## INSTRUCTIONS:

Write your answers in the space provided immediately after each question. You may use the back of each page. The duration of this exam is 2 hours and 15 minutes.

## Name:

Degree:

## Group:

## 1. (20 points)

In the aeronautical industry the firm RINOSA can choose between expanding or not expanding, E or NE. The payoffs of this possible choice depend whether the competing firm KASA innovates its product or not, I or NI. RINOSA knows the final costs of production whether it expands or does not expand. These costs can be low or high. However, KASA does not know with certainty the costs that its competitor incurs. The probability that these costs are low is denoted by $p$, where $0 \leq p \leq 1$. The decisions to expand or not, and to innovate or not are taken simultaneously, and the payoffs of each firm are given as follows:


Low Costs


High Costs
(a) (5 points) Suppose $p=1$. Calculate all the Nash Equilibria of the game. Do/es the equilibrium/a that you have found maximize social utility?
(b) (15 points) Suppose $p \in(0,1)$. Calculate the Bayesian Nash Equilibria of the game as a function of the value of $p$.

## SOLUTIONS:

(a) $\operatorname{PSNE}=\{(E, N I)\}$ which is the social optimum.

There is no mixed strategy equilibrium, since the equilibrium in pure strategies is the only rationalizable strategy profile.
(b) First, note that RINOSA has a strictly dominated strategy when it has low cost: it will never choose NE. Its best response are given by:

$$
\begin{aligned}
& B R_{R}(I)=E E \\
& B R_{R}(N I)=E E \text { and } E N E
\end{aligned}
$$

KASA's best response to the remaining strategies from RINOSA:

|  | $E E$ | $E N E$ |
| :---: | :---: | :---: |
| $I$ | $4 p+5(1-p)$ | $\underline{10-6 p}_{p \leq 5 / 6}$ |
| $N I$ | $5 p+6(1-p)$ | $\underline{5}_{p \geq 5 / 6}$ |
|  |  |  |

Therefore, if $p \geq 5 / 6$

$$
B N E=\{(E N E, N I),(E E, N I)\}
$$

and if $p<5 / 6$

$$
B N E=\{(E E, N I)\}
$$

## 2. ( 25 points)

The following figure illustrates the game tree of a dynamic game $G$ between two players:

(a) (10 points) How many information sets does Player 1 have? What about Player 2? Calculate the Subgame Perfect Nash Equilibrium/a (SPNE) for the game.

Consider now that in the second time Player 1 has to choose an action, he doesn't know the action taken by Player 2 .
(b) (15 points) Find all the Nash Equilibria (pure and mixed) for the subgame(s) different than the whole game. Find analogously all the Subgame Perfect Nash Equilibria for the whole game.

## SOLUTIONS:

(a) Player 1 has three information sets, while Player 2 has only one.

$$
S P N E=\{(S x y, D)\}
$$

(b) Now, Player 1 has two information sets, while Player 2 has only one.


Subgame:

|  | $L$ | $D$ |
| :---: | :---: | :---: |
| $x$ | $\underline{6}, \underline{3}$ | $-2,-2$ |
| $y$ | $-3,-3$ | $\underline{3}, \underline{6}$ |
|  |  |  |

PSNE subgame $=\{(x, L),(y, D)\}$
Mixed strategies:
$6 q-2(1-q)=-3 q+3(1-q) \quad \Rightarrow \quad q=\frac{5}{14}$
$3 p-3(1-p)=-2 p+6(1-p) \quad \Rightarrow \quad p=\frac{9}{14}$
MSNE subgame $=\left\{\left(\frac{9}{14} x+\frac{5}{14} y, \frac{5}{14} L+\frac{9}{14} D\right)\right\}$
$U_{1}(x, L)=6>4, U_{1}(y, D)=3<4$ and $U_{1}\left(\frac{9}{14}, \frac{5}{14}\right)=\frac{5}{14} 6+\frac{9}{14}(-2)=\frac{6}{7}<4$. So,

$$
S P N E=\left\{(T x, L),(S y, D),\left(S, \frac{9}{14} x+\frac{5}{14} y ; \frac{5}{14} L+\frac{9}{14} D\right)\right\}
$$

