OPTIMIZATION
Question Bank 2019-2020 ordinary call

1. Block I. Introduction

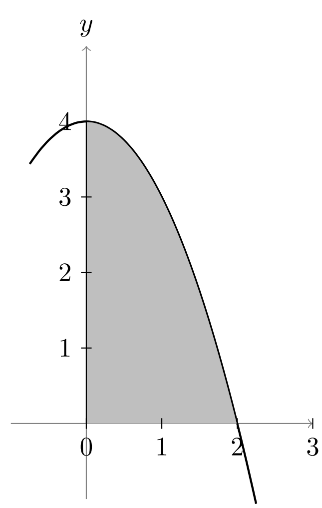
1. Consider the set $S = \{(x, y) \in \mathbb{R}^2_+ : x \geq 0, y \geq 0, y \leq -x^2 + 4\}$ and the function $f(x, y) = \mathrm{e}^{-x+y}$.

(a) (4 points) Draw the set and discuss whether the function f and the set S satisfy the assumptions of the Weierstrass Theorem.

(b) (6 points) Draw the level curves of f on the set, one of them being the curve that passes through $(0, 0)$, and compute its level. Show the direction in which f increases/decreases and determine (if they exist) the global extrema of f on the set S.

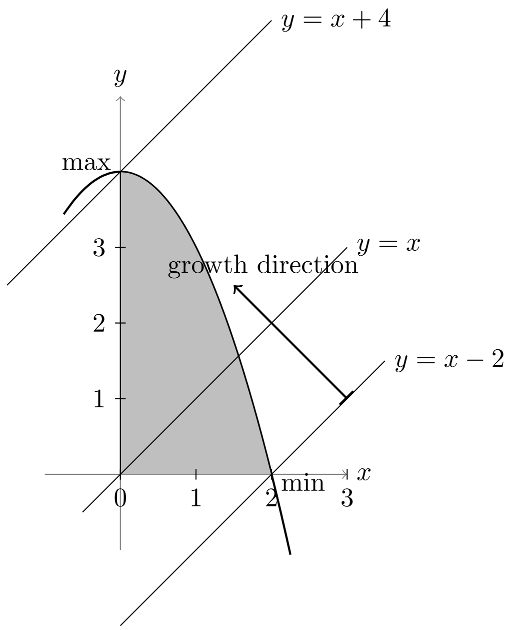
Solution

(a) The set S is the shadow region sketched in the figure below. It is closed since it contains its boundary, and it is bounded, hence compact. f is continuous in $\mathbb{R}^2$, so it is continuous on S. This means that the conditions of the Weierstrass Theorem are satisfied and there is a global maximum and a global minimum of the function f on the set S.



(b) Level curves are given by the family of equations $e^{-x+y} = c$, as $c \in \mathbb{R}$ or, equivalently $-x + y = \ln c$, $c > 0$, hence they are lines of slope equal to 1 and that grow in the direction $(-1, 1)$ (shown in the graph below). The highest level is attained on the set S at the intersection of $x = 0$ and $y = -x^2 + 4$, which is the point $(0, 4)$ and corresponds to the level $c = e^4$. The minimum is attained at the point $(2, 0)$, with level $c = e^{-2}$. Hence the global maximum is $(0, 4)$, with maximum value e^4 and the global minimum is $(2, 0)$, with minimum value e^{-2}.

The level curve passing through $(0, 0)$ has level $c = e^0 = 1$, thus it is the line $-x + y = \ln 1 = 0$.



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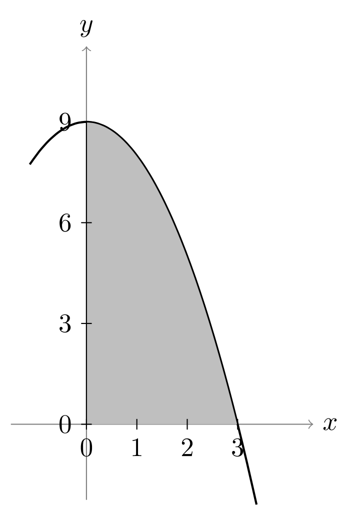
2. Consider the set $S = \{(x, y) \in \mathbb{R}^2_+ : x \geq 0, y \geq 0, y \leq -x^2 + 9\}$ and the function $f(x, y) = \mathrm{e}^{-x+y}$.

(a) (4 points) Draw the set and discuss whether the function f and the set S satisfy the assumptions of the Weierstrass Theorem.

