Game Theory: Additional Exercises

Problem 1. Consider the following scenario. Players 1 and 2 compete in an auction for a valuable object, for example a painting. Each player writes a bid in a sealed envelope, without knowing the other player's bid. The bidding starts from 0, are multiples of 100 euros and the maximum that you can bid is 500 euros. The object's value is 400 euros for Player 1 and 300 euros for Player 2. The player making the highest bid wins the auction. In case of a tie, suppose that Player 1 wins the object. The winner pays a price P that we explain below. That is, if the value of the object is υ_i for Player $i = 1, 2$ and Player i is the winner, his payoff or utility is $\upsilon_i - P$ and the payoff or the utility of the loser is equal to zero.

Consider the following two cases:

(i) First price auction: the winner pays a price P equal to the bid he does.

(ii) Second-price auction: the winner pays a price P equal to the bid of the player that lost.

Answer the following questions for both cases:

Write both of the games in strategic form.

What are the rationalizable strategies?

Find the Nash equilibria in pure strategies.

Problem 2. Consider the following Cournot game. The inverse demand function of the market is

$$p(q_1, q_2) = 2 - q_1 - q_2$$

and the unit cost is $c = 1$ for both of the firms, which as usual simultaneously choose q_1 and q_2 to maximize their payoffs given by

$$U_i(q_1, q_2) = (1 - \alpha)\,\pi_i + \alpha q_i$$

where π_i are firms' profits $i = 1, 2$. Find the pure strategy Nash equilibria.

Problem 3. Consider the following games in the normal form (for the second one assume that $0 < c < 1$):

	H	D
H	$0,0$	$6,1$
D	$1,6$	$3,3$

	S	W
S	$0,0$	$0,-c$
W	$-c,0$	$1-c,1-c$

	L	M	R
U	$1,3$	$0,0$	$2,-1$
M	$1,0$	$4,2$	$0,-2$
D	$0,1$	$0,1$	$0,0$

Find all the pure and mixed strategy Nash equilibria of the three games.

Problem 4. The following stage game is repeated twice.

	L	M	R
L	$1,1$	$5,0$	$1,0$
M	$0,5$	$4,4$	$0,0$
R	$0,1$	$0,0$	$3,3$

Which of the *symmetric* strategy profiles that is explained below form a Nash Equilibrium? Which one is Subgame Perfect Nash Equilibrium? Explain.

(i) Play M in period 1; if (M,M) is played in the first period, play R in the second period. In any other case, play L.

(ii) Play M in period 1; if the opponent plays M in the first period, play R in the second period. In any other case, play L.

(iii) Play M in period 1. If any player plays M in the first period, play R in the second period. If both of the players choose L or R in the first period, play L in the second period.

Problem 5. Two people are involved in a dispute. Person 1 does not know if the person 2 is weak or strong and believes that he is strong with probability equal to α. Person 2 however is fully informed about the characteristics of the person 1. Each individual can choose between "fight" or "surrender". Surrender gives a payment of zero, regardless of what the opponent does, while fighting gives a payment of 1 if and only if the opponent surrenders. If two people fight, the payments are $(-1,1)$ if the person 2 is strong, and $(1,-1)$ if the person 2 is weak.

Formulate the situation as a Bayesian game and find all the Bayesian Nash equilibria when $\alpha < 1/2$ and where $\alpha > 1/2$.

Problem 6. María is deciding whether to stay home or call her boyfriend. If she stays home she and her boyfriend a payoff of 2. If she calls her boyfriend, a decision-making process will begin about which movie to see, known as the Battle of the Sexes, and whose normal form is

	Kill Bill	The Pursuit of Happiness
Kill Bill	$1,3$	$0,0$
The Pursuit of Happiness	$0,0$	$3,1$

María chooses among rows. Represents the entire game (including the decision to call or not), in extensive and normal form, and find all Nash equilibria (pure strategy)and subgame perfect Nash equilibria (in pure and mixed).

Problem 7. Two players must share 10 euros. Player 1 makes a proposal to divide the money in *integers*. In other words, you can suggest any integer between 0 and 10. Player 2 can accept or reject the proposal. If accepted, the 10 euros are divided according to the proposed deal, and if rejected both players get zero. Suppose m_i is the money assigned to player $i = 1, 2$ in the suggested division. Suppose that the utility of player i is: $m_i - cm_{-i}$ for $0 \leq c < 1$.

