UC3M
Mathematical Optimization for Economics
Final Exam (retake), 23 June 2021

Niu: ____________ Group: ______

Name: ______________________________

Question:	1	2	3	4	Total
Points:	12	18	15	15	60
Score:					

Instructions:

- **DURATION OF THE EXAM: 90'**.
- Calculators are **NOT** allowed.
- **Turn off** your smart phone.
- **DO NOT UNSTAPLE** the exam.
- Please show a valid ID to the professor if required.
- Read the exam carefully. The exam has 4 questions, for a total of 60 points.
- Justify all your answers.

$\boxed{1}$

Consider the function $f(x,y) = \dfrac{1}{y - \frac{x}{2} + 1}$ and the set

$$A = \{(x,y) \in \mathbb{R}^2 \,:\, x \geq 0,\ y \geq 0,\ y \leq 3,\ x - 2 \leq y \leq x + 1\}.$$

(a) (6 points) Draw the set A and discuss whether the function f and the set A satisfy the assumptions of the Theorem of Weierstrass. Can you ensure the existence of global extremes of f in A?

(b) (6 points) Draw some of the level curves of f on the plane, showing the directions in which f increases/decreases and determine (if they exist) the global extrema of f on A. In case they do not exist, justify your answer.

$\boxed{2}$

Consider the function $f(x, y) = (x^2 - x + 1)e^{x+ay^2}$, where a is an unknown parameter, different from 0.

(a) (6 points) Find the critical points of f.

(b) (6 points) For each of the critical points found in the item above, determine the range of values of the parameter a for which the critical point considered is

- A local maximum.
- A local minimum.
- A saddle point.

(c) (6 points) Prove that the function f has no global maximum or global minimum.
Hint: Consider the function $g(x) = f(x, 0) = (x^2 - x + 1)e^x$ *and calculate* $\lim_{x\to\pm\infty}(x^2 - x + 1)e^x$.

$\boxed{3}$

Consider the problem of Lagrange:

$$\text{Opt } f(x,y,z) = -x\ln x - y\ln y - z\ln z \quad \text{s.t.: } x+y+z=1$$

(a) (3 points) Obtain the Lagrange equations.

(b) (3 points) Find the critical points of the Lagrangian.

(c) (3 points) Classify the critical points found in the item above. Are they global extrema of f subject to the constraint? Justify your answer.

(d) (6 points) Suppose that the constraint changes to $x+y+z=0.85$, that is, the problem becomes

$$\text{Opt } f(x,y,z) = -x\ln x - y\ln y - z\ln z \quad \text{s.t.: } x+y+z=0.85.$$

Without solving the problem again, calculate approximately the optimal value of $f(x,y,z)$ after this change. This implies to calculate approximately the value of the multiplier, knowing that $\ln 3 \approx 1.1$.

$\boxed{4}$

Consider the function $f(x, y) = (x-3)^2 + (y-4)^2$ defined on the set

$$A = \{(x, y) \,:\, x^2 + y^2 \leq 1\}.$$

(a) (6 points) Establish the Kuhn–Tucker necessary optimality conditions to the problem

$$\max\ (x-3)^2 + (y-4)^2 \quad \text{subject to } x^2 + y^2 \leq 1.$$

(b) (6 points) Find all the solutions of the Kuhn–Tucker conditions established in part (a).

(c) (3 points) Find the global maximum of f on A.