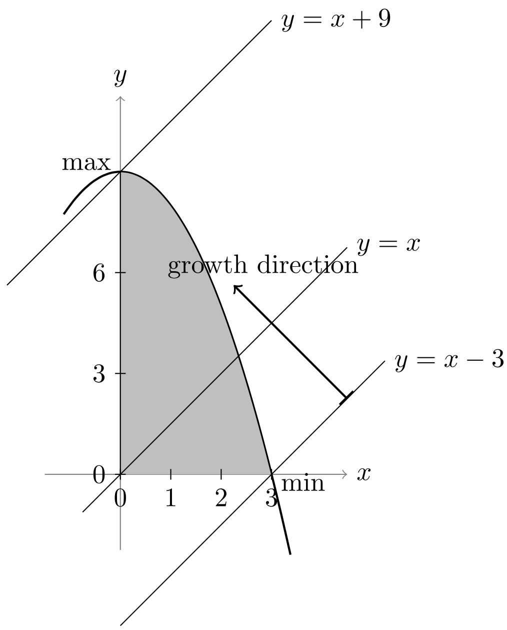
(b) (6 points) Draw the level curves of f on the set, one of them being the curve that passes through $(0, 0)$, and compute its level. Show the direction in which f increases/decreases and determine (if they exist) the global extrema of f on the set S.

Solution

(a) See Problem 1(a) above and the figure below.



(b) See Problem 1(b) above, with the following modifications: the maximum is attained at (0,9), with maximum value e^9, the minimum is attained at (3,0), with minimum value e^{-3}. See the figure below.



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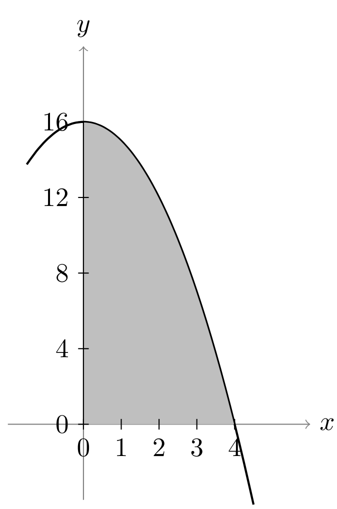
3. Consider the set $S = \{(x, y) \in \mathbb{R}^2_+ : x \geq 0, y \geq 0, y \leq -x^2 + 16\}$ and the function $f(x, y) = \mathrm{e}^{-x+y}$.

(a) (4 points) Draw the set and discuss whether the function f and the set S satisfy the assumptions of the Weierstrass Theorem.

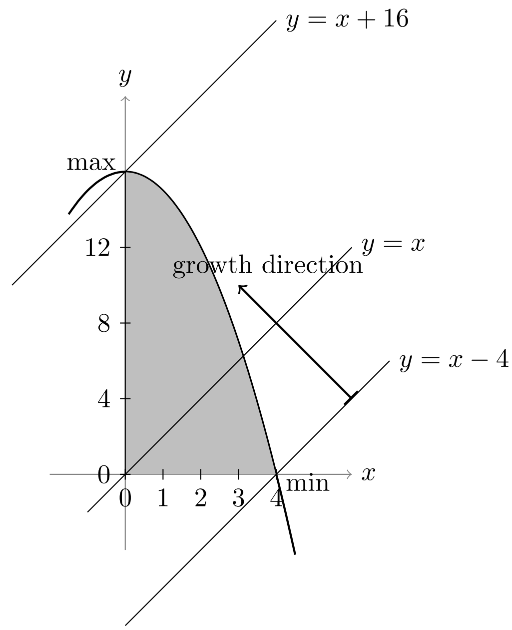
(b) (6 points) Draw the level curves of f on the set, one of them being the curve that passes through $(0, 0)$, and compute its level. Show the direction in which f increases/decreases and determine (if they exist) the global extrema of f on the set S.

Solution

(a) See Problem 1(a) above and the figure below.



(b) See Problem 1(b) above, with the following modifications: the maximum is attained at (0,16), with maximum value e^{16}, the minimum is attained at (4,0), with minimum value e^{-4}. See the figure below.


$y = x + 16$
y
max
12
8
4
0
$y = x$
growth direction
$y = x - 4$
x
0
1
2
3
4
min

1. A computer company sells 3 kind of laptops (A, B, C) in amounts x, y and z, respectively. The cost of producing one laptop A is 1, the cost of producing one laptop B is 2 and the cost of producing one laptop C is 1. There is also a fixed cost of production of 10 for all the production just in case the firm produces a positive quantity of at least one of the three kinds of laptops. If the firm does not produce anything, there are no fixed costs. The selling price for products are given by the functions:

for laptop A is $p(x) = 3 - x$,

for laptop B is $q(y) = 12 - y$,

for laptop C is $r(z) = 1 - z$.