(i) Draw the extensive form of the game.

(ii) Find the subgame perfect Nash equilibrium/a for c = 0, for c = 0.25 and for c = 0.5.

(iii) How does the solution depend on c?

Problem 8. Consider the following Bayesian game. The type of the column player is known, however, the row player can be of type 1 (with probability 0.9) or type 2. Obviously, the row player know his own type but the column player doesn't know the type of row player. If they type of the row player is 1, the payoff matrix is as follows:

	L	R
U	$2, 2$	$-2, 0$
D	$0, -2$	$0, 0$

On the other hand, if type 2, the payoff matrix is

	L	R
U	$0, 2$	$1, 0$
D	$1, -2$	$2, 0$

(i) Describe all the pure strategies for both of the players.

(ii) Find all the Bayesian Nash Equilibria in pure strategies.

Problem 9. The total revenue function of a firm depends on the number of workers hired. A union that represents workers makes an offer to the company for a wage $w \in [0, +\infty)$. The company, after observing the proposed salary, decides whether to accept or reject it. If the company accepts the offer, it then chooses the number of workers L to employ. If rejected, does not employ anyone and company revenues are equal to zero. The payoff function of the firm is

$$\pi(w, L) = IT(L) - wL,$$

where

$$IT(L) = L^{1/2}.$$

The payoff function of the labor union is given by:

$$u(w, L) = (w - 1)L$$

Both payoff functions is equal to zero if the firm rejects the wage offer. Find the Subgame Perfect Nash Equilibrium/a in pure strategies.

Problem 10. Two individuals are involved in a relationship with positive synergies: if both put more effort into their relationship, both are better. Let's be more specific, a level of effort is a non-negative number and the payoff function of player 1 is $e_1(1 + e_2 - e_1)$ where e_i is the effort level of player $i = 1, 2$. For Player 2, the cost of effort is: (i) either the same as for player 1 and therefore its payoff function is given by $e_2(1 + e_1 - e_2)$, or (ii) to exercise effort is very costly for him, in which case the payment function is $e_2(1 + e_1 - 2e_2)$.

Player 2 knows its payoff function (and therefore if the cost of effort is 1 or 2) and also knows the payoff function of player 1. The latter however does not know the cost of effort for player 2. He believes that player 2 has a cost of effort that is low (i.e. 1) with probability $p \in (0, 1)$. Find all Bayesian Nash equilibria in pure strategies in terms of p.

Problem 11.

A thief (L) has seen a possible victim and is deciding whether to attack (A) or to pass (P). If he attacks, the victim (V) has to decide whether to defend (D) or surrender (R). If he doesn't attack, both players get a zero payoff. If the thief attacks and the victim surrenders, the thief obtains a quantity of euros v of the victim; but if this is defended the thief obtains only $v/2$ Euros from the victim. When the victim defends, a violent dispute occurs and both the thief and the victim suffer a cost for the fight (damage, etc) which we call c. Assume that $c > v/2$.

(i) Find all the Nash equilibria in pure strategies.

(ii) Find all the Subgame Perfect Nash equilibria in pure strategies.

Now consider the repeated version of this game with a discount factor for both players.

(iii) Find subgame perfect equilibrium/a if the game is repeated a finite number of times T and $\delta = 1$.

(iv) Consider now that the game is repeated an infinite number of times $T = \infty$ and $\delta \in (0, 1)$. Check whether there are conditions under which the following strategies form a subgame perfect NE:

- *Thief*: Start by playing P and will continue playing P unless in the past (A, R) has resulted, in that case will play A.
- *Victim*: In case of an attack, will play R if in the past (A, R) has resulted. Otherwise, the victim will choose D.

Problem 12. Consider a market with the inverse demand $P(Q) = 100 - 2Q$ (where Q is the aggregate quantity produced in the industry) in which two companies, 1 and 2, operate. These firms compete by choosing the quantity to be produced. For historical reasons the company 1 is generally recognized as the industry leader, so that firm 2 chooses its quantity produced after observing firm 1's decision. The cost functions of both companies are: for 1 $C(q_1) = 4q_1$ and for 2 $C(q_2) = 2q_2$.

i) Find the subgame perfect Nash equilibrium (SPNE) of this game.

ii) Would firm 1 like to leave its status as the leader in this industry and simultaneously compete with firm 2? (Justify your answer by comparing the benefits in each case).

iii) Now suppose that firms 1 and 2 are facing the threat of entry by another firm, company 3, which has the following cost structure: $C(q_3) = q_3$. If enters, company 3 would become a follower, simultaneously competing with firm 2 after they both observe the quantity of firm 1. Find the new SPNE. What is the maximum quantity that firm 3 would be willing to pay firm 1 to exchange their positions (i.e., so that 3 was the new leader and 1 became the follower)? Would firm 1 accept this amount?