## 3. (30 points)

Two firms, $A$ and $B$, provide similar services. Both firms can choose the quality of the products they offer. Let $s_{A}$ be the quality of $A$ and $s_{B}$ the quality of $B\left(s_{i} \in[0,5]\right.$ for $\left.i=A, B\right)$. The total market size (i.e., total revenue that can be obtained) is $200 €$. Firm $A$ 's revenue is $R_{A}\left(s_{A}, s_{B}\right)=200[0.5+$ $\left.0.05\left(s_{A}-s_{B}\right)\right]+2 s_{A} s_{B}$ and firm $B$ 's revenue is $R_{B}\left(s_{A}, s_{B}\right)=200\left[0.5+0.05\left(s_{B}-s_{A}\right)\right]-2 s_{A} s_{B}$. The cost to firm $A$ in choosing $s_{A}$ is $s_{A}^{2}$, while the cost to firm $B$ in choosing $s_{B}$ is $0.5 s_{B}^{2}$.
(a) (12 points) If the two firms compete against each other by choosing their qualities simultaneously, what will be equilibrium qualities? How much profit will each firm get?
(b) (10 points) Now before the two firms choose their qualities, firm $A$ can launch an advertisement campaign that costs $C$ euros. The advertisement will lead to a small change in the consumer tastes. To be precise, in the case of advertisement campaign, revenues of the firm $A$ become $R_{A}\left(s_{A}, s_{B}\right)=200\left[0.6+0.05\left(s_{A}-s_{B}\right)\right]+2 s_{A} s_{B}$ and of firm $B$ becomes $R_{B}\left(s_{A}, s_{B}\right)=$ $200\left[0.4+0.05\left(s_{B}-s_{A}\right)\right]-2 s_{B} s_{A}$. For which values of $C$ the firm $A$ will decide to undertake the advertisement campaign in the SPNE? Explain your answer. Calculate the profits of both firms if $A$ decides to launch the campaign and $C=45$.
(c) (8 points) Following from part (b), let $C=45$. Suppose now that firm $B$ incurs costs of 40 in order to enter the market. The sequence of actions is the following. First, firm $A$ decides whether it should launch the advertisement campaign or not. After observing $A$ 's choice, $B$ decides whether to enter the market or not. If firm $B$ does not enter $A$ gets the whole market, that is, $R_{A}\left(s_{A}\right)=200$. If firm $B$ enters, both compete like in part (a) if $A$ does not launch any campaign, or like in (b) if it does. In the SPNE: (i) should firm $A$ spend money on advertisement? (ii) should firm $B$ enter the market? Explain your answer. (iii) What is the profit of each firm in the SPNE?

## SOLUTIONS:

(a) To find the best response:

$$
\begin{aligned}
\frac{\partial}{\partial s_{A}}\left[200\left(0.5+0.05\left(s_{A}-s_{B}\right)\right)+2 s_{A} s_{B}-s_{A}^{2}\right] & \rightarrow s_{A}=5+s_{B} \\
\frac{\partial}{\partial s_{B}}\left[200\left(0.5+0.05\left(s_{B}-s_{A}\right)\right)-2 s_{A} s_{B}-0.5 s_{B}^{2}\right] & \rightarrow s_{B}=10-2 s_{A}
\end{aligned}
$$

Solving the system, we find

$$
N E=\left(s_{A}=5, s_{B}=0\right)
$$

Then, plugging the qualities in the profit functions, we have:

$$
\Pi_{A}=125 \text { and } \Pi_{B}=50
$$

(b) If $A$ chooses to launch the advertisement campaign, the equilibrium qualities do not change. So now, $A$ 's profits would become:

$$
\Pi_{A}=200(0.6+0.05 \times 5)-25-C=145-C
$$

Thus, firm $A$ will undertake the advertisement campaign if $145-C \geq 125$, i.e., if $C \leq 20$.

If $C=45$ and firm $A$ launches the campaign, the profits would be:

$$
\Pi_{A}=100 \text { and } \Pi_{B}=30
$$

(c) If $A$ launches the campaign and $B$ enters, $B$ 's profits would be $\Pi_{B}=30-40=-10$. So, in case of advertisement, $B$ will not enter the market. In that case, $A$ would choose $s_{A}=0$ and get profits $\Pi_{A}=200-45=155$.