(a) (3 points) Find the cost function, the revenue function and the profit function of the company when the firm produces a positive quantity of at least one of the three kinds of laptops.

(b) (3 points) Find and classify the critical points of the profit function.

(c) (4 points) Suppose the fixed costs of the company were lower, how would that affect the decision company regarding the production? What about if they were higher?

Solution

(a) The cost function when the firm produces something is $C(x, y, z) = x + 2y + z + 10$ and the revenue function is $R(x, y, z) = x(3 - x) + y(12 - y) + z(1 - z)$. Hence, the profit function is

$$B(x, y, z) = R(x, y, z) - C(x, y, z) = x(3-x) + y(12-y) + z(1-z) - (x+2y+z+10) = 2x - x^2 + 10y - y^2 - z^2 - 10.$$

(b) The gradient of the profit function is $\nabla B(x, y, z) = (2 - 2x, 10 - 2y, -2z)$. Making $\nabla B(x, y, z) = 0$:

$$\begin{aligned} 2 - 2x &= 0 \\ 10 - 2y &= 0 \\ -2z &= 0. \end{aligned}$$

The solution of that system is the unique critical point: $x = 1, y = 5, z = 0$. The Hessian matrix of f

$$Hf = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

is negative definite. Thus, the profit function B is strictly concave and hence it attains a unique global maximum at the point $(1, 5, 0)$.

(c) The decision of the firm of producing $x = 1$, $y = 5$ and $z = 0$ continues to be optimal as soon as the optimal profits are non-negative. Since profits are $B^* = B(1, 5, 0) = 16$, this means that the optimal choice does not change when fixed costs diminish or if they increase by an amount not greater than 16. When fixed costs increase more than 16 monetary units, profits become negative, and the firm's optimal choice is to produce nothing. If the increment is exactly 16, then the firm is indifferent between producing or not.

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2. A computer company sells 3 kind of laptops (A, B, C) in amounts x, y and z, respectively. The cost of producing one laptop A is 1, the cost of producing one laptop B is 2 and the cost of producing one laptop C is 1. There is also a fixed cost of production of 10 for all the production just in case the firm produces a positive quantity of at least one of the three kinds of laptops. If the firm does not produce anything, there are no fixed costs. The selling price for products are given by the functions:

for laptop A is $p(x) = 5 - x$,

for laptop B is $q(y) = 22 - y$,

for laptop C is $r(z) = 1 - z$.

(a) (3 points) Find the cost function, the revenue function and the profit function of the company when the firm produces a positive quantity of at least one of the three kinds of laptops.

(b) (3 points) Find and classify the critical points of the profit function.

(c) (4 points) Suppose the fixed costs of the company were lower, how would that affect the decision company regarding the production? What about if they were higher?

Solution

See Problem 1 above, considering the new profit function

$$B(x, y, z) = x(5 - x) + y(22 - y) + z(1 - z) - C(x, y, z).$$

The solution is $(x^*, y^* z^*) = (2, 10, 0)$ with maximum profits $B^* = 94$. This is the threshold such that if the fixed costs exceeds this amount, the firm stops producing.

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3. A computer company sells 3 kind of laptops (A, B, C) in amounts x, y and z, respectively. The cost of producing one laptop A is 1, the cost of producing one laptop B is 2 and the cost of producing one laptop C is 1. There is also a fixed cost of production of 10 for all the production just in case the firm produces a positive quantity of at least one of the three kinds of laptops. If the firm does not produce anything, there are no fixed costs. The selling price for products are given by the functions:

for laptop A is $p(x) = 7 - x$,

for laptop B is $q(y) = 12 - y$,

for laptop C is $r(z) = 1 - z$.

(a) (3 points) Find the cost function, the revenue function and the profit function of the company when the firm produces a positive quantity of at least one of the three kinds of laptops.

(b) (3 points) Find and classify the critical points of the profit function.

(c) (4 points) Suppose the fixed costs of the company were lower, how would that affect the decision company regarding the production? What about if they were higher?

Solution

See Problem 1 above, considering the new profit function

$$B(x, y, z) = x(7 - x) + y(12 - y) + z(1 - z) - C(x, y, z).$$

The solution is $(x^*, y^* z^*) = (3, 5, 0)$ with maximum profits $B^* = 24$. This is the threshold such that if the fixed costs exceeds this amount, the firm stops producing.

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3. Block III. Lagrange

1. Maximize and minimize (when possible) the function $f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2$ when $(x,y,z) \in A$, where $A = \{(x,y,z) : x+y+z = 1\}$. In order to do it you are asked to:

(a) (3 points) Obtain Lagrange function and the equations of its critical points.