Problem 13. Consider the following stage games. Find, if possible, strategies for the players and the conditions for the discount rate $\delta \in (0,1)$ to sustain the strategy profile (U, L) as a Subgame Perfect Nash equilibrium.

$$\begin{bmatrix} & L & R \\ U & 2,2 & 0,4 \\ D & 4,0 & 1,1 \end{bmatrix}; \begin{bmatrix} & L & R \\ U & 3,4 & 0,7 \\ D & 5,0 & 1,2 \end{bmatrix}; \begin{bmatrix} & L & R \\ U & 3,2 & 0,1 \\ D & 7,0 & 2,1 \end{bmatrix}$$

Problem 14. Three flatmates, A, B and C are arguing about splitting a pizza, whose size we normalize to 1. At the end, A whom studied game theory proposes the following rule:

STAGE 1:

i. A splits the pizza in two parts.

ii. B chooses one of the parts and A eats the part of the pizza that B *did not* choose.

STAGE 2:

i. B splits the remaining part of the pizza in two parts.

ii. C chooses one of the parts and B eats the part of the pizza that C *did not* choose.

STAGE 3:

i. C eats whatever is left of the pizza.

Let's a, b and c denote, the fraction of the pizza that A, B and C ate respectively. Given that there is no pizza left: $a + b + c = 1$. The payoffs of the players are simply the fraction of the pizza that they managed to eat and the discount factor is equal to 1 for all of them. Respond:

(i) Suppose that at the beginning of stage 2, there is still $4/5$ of the pizza. How should B slice the pizza?

(ii) More generally: suppose that A ate a, leaving $1 - a$ for the start of period 2. How should B slice the pizza?

(iii) Derive the fraction of pizza that each player eat in equilibrium using Backward induction.

Problem 15.

A and B are considering to form a joint venture (Joint Venture, JV) that would bring a profit of 100 euros. If they don't work together, the JV cannot be carried out because their ideas are unique and complementary. If the JV does not take place, each has different employment options. A can work with his cousin and earn 20 euros while B has a job offer that will bring a profit of 40 euros. Before deciding whether or not to carry out the JV, they must agree on how to share the future profits of 100 euros. Both firms have a discount factor equal to 0.9. Answer:

(i) Suppose B makes an offer of revenue sharing to A. Then A must decide whether to accept B's offer or to reject it and go to work with his cousin. Remember that if A rejects the JV doesn't happen. Which division should B propose? Write the strategies that constitute a subgame perfect Nash equilibrium.

(ii) Suppose now that there are two phases or periods. In the first, B makes a request seeking to appropriate the benefits. Now A has three options: (1) accept, (2) break the relationship (in which case both will proceed to get their work described before), or (3) continue negotiating. When A decides to continue negotiating, the second stage begins, A makes the request and B decides whether to: (1) accept, or (2) end the relationship.

(i) Draw the extensive form game carefully.

(ii) Find the subgame perfect Nash equilibrium.

Problem 16. Consider the following simultaneous game between Player 1(whom chooses among the rows in the matrix) and Player 2:

$$\begin{array}{ccc} & C & D \\ C & 2,3 & c,d \\ D & a,b & 1,1 \end{array}$$

where a, b, c, d are real numbers.

(i) In which interval should a, b, c, d be so that C is a strictly dominated strategy for both of the players. In that case, what would be the Nash equilibrium?

(ii) In which interval should a, b, c, d be so that the following Nash equilibria in pure strategies exist: (C, D) and (D, C). Assume that a, b, c, d take one of these values. Check if there exist any mixed strategy Nash equilibrium.

(iii) In which interval should a, b, c, d be so that there is a unique Nash equilibrium in pure strategies that is equal to (C, C)?

(iv) Find the conditions about the values of a, b, c, d so that all the strategy profiles (C, C), (C, D), (D, C), (D, D) are Nash equlibrium?