UC3M

Mathematical Optimization for Economics

Final Exam, 20 May 2022 **Solutions**

$\boxed{1}$

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be the function $F(x, y) = ax + y$, where $a \neq 0$. Consider the order of Pareto defined on the set

$$A = \left\{(x, y) \in \mathbb{R}^2 : y \geq x^2 - 2x;\ y \leq x + 4;\ x \geq 0\right\}.$$

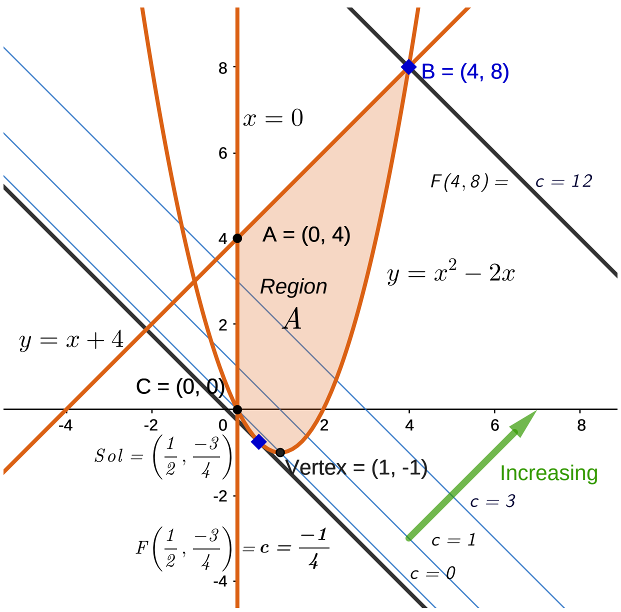
(a) (5 points) Draw the set A. Calculate, if they exist, the maximal and minimal elements, the maximum and the minimum of A. Justify your answers.

(b) (5 points) Suppose $a = 1$. Draw the curves of level $c = 0, 1, 3$ of F. Represent the increasing direction of the function. Calculate, if they exist the global maximum and global minimum of F on A.

(c) (5 points) Find the range of values of a such that the global maximum of F on A is attained at the point $(0, 4)$.

Solution:

(a)



Maximum(A) = Maximals$(A) = B(4, 8)$
Minimals$(A) = \{(x, y) : y = x^2 - 2x;\ 0 \leq x \leq 1\}$
Minimum$(A) = \emptyset$

(b) For $a = 1$, the function is $F(x, y) = x + y$. The level curves of the function $x + y = c$ for $c = 0, 1, 3$ and the direction for which the function is increasing are represented in the figure above. The function attains its maximum at the point $B(4, 8)$ with value $F(4, 8) = 12$. The minimum is attained at a point that belongs to the set of minimals, to calculate it we equalize the derivatives of the boundary curve with the set of level curves to find the point of tangency. $y = x^2 - 2x \to y' = 2x - 2$ and $x + y = c \to y = -x - 1 \to y' = -1$ then $2x - 2 = -1 \to x = \frac{1}{2}$,which correspond to the point in the parabola with $y = \frac{-3}{4}$, so the global minimizer is the point Sol $= (\frac{1}{2}, \frac{-3}{4})$ and the minimum value is $F(\frac{1}{2}, \frac{-3}{4}) = -\frac{1}{4}$.

(c) Note that the boundary AB of the set A is a segment of slope 1 and that the boundary OA is a vertical segment. For the point $A(0, 4)$ to be the global maximum, the gradient of F, $\nabla F = (a, 1)$, has to point NW, hence $a < 0$; thus, the slope of the level curves, a, must belong to the interval $(-\infty, -1]$.

$\boxed{2}$

A monopolistic firm produces two goods A and B, of which it sells x and y units per day, respectively. The cost function is given by

$$C(x, y) = x^2 + 4y^2 + 2xy - 20x + 30.$$

The unitary prices of the goods A and B are

$$p_A(x, y) = 60 - x - ay,$$
$$p_B(x, y) = 80 - 4y - ax,$$

respectively, where a is an unknown parameter.

(a) (5 points) Find the range of values of a for which the profit function of the firm is a concave function.

(b) (5 points) Let $a = 1$. Find the values of x and y which maximize the firm's profits.