(b) (4 points) Obtain the critical points of the Lagrange function.

(c) (3 points) Justify whether those points are maximizers or minimizers of the function. In case there is no maximum or no minimum explain why.

Solution

(a) The Lagrange function is

$$L(x,y,z,\lambda) = (x-1)^2 + (y-2)^2 + (z-3)^2 + \lambda(1-x-y-z).$$

The critical points of the Lagrange function satisfy the system of equations

$$\frac{\partial L}{\partial x} = 2(x-1) - \lambda = 0 \tag{1}$$

$$\frac{\partial L}{\partial y} = 2(y-2) - \lambda = 0 \tag{2}$$

$$\frac{\partial L}{\partial z} = 2(z-3) - \lambda = 0 \tag{3}$$

$$\frac{\partial L}{\partial \lambda} = 1 - x - y - z = 0. \tag{4}$$

(b) From equations (1), (2) and (3) we obtain $\lambda = 2(x-1) = 2(y-2) = 2(z-3)$, from which we get $y = x+1$, $z = x+2$. Plugging these into (4) we obtain $x = -2/3$. Hence the critical point is $(-2/3, 1/3, 4/3)$.

(c) The critical point obtained in (b) is the unique global minimizer of f subject to the constraint, since that f is strictly convex and the feasible set is convex.

There are no global maximizers, since the function takes arbitrary large values on the feasible set. For instance, $(1, n, -n)$ is feasible for any integer n and $f(1,n,-n) = (n-2)^2 + (n+3)^2$ tends to ∞ as $n \to \infty$.

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2. Maximize and minimize (when possible) the function $f(x,y,z) = (x-2)^2 + (y-3)^2 + (z-1)^2$ when $(x,y,z) \in A$, where $A = \{(x,y,z) : x+y+z = 1\}$. In order to do it you are asked to:

(a) (3 points) Obtain Lagrange function and the equations of its critical points.

(b) (4 points) Obtain the critical points of the Lagrange function.

(c) (3 points) Justify whether those points are maximizers or minimizers of the function. In case there is no maximum or no minimum explain why.

Solution

See Problem 1 above, with the new Lagrangian

$$L(x,y,z,\lambda) = (x-2)^2 + (y-3)^2 + (z-1)^2 + \lambda(1-x-y-z).$$

The problem admits $(1/3, 4/3, -2/3)$ as the unique global minimum and the there is no maximum.

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3. Maximize and minimize (when possible) the function $f(x,y,z) = (x-3)^2 + (y-2)^2 + (z-1)^2$ when $(x,y,z) \in A$, where $A = \{(x,y,z) : x+y+z = 1\}$. In order to do it you are asked to:

(a) (3 points) Obtain Lagrange function and the equations of its critical points.

(b) (4 points) Obtain the critical points of the Lagrange function.

(c) (3 points) Justify whether those points are maximizers or minimizers of the function. In case there is no maximum or no minimum explain why.

Solution

See Problem 1 above, with the new Lagrangian

$$L(x,y,z,\lambda) = (x-3)^2 + (y-2)^2 + (z-1)^2 + \lambda(1-x-y-z).$$

The problem admits $(4/3, 1/3, -2/3)$ as the unique global minimum and the there is no maximum.

4. Block IV. Interpretation of the Multiplier

1. A consumer can choose between good A whose price is 4€ and good B whose price is 1€. The consumer has a monthly budget of 72€to spend entirely in both goods. The utility function is $U(x, y) = 8x^{\frac{1}{2}}y^{\frac{1}{2}}$, where x is the number of units consumed of good A and y is the number of units consumed of good B. If you cannot buy negative units, it is asked:

(a) (3 points) If the consumer obtains the maximum utility buying 9 units of good A and 36 units of good B, what is the value of the multiplier?

(b) (3 points) How much does that maximum utility vary approximately if the allocated monthly budget for buying both goods is now of 73.50€? (*Remark:* if you cannot solve part a) take that the multiplier is equal to 4)

(c) (4 points) Suppose now the same problem where the utility function is $U(x, y) = f(x + y)$, where f is a strictly concave function that satisfies $f'(27) = 0$. At what points (x, y) does this consumer maximize her utility?

