UC3M
Mathematical Optimization for Economics
Final Exam, 21 June 2022

Niu: ____________ Group: ______

Name: ______________________________

Question:	1	2	3	Total
Points:	20	20	20	60
Score:				

Instructions:

- DURATION OF THE EXAM: 90'.
- Calculators are NOT allowed.
- Turn off your smart phone.
- DO NOT UNSTAPLE the exam.
- Please show a valid ID to the professor if required.
- Read the exam carefully. The exam has 3 questions, for a total of 60 points.
- Justify all your answers.

$\boxed{1}$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = \ln(x + y - 2)$. Consider the order of Pareto defined on the set

$$A = \left\{(x, y) \in \mathbb{R}^2 : y \leq 6 - (x - 2)^2;\ y \geq 2 - x;\ x \leq 2\right\}.$$

(a) (5 points) Draw the set A, justifying your answer.

(b) (5 points) Calculate, if they exist, the maximal and minimal elements, the maximum and the minimum of A. Justify your answers.

(c) (5 points) Justify whether the function f and the set A satisfy the hypotheses of the Theorem of Weierstrass.

(d) (5 points) Draw the curves of level $c = \ln \frac{1}{2}$, $c = 1$ and $c = \ln 6$ of the function f, showing the increasing direction and decreasing directions of the function. Calculate, if they exist the global maximum and global minimum of f on A. In case that some of them do not exist, say the reason.

2

A state-owned company managed by the Government of country $\mathbb{G}$ produces two goods A and B, of which it sells x and y units per day, respectively. The cost function is given by

$$C(x, y) = xy + 2x^2 + y^2.$$

The unitary price of good A is $p_A(x, y) = 3 - 2x - \alpha y$, where $\alpha > 0$ is a positive parameter. The unitary price of good B is constant and equal to 1, that is, $p_B(x, y) = 1$.

(a) (5 points) Find the range of values of α for which the profit function of the firm is a concave function.

(b) (5 points) Let $\alpha = 1$. Find the values of x and y which maximize the firm's profits.

(c) (5 points) Let $\alpha = 1$. The Government requires the firm managers to produce goods A and B in quantities such that the unitary price of good A is 2, that is, $p_A(x, y) = 3 - 2x - y = 2$. Find the values of x and y which maximize the firm's profits under this constraint.

(d) (5 points) Let $\alpha = 1$. Suppose now that the Government requires the firm managers to produce goods A and B in quantities such that the unitary price of good A is $2 + \frac{1}{6}$, that is, $p_A(x, y) = 3 - 2x - y = 2 + \frac{1}{6}$. Without solving the new Lagrange problem, give an estimate of the increment (positive or negative) of the optimal profits of the firm with respect to the case solved in part (c) above when the unitary price of A was fixed to 2. Base your answer in the value of the multiplier found in part (c).

$\boxed{3}$

Consider the Lagrange problem

$$\begin{array}{ll} \text{opt} & 2x^2 + y^2 - z^2, \\ \text{s.t.} & x^2 - y^2 = 1 \\ & x^2 + y^2 + z^2 = 4. \end{array}$$

(a) (5 points) Study the regularity condition.

(b) (10 points) Find all points that satisfy the Lagrange necessary conditions.
Hint: Note that points $(0, y, z)$ *with null first coordinate are not feasible, as they cannot satisfy the first constraint.*

(c) (5 points) Justify that the problem admits global solutions and find them.
Hint: Apply the Theorem of Weierstrass and part (b) above.
