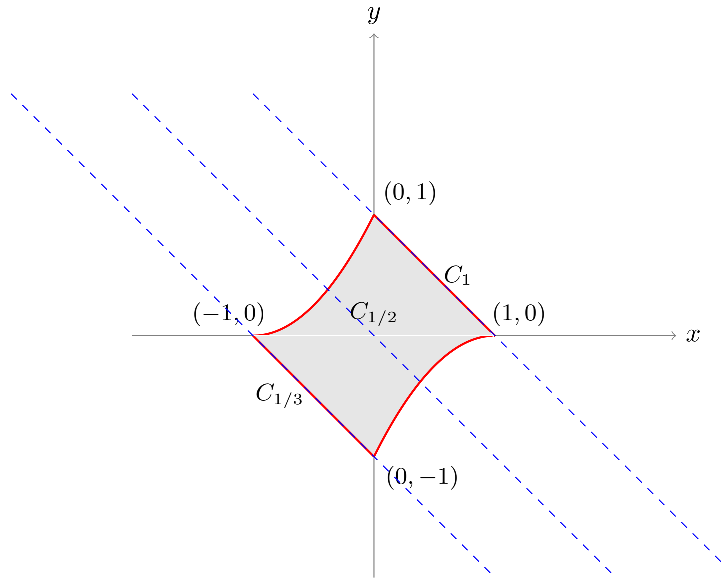
Manipulating a bit, the level curves are parallel lines

$$2 - (x+y) = \frac{1}{k},$$

that is,

$$x + y = 2 - \frac{1}{k}.$$

When k grows, that is, when the function f grows, the level curves move to the right in the plane OXY, and when k decreases, that is, when the function f decreases, the level curves move to the left in the plane OXY. See the companion figure. The global maxima of f on A is the whole segment joining $(0,1)$ and $(1,0)$, with value 1, and the global minima is the whole segment joining $(-1,0)$ and $(0,-1)$, with value $\frac{1}{3}$.



$\boxed{2}$

Let the function $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x, y) = y(2x^2 + y^2 + 2xy + 2y + 2).$$

(a) (10 points) Find the critical points of f.

(b) (10 points) Classify the critical points found in part (a) above.

Solution:

(a) The critical points (x, y) satisfy the system $\nabla f(x, y) = (0, 0)$, that is

$$\begin{cases} y(4x + 2y) = 0 \\ y(2y + 2x + 2) + (2x^2 + y^2 + 2xy + 2y + 2) = 0. \end{cases}$$

Suppose that $y = 0$. Then the second equation is $2x^2 + 2 = 0$, which does not have solution.

Suppose that $4x + 2y = 0$, that is, $y = -2x$. Then the second equation becomes

$$-2x(-4x + 2x + 2) + (2x^2 + 4x^2 - 4x^2 - 4x + 2) = 6x^2 - 8x + 2 = 0.$$

This quadratic equation admits solutions $x = 1$ and $x = \frac{1}{3}$.

Thus, f has two critical points, $(1, -2)$ and $(\frac{1}{3}, -\frac{2}{3})$.

(b) The Hessian matrix of f is

$$Hf(x, y) = \begin{pmatrix} 4y & 4x + 4y \\ 4x + 4y & 4x + 6y + 4 \end{pmatrix}.$$

At the point $(1, -2)$, $Hf(1, -2) = \begin{pmatrix} -8 & -4 \\ -4 & -4 \end{pmatrix}$, which is negative definite, thus $(1, -2)$ is a local maximum of f.

At $(1/3, -2/3)$

$$Hf(1/3, -2/3) = \frac{1}{3}\begin{pmatrix} -8 & -4 \\ -4 & 4 \end{pmatrix},$$

which is indefinite, thus $(1/3, -2/3)$ is a saddle point of f.

$\boxed{3}$

Let the Lagrange optimization problem given by

$$\begin{aligned} &\text{optimize} \quad f(x,y) := 15x + 3y \\ &\text{subject to:} \quad g(x,y) := 5x + xy + 3y = 30. \end{aligned}$$

(a) (5 points) Check that the regularity condition holds. Write the Lagrangian and the Lagrange equations.