Solution

(a) The optimization problem is the following:

$$\max 8x^{1/2}y^{1/2} \quad \text{subject to: } 4x + y = 72.$$

Its Lagrangian function is $L(x, y, \lambda) = 8x^{1/2}y^{1/2} + \lambda(72 - 4x - y)$ and the critical points satisfy

$$\begin{aligned} \frac{\partial L}{\partial x} &= 4\frac{y^{1/2}}{x^{1/2}} - 4\lambda = 0 \\ \frac{\partial L}{\partial y} &= 4\frac{x^{1/2}}{y^{1/2}} - \lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= 72 - 4x - y = 0 \end{aligned}$$

Since $x = 9$, $y = 36$ is solution of the consumer's problem, we plug these values into the first equation ot the system above to find the value of the multiplier $\lambda = \frac{36^{1/2}}{9^{1/2}} = \frac{6}{3} = 2$.

(b) The multiplier is the variation of the optimal utility against infinitesimal changes in consumer's budget (in more precise terms: it is the derivative of the value function or indirect utility function when the consumer's budget is 72). When the change is discrete, the multiplier is only an approximation to that variation. In this case the change is a positive increase, $\Delta b = 73.50 - 72 = 1.5$ euros. Thus the optimal utility change is a positive increase of approximately

$$\lambda \times \Delta b = 2 \times 1.5 = 3.$$

(c) The Lagrange function for the new problem is

$$L(x, y, \lambda) = f(x + y) + \lambda(72 - 4x - y)$$

and the Lagrange system identifying critical points becomes

$$\begin{aligned} \frac{\partial L}{\partial x} &= f'(x + y) - 4\lambda = 0 \\ \frac{\partial L}{\partial y} &= f'(x + y) - \lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= 72 - 4x - y = 0 \end{aligned}$$

from the first and the second equations we obtain that the critical points (x, y, λ) satisfy $f'(x+y) = \lambda = 0$ and $4x + y = 72$. Since f is strictly concave, and $f'(27) = 0$, 27 is the only point that makes null f'. Hence the critical points of the Lagrangian are of the form $(x, y, 0)$ with $x+y = 27$ and $4x+y = 72$. The solution of these two equations is $x = 15$ and $y = 12$. Since the problem is convex, this is the solution of the new consumer's problem.

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2. A consumer can choose between good A whose price is 1€ and good B whose price is 4€. The consumer has a monthly budget of 72€to spend entirely in both goods. The utility function is $U(x, y) = 8x^{\frac{1}{2}}y^{\frac{1}{2}}$, where x is the number of units consumed of good A and y is the number of units consumed of good B. If you cannot buy negative units, it is asked:

(a) (3 points) If the consumer obtains the maximum utility buying 36 units of good A and 9 units of good B, what is the value of the multiplier?

(b) (3 points) How much does that maximun utility vary approximately if the allocated monthly budget for buying both goods is now of 71.50€? (*Remark:* if you cannot solve part a) take that the multiplier is equal to 4)

(c) (4 points) Suppose now the same problem where the utility function is $U(x, y) = f(x + y)$, where f is a strictly concave function that satisfies $f'(27) = 0$. At what points (x, y) does this consumer maximize her utility?

Solution

See Problem 1 above. The problem becomes:

$$\max 8x^{1/2}y^{1/2} \quad \text{subject to: } x + 4y = 72.$$

(a) $\lambda = 2$.

(b) The variation of the optimal utility is approximately $\lambda \times (71.50 - 72) = 2 \times (-0.50) = -1$.

(c) Using the equations $x + 4y = 72$ and $x + y = 27$ we obtain $x = 12$ and $y = 15$.

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3. A consumer can choose between good A whose price is 4€ and good B whose price is 1€. The consumer has a monthly budget of 72€to spend entirely in both goods. The utility function is $U(x, y) = 12x^{\frac{1}{2}}y^{\frac{1}{2}}$, where x is the number of units consumed of good A and y is the number of units consumed of good B. If you cannot buy negative units, it is asked:

(a) (3 points) If the consumer obtains the maximum utility buying 9 units of good A and 36 units of good B, what is the value of the multiplier?

(b) (3 points) How much does that maximun utility vary approximately if the allocated monthly budget for buying both goods is now of 73€? (*Remark:* if you cannot solve part a) take that the multiplier is equal to 4)

(c) (4 points) Suppose now the same problem where the utility function is $U(x, y) = f(x + y)$, where f is a strictly concave function that satisfies $f'(24) = 0$. At what points (x, y) does this consumer maximize her utility?

Solution

See Problem 1 above. The problem becomes:

$$\max 12x^{1/2}y^{1/2} \quad \text{subject to: } 4x + y = 72.$$

(a) $\lambda = 3$.

(b) The variation of the optimal utility is approximately $\lambda \times (73 - 72) = 3$.

(c) Using the equations $4x + y = 72$ and $x + y = 24$ we obtain $x = 16$ and $y = 8$.