(c) (5 points) Let $a = 1$. A new regulation requires to sell the products in packages formed by 1 unit of good B and 2 units of good A. Find the values of x e y which maximize the firm's profits.

Solution:

(a) Return function:

$$R(x, y) = x \cdot p_A + y \cdot p_B = 60x - x^2 - axy + 80y - 4y^2 - axy$$

Profits:

$$\begin{aligned}\Pi(x, y) &= R(x, y) - C(x, y) \\ &= 60x - x^2 - axy + 80y - 4y^2 - axy - (x^2 + 4y^2 + 2xy - 20x + 30) \\ &= -2x^2 - 8y^2 + 80x + 80y - (2 + 2a)xy - 30\end{aligned}$$

Hessian matrix of the profit function:

$$\frac{\partial \Pi}{\partial x} = -4x + 80 - (2 + 2a)y \Rightarrow \begin{cases} \dfrac{\partial^2 \Pi}{\partial x^2} = -4 \\ \\ \dfrac{\partial^2 \Pi}{\partial y \partial x} = -(2 + 2a) \end{cases}$$

$$\frac{\partial \Pi}{\partial y} = -16y + 80 - (2 + 2a)x \Rightarrow \frac{\partial^2 \Pi}{\partial y^2} = -16$$

Thus

$$\mathrm{H}\Pi(x, y) = \begin{pmatrix} \dfrac{\partial^2 \Pi}{\partial x^2} & \dfrac{\partial^2 \Pi}{\partial x \partial y} \\ \\ \dfrac{\partial^2 \Pi}{\partial y \partial x} & \dfrac{\partial^2 \Pi}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -4 & -(2 + 2a) \\ \\ -(2 + 2a) & -16 \end{pmatrix}$$

Sign of the principal minors:

- $D_1 = -4 < 0$;
- $D_2 \geq 0$ iff $64 - (2 + 2a)^2 \geq 0$ iff $|2 + 2a| \leq 8$ iif $-8 \leq 2 + 2a \leq 8$. Hence, $D_2 \geq 0$ iff $a \in [-5, 3]$.

In sum, Π is concave iff $a \in [-5, 3]$.

(b) By (a) above, the local extrema of Π satisfy with $a = 1$

$$-4x + 80 - 4y = 0$$
$$-16y + 80 - 4x = 0.$$

The solution is $(x, y) = (20, 0)$. Since $a = 1$ is in the region of the parameter a where Π is concave, (20,0) is the global maximum of Π.

(c) Let $a = 1$. Taking into account the constraint, the problem becomes

$$\max \;\; \Pi(x, y) = -2x^2 - 8y^2 + 80x + 80y - xy - 30 \quad \text{s.a}: \; x - 2y = 0.$$

The regularity condition is fulfilled since the gradient of the constraint, $(1, -2)$, never vanishes.

Lagrangian:

$$L(x, y, \lambda) = -2x^2 - 8y^2 + 80x + 80y - 6xy - 30 + \lambda(-x + 2y).$$

Lagrange's equations:

$$\frac{\partial L}{\partial x} = 0 \Rightarrow -4x + 80 - 4y - \lambda = 0$$
$$\frac{\partial L}{\partial y} = 0 \Rightarrow -16y + 80 - 4x + 2\lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x = 2y.$$

Solving for λ we find

$$\lambda = -4x + 80 - 4y = 8y - 40 + 2x \quad \Rightarrow \quad 6x + 12y = 120.$$

Plugging $x = 2y$ into this equation, we obtain $24y = 120$, thus $y = 5$ and then $x = 10$.

Since the profit function is concave when $a = 1$ and the constraint is linear, the critical point $(10, 5)$ is the unique global maximum of the problem.

$\boxed{3}$

Consider the problem of Lagrange:

$$\text{Optimize } f(x,y) = xy - 3x - 6y$$
$$\text{subject to: } g(x,y) = 2x + 4y = 40.$$

(a) (5 points) Find all critical points of the problem.

(b) (5 points) Find all local extrema of $f(x,y)$ subject to the constraint. Justify whether the local extrema are global extrema.

(c) (5 points) Suppose that $f(x,y)$ is the profit function of a firm and that $2x+4y=40$ is the budget constraint, both in thousands of euros.
Approximately, what would be the added benefit of increasing the company's funds by 1,000 euros?