(b) (10 points) Solve the Lagrange equations, finding all critical points (x^*, y^*, λ^*) of the Lagrangian.

(c) (10 points) Classify the critical points (x^*, y^*, λ^*) obtained in part (b) above (local maximum, local minimum, or saddle points).

Solution:

(a) Lagrangian is $L(x, y, \lambda) = 15x + 3y - \lambda(5x + xy + 3y - 30)$. Note that

$$\nabla g(x,y) = (5+y, 3+x) = (0,0) \quad \text{iff} \quad (x,y) = (-3,-5).$$

This point is not feasible, thus the Theorem of Lagrange can be applied.

Lagrange's equations

$$\begin{cases} \frac{\partial L}{\partial x}(x,y,\lambda) &= 15 - \lambda(5+y) = 0 \\ \frac{\partial L}{\partial y}(x,y,\lambda) &= 3 - \lambda(3+x) = 0 \\ \frac{\partial L}{\partial \lambda}(x,y,\lambda) &= -(5x + xy + 3y - 30) = 0 \end{cases}$$

(b) The first two equations allow to equate $\lambda = \frac{15}{5+y} = \frac{3}{x+3}$. Then we obtain a relation between x and y, for instance $y = 5x + 10$, which can be substituted into the third equation to obtain the quadratic

$$5x^2 + 30x = 0.$$

The solutions are $x = 0$ and $x = -6$. It is easy to get the two critical points

$$(x^*, y^*, \lambda^*) = (0, 10, 1) \text{ and } (x^{**}, y^{**}, \lambda^{**}) = (-6, -20, -1).$$

(c) The Hessian matrix of the Lagrangian with respect to variables (x, y) is

$$H_{(x,y)}L(x,y,\lambda) = \begin{pmatrix} 0 & -\lambda \\ -\lambda & 0 \end{pmatrix},$$

which is indefinite.

The tangent space at a feasible point (x, y) is given by $\nabla g(x, y) \cdot (h, k) = 0$, that is

$$(5+y)h + (3+x)k = 0. \tag{1}$$

At $(x^*, y^*) = (0, 10)$, the equation is $15h + 3k = 0$, or $k = -5h$. Since

$$H_{(x,y)}L(0,10,1) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

this quadratic form restricted to $15h + 3k = 0$ is $10h^2$, positive definite, thus $(0, 10)$ is a local minimum.

At $(x^*, y^*) = (-6, -20)$, the equation of the tangent plane (1) is $-15h - 3k = 0$, or $k = -5h$ (the same as above). Since

$$H_{(x,y)}L(-6,-20,-1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

this quadratic form restricted to $15h + 3k = 0$ is $-10h^2$, negative definite, thus $(-6, -20)$ is a local maximum.

$\boxed{4}$

Let us consider the Kuhn–Tucker optimization problem given by

$$\max \ f(x,y) := x^3 + y^2$$

$$\text{subject to: } g(x,y) := x^2 + y^2 \leq 1.$$

(a) (5 points) Draw the feasible set S. Justify whether S and the function f satisfy the conditions in Weierstrass Theorem. Check that all points of the feasible set are regular points.

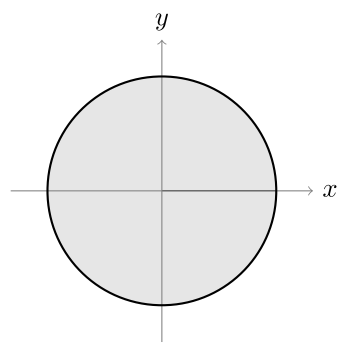
(b) (10 points) Write the necessary Kuhn-Tucker conditions for this problem (equalities and inequalities). Find all points which satisfy the Kuhn–Tucker conditions, multiplier included in each case.

(c) (10 points) Find the solution(s) to the Kuhn-Tucker problem.