If $A$ chooses do not advertise and $B$ enters, the firms' profits, in equilibrium, would be $\Pi_{A}=125$ and $\Pi_{B}=50-40=10$.

Therefore, by backwards induction, we conclude that firm $A$ will advertise (since $155>125$ ) and $B$ will not enter.

The firms' profits will be $\Pi_{A}=155$ and $\Pi_{B}=0$.

## 4. ( 25 points)

Three neighborhoods have to choose the common municipal budget $x$ from the set of whole numbers $X=\{0,1,2, \ldots, 99,100\}$. The blue neighborhood is represented by agent B , the white neighborhood by agent W and the green neighborhood is represented by agent G , whose utility functions are:

$$
U_{B}(x)=100-x, \quad U_{W}(x)=x, \quad U_{G}(x)=\left\{\begin{array}{cll}
50 & \text { if } \quad x \leq 50 \\
x & \text { if } & x \geq 50
\end{array}\right.
$$

The current budget of the municipality is $\widehat{x}=70$.
City rules are such that, first, agent B proposes a budget of $x_{1} \in X$. Observing $x_{1}$, agent W proposes a budget of $x_{2} \in X$. Observing both options, all agents choose the budget using a simple majority rule in a two step vote. In the first round, they choose between budgets $x_{1}$ or $x_{2}$ and in the second round, the winner of the first round competes against initial budget $\widehat{x}$. The winner of the second vote will be the implemented budget of the municipality.
(a) (12 points) Indicate which budget will win the second round if they choose the budget 20 in the first round. What if they choose 60 ? What if they choose 80 ? In each of these three cases, describe a Nash equilibrium in which no voter uses a weakly dominated strategy that support the indicated results.

Using your answers above, describe the budget that is to be chosen in the second round, depending on the winner of the first round. That is, if we call $y$ the winning budget of the first round, and if we call $z$ the winning budget of the second round, find the function $z=f(y)$.
(b) (5 points) If B has proposed $x_{1}=0$, what will agent W propose as his budget $x_{2}$, so that $x_{2}$ win against $x_{1}$ in the first round of the vote and also win against $\widehat{x}$ in the second round of the vote? Repeat the exercise for $x_{1}=80$.
Using your answers above, find the best response of agent W for each budget $x_{1}$ proposed by agent B, i.e. the value of $x_{2} \underline{\text { for all values of }} x_{1}$.
(c) (8 points) Given your answers to parts (a)-(b), indicate which budget shall be adopted in a SPNE. Note: You only need to find the equilibrium path, i.e. the proposed equilibrium budgets $x_{1}$ and $x_{2}$ and subsequent votes in each round, given that no player can use weakly dominated strategies.

## SOLUTIONS:

(a) If 20 is chosen in the first round, 70 wins in the second round. $N E=(20,70,70)$.

If 60 is chosen in the first round, 70 wins in the second round. $N E=(60,70,70)$.
If 80 is chosen in the first round, it wins again in the second round. $N E=(70,80,80)$.
In general, we have

$$
z(y)=\left\{\begin{array}{cc}
70 & y \leq 70 \\
y & y \geq 70
\end{array}\right.
$$

(b) If $B$ proposes $x_{1}=0, W$ proposes $x_{2} \geq 70$ so to win in both rounds. If $x_{1}=80, W$ proposes $x_{2} \geq 80$ so to win in both rounds. If $W$ wants also to maximize his utility, he proposes $x_{2}=100$.

In fact,

$$
B R_{W}\left(x_{1}\right)= \begin{cases}100 & \text { for all } x_{1} \\ x_{2} \in X & \text { if } x_{1}=100\end{cases}
$$

(c) $B$ proposes any $x_{1}$ since in equilibrium, the implemented budget is always $x=100$. Then, $W$ proposes $x_{2}=100$, and the votes in first and second rounds are $\left(x_{1}, 100,100\right)$ and $(70,100,100)$ respectively.