Solution:

(a) Lagrangian:

$$L(x,y,\lambda) = xy - 3x - 6y + \lambda(40 - 2x - 4y).$$

Lagrange's equations:

$$\frac{\partial L}{\partial x}(x,y,\lambda) = y - 3 - 2\lambda = 0$$
$$\frac{\partial L}{\partial y}(x,y,\lambda) = x - 6 - 4\lambda = 0$$
$$\frac{\partial L}{\partial \lambda}(x,y,\lambda) = 40 - 2x - 4y = 0$$

Solving from λ in the first and the second equation and equating the result, we obtain

$$2y - 6 = 4\lambda = x - 6 \Longrightarrow 2y = x$$

and plugging this into the third equation

$$4x = 8y = 40 \Longrightarrow (x^*, y^*) = (10, 5), \quad \text{with } \lambda = 1.$$

(b) To know whether $(10,5)$ is a local maximum, minimum or a saddle point, we calculate the Hessian matrix of L with respect to x, y. The matrix coincides with the Hessian matrix of $f(x,y)$ since the constraint is linear. Thus

$$\mathrm{H}f(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The tangent space is $\{(v,w) : 2v + 4w = 0\} = \{(2v, -v)\}$, thus

$$(2v, -v)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2v \\ -v \end{pmatrix} = -4v^2 < 0,$$

if $(2v,-v) \neq (0,0)$. Thus, the Hessian matrix restricted to the tangent space is negative definite, and then $(10,5)$ is a local maximum. It is clear then that there is no local/global minimum of f subject to the constraint. To see that $(10,5)$ is a global maximum indeed, we substitute the constraint $x = 20 - 2y$ into f, to obtain the function

$$F(y) = f(20-2y, y) = (20-2y)y - 3(20-2y) - 6y = -2y^2 + 20y - 60$$

which is strictly concave; thus, the critical point $y = 5$ is the global maximum.

(c) The Lagrange multiplier is $\lambda = 1$. Hence, denoting

$$V(b) = \max\Big(xy - 3x - 6y\Big)$$
$$\text{subject to: } 2x + 4y = b,$$

we see that $\triangle V \approx \lambda \triangle b$, hence $\triangle V \approx 1.000$ euros.

$\boxed{4}$

Consider the Kuhn–Tucker problem

$$\begin{aligned} \max \quad & x + 2y \\ \text{s.t.} \quad & x^4 + 2y^4 \leq 3. \end{aligned}$$

(a) (10 points) Find all points that satisfy the Kuhn–Tucker conditions.

(b) (5 points) Justify that the problem admits global solutions and find them.

Solution:

(a) All points are regular, since the only point that makes trivial the gradient of $g(x, y) = x^4 + 2y^4$, $\nabla g(x, y) = (4x^3, 8y^3)$, is $(0, 0)$, which does not saturate the constraint.

Lagrangian: $L(x, y, \lambda) = x + 2y + \lambda(3 - x^4 - 2y^4)$.

K–T necessary conditions:

$$1 - 4\lambda x^3 = 0 \tag{1}$$
$$2 - 8\lambda y^3 = 0 \tag{2}$$
$$\lambda(3 - x^4 - 2y^4) = 0 \tag{3}$$
$$3 - x^4 - 2y^4 \geq 0 \tag{4}$$
$$\lambda \geq 0 \tag{5}$$

From (1) and (2), $x \neq 0$ and $y \neq 0$. Thus

$$\lambda = \frac{1}{4x^3} = \frac{2}{8y^3},$$

which implies $x = y$ and $\lambda \neq 0$; in particular this implies by (3) that $3 - x^4 - 2y^4 = 0$. Plugging $x = y$ into this equality we obtain

$$3 - 3x^4 = 0 \Rightarrow x = \pm 1.$$

Thus we have found the solution $(1, 1)$ with $\lambda = \frac{1}{4}$, which fulfills all K–T conditions, and $(-1, -1)$, which does not meet (5), since $\lambda = -\frac{1}{4} < 0$.

(b) It is a convex problem, since the objective function is concave ant the constraint is given by a convex function, thus the K–T conditions are sufficient (Theorem 3.1) and the point found is a global maximum. The Theorem of Weierstrass could also be invoked to obtain the same result, since the feasible set is compact.