Solution:

(a)

Let $f(x,y) = x^3+y^2$, which is obviously a continuous function, because it is a polynomial. The feasible set is a disk with center (0,0) and radius 1, which is compact, since it contains its boundary points and it is bounded. Thus the conditions of Weierstrass' Theorem are fulfilled. See the figure at the right.
Note that all feasible points are regular:
(i) All interior points are regular, by definition;
(ii) $\nabla(x^2+y^2) = (2x, 2y)$, that never vanishes at points $x^2+y^2 = 1$;.



(b) The Lagrangian is $L(x,y,\lambda) = x^3 + y^2 + \lambda(1 - x^2 - y^2)$. The Kuhn-Tucker conditions are formed by the system of equations

$$\text{KT} \quad \left\{ \begin{array}{rcl} \frac{\partial L}{\partial x}(x,y,\lambda,\mu) & = & x(3x - 2\lambda) = 0 \\ \frac{\partial L}{\partial y}(x,y,\lambda,\mu) & = & 2y(1-\lambda) = 0 \\ \lambda\frac{\partial L}{\partial \lambda}(x,y,\lambda,\mu) & = & \lambda(1 - x^2 - y^2) = 0 \end{array} \right.$$

and the inequalities $\lambda \geq 0$, $x^2 + y^2 \leq 1$.
Let us solve the KT system

$$\left\{ \begin{array}{lr} (1) & x(3x - 2\lambda) = 0 \\ (2) & 2y(1-\lambda) = 0 \\ (3) & \lambda(1 - x^2 - y^2) = 0 \end{array} \right.$$

and afterwards check the inequalities on the solutions.

Case 1: $(0,0,0)$ is obviously a solution

Case 2: $x = 0$, $y \neq 0$: then (2) implies $\lambda = 1$ and (3) implies $y^2 = 1$, or $y = \pm 1$. Thus we obtain

$$(0, \pm 1, 1).$$

Case 3: $x \neq 0$ and $y = 0$: then (1) implies $3x - 2\lambda = 0$, thus $\lambda = \frac{3}{2}x \neq 0$ and (3) implies $x^2 = 1$, or $x = \pm 1$. We obtain two solutions of the system

$$(1, 0, \frac{3}{2}) \quad \text{and} \quad (-1, 0, -\frac{3}{2}).$$

Case 4: $x \neq 0$ and $y \neq 0$: then (1) implies $3x - 2\lambda = 0$ and (2) $\lambda = 1$; thus, $x = \frac{2}{3}$ and from (3), $(2/3)^2 + y^2 = 1$ implies $y^2 = 1 - (4/9) = 5/9$, thus $y = \pm\sqrt{5/9}$. We obtain two solutions

$$(\frac{2}{3}, \sqrt{\frac{5}{9}}, 1) \quad \text{and} \quad (\frac{2}{3}, -\sqrt{\frac{5}{9}}, 1)$$

Now, we check the inequalities $\lambda \geq 0$ and $x^2 + y^2 \leq 1$ on the solutions found.

They are fulfilled in Case 1 and Case 2.

In Case 3, only $(1, 0, \frac{3}{2})$ pass the test, as the other point has a negative multiplier.

In Case 4, both solutions pass the test.

Thus we have 6 points satisfying KT conditions, which are listed in (c) below.

(c) By part (a), the problem admits solution. By KT Theorem, the solutions must satisfy the necessary KT conditions. We have found the candidates $(0, 0)$, $(0, \pm 1)$, $(1, 0)$ and $(\frac{2}{3}, \pm\sqrt{\frac{5}{9}})$ in part (b), which are marked in the figure below. Points $(0, \pm 1)$ and $(1, 0)$ are global maximum.

$$f(0,0) = 0, \quad f(\frac{2}{3}, \pm\sqrt{\frac{5}{9}}) = \left(\frac{2}{3}\right)^3 + \sqrt{\frac{5}{9}}^2 = \frac{8}{27} + \frac{5}{9} = \frac{23}{27} < 1, \quad f(0, \pm 1) = f(1, 0) = 1.$$