P1	P2	P3	P4

GAME THEORY EXAM
(with SOLUTIONS)
January 2010

INSTRUCTIONS:

Write your answers in the space provided immediately after each question. You may use the back of each page. The duration of this exam is **2 hours**.

Name:
Degree:
Group:

1. (15 points)

Two neighbors are planning to construct a community swimming pool whose cost is 20 units (thousands of Euros) and the construction of which both neighbors value at 30 units each. They must come to an agreement by the following method. Each sends a closed envelope to a mediator containing their decision for or against building the swimming pool. If both are in favor, they share the cost equally between them; if just one is in favor that one must pay the whole cost; and if neither is in favor, the swimming pool will not be built.

a) (5 points) Represent this game in normal form and calculate all the Nash equilibria (in both pure and mixed strategies).

b) (10 points) Assume that Neighbor 1 values the swimming pool at 30 units and this is public information, while the valuation of Neighbor 2 is private information and could be either 30 or 10 units. Calculate the Bayesian Nash equilibria of this new game, given that Neighbor 2's valuation is equal to 30 units with probability γ.

SOLUTIONS:

(a)

	F	A
F	$20, 20$	$\underline{10}, \underline{30}$
A	$\underline{30}, \underline{10}$	$0, 0$

pure strategy NE: $\{(F, A), (A, F)\}$

mixed strategy NE: $\{\frac{1}{2}F + \frac{1}{2}A, \frac{1}{2}F + \frac{1}{2}A\}$

$$\begin{aligned} EU_1(F) &= 20q + 10(1-q) = EU_1(A) = 30q \rightarrow q = 1/2 \\ EU_2(F) &= 20p + 10(1-p) = EU_2(A) = 30p \rightarrow p = 1/2 \end{aligned}$$

$20q + 10(1-q) = 30q$, Solution: $\frac{1}{2}$

(b)

$V = 30$	F	A
F	$20, 20$	$10, \underline{30}$
A	$30, \underline{10}$	$0, 0$

$V = 10$	F	A
F	$20, 0$	$10, \underline{10}$
A	$30, -10$	$0, \underline{0}$

- Player 2's best response:

$$\begin{aligned} BR_2(F) &= AA \\ BR_2(A) &= FA \end{aligned}$$

- Player 1's best response:

	FA	AA
F	$20\gamma + 10(1-\gamma) = 10\gamma + 10$	$\underline{10}$
A	30γ	0

$$BR_1(FA) = \begin{cases} \text{F} & \text{if} \quad \gamma \leq 1/2 \\ \text{A} & \text{if} \quad \gamma \geq 1/2 \end{cases}$$

-BNE:

$$BNE = \begin{cases} \text{(F,AA)} & \text{for all } \gamma \\ \text{(A,FA)} & \text{if } \gamma \geq 1/2. \end{cases}$$

2. (25 points)

Miguel and Antonio face the following situation before canoeing down the river Sella. Miguel must choose whether to go down the river (D) or not go down (ND). If Miguel chooses not to go down the river, he gets a payoff of $2A$ and Antonio gets $3A$. If he chooses to go down the river, both of them must simultaneously choose the type of canoe each will use, either rigid (R) or flexible (F). The payoff matrix of the simultaneous game is

$Mig\backslash Ant$	R	F
R	$2,0$	$1,-2$
F	$0,-1$	$3,4$

a) (10 points) Find all the Nash equilibria (in both pure and mixed strategies) of the simultaneous game that starts if Miguel chooses to go down the river (D), and the utility that Miguel and Antonio would get in each of these Nash equilibria.

b) (5 points) Find all the payoffs for Miguel associated with the action ND (all the values for A) such that Miguel will always choose D in all of the SPNE (Subgame Perfect Nash Equilibria) of the game. Write down all the SPNE associated with these values of A.

c) (10 points) Now suppose that Antonio gets to observe the type of canoe that Miguel has chosen, before choosing his own. Find all the values of A such that in all the SPNE of this new game Miguel chooses to go down the river Sella (D).

SOLUTIONS:

(a)

$Mig\backslash Ant$	R	F
R	$\underline{2},\underline{0}$	$1,-2$
F	$0,-1$	$\underline{3},\underline{4}$

pure strategy NE: (R,R) and (F,F) with payoffs (2,0) and (3,4)

$$\begin{aligned} 2q+1-q &= 3-3q \\ -1+p &= -2p+4-4p \end{aligned}$$

mixed strategy NE: $\left(\frac{5}{7}R+\frac{2}{7}F;\frac{1}{2}R+\frac{1}{2}F\right)$ with payoffs $\left(1.5,-\frac{2}{7}=-0.28\right)$

Miguel's utility: 2, 3 and 1.5

Antonio's utility: 0, 4 and -0.28

(b)

For $A<0.75$ ($2A<1.5$) Miguel always choose D.

Thus, if $A<0.75$ the SPNE are: (DR,R) with payoffs $(2,0)$;

(DF,F) with payoffs $(3,4)$; and

$\left(D\frac{5}{7}R+\frac{2}{7}F;\frac{1}{2}R+\frac{1}{2}F\right)$ with payoffs $(1.5,-0.28)$

(c)

$A < 1.5$: In equilibrium, Antonio chooses RF with payoffs (2, 0) and (3, 4). Miguel, in his second info set chooses F since $3 > 2$, and will always choose D in his first info set iff $2A < 3$.

3. (30 points)

The firms *Andesa* and *Bertola* (A and B) are going to compete in a Cournot game in the production of solar panels from June 2010 on. They must decide simultaneously which of the two available technologies in the market they will use. One of the technologies is free, and would allow them to produce solar panels at a cost of 100 per unit; the other technology costs 2500 and would allow them to produce the solar panels at a cost of 50 per unit. The decisions taken by each of the firms will be announced the first of May. The market demand function for solar panels is $P(Q) = 200 - Q$, where $Q = q_A + q_B \in [0, 200]$.

a) (18 points) Find all/the SPNE (Subgame Perfect Nash Equilibria/um) in pure strategies of this game, and write them/it down.

b) (12 points) Suppose that both firms have adopted the free technology. The 1st of June the aim of firm A continues to be maximizing profits, but firm B is no longer interested in profits but in the difference between it's production and the production of firm A. More concretely, the new utility function of B is given by

$$U_B(q_A, q_B) = (q_B - q_A)^2.$$

Find the SPNE of the new game given that firm B is now the market leader, so that firm A observes how much firm B chooses to produce before having to decide its own production. How much does each firm produce in SPNE?

SOLUTIONS:

Note: We can identify the actions in the first stage of the game either by the marginal costs associated with the two technologies (50 and 100), or directly by the technologies (Free and NotFree). Here, we are going to use the first option.

(a)

Stage 2: $\max_{q_i} (200 - q_i - q_j - c_i)q_i$

$$q_i = \frac{200 - c_i - q_j}{2}$$

$$\begin{aligned} EN \quad &: \quad q_i = \frac{200 - 2c_i + c_j}{3} \\ \pi_i^{EN} \quad &= \quad q_i^2 \end{aligned}$$

There are 4 subgames in the stage 2, corresponding to the 4 possible outcomes from the stage 1: (50,50), (50, 100), (100,50) and (100,100). The NE of these subgames are respectively:

$$\left\{ (50, 50), (\frac{200}{3}, \frac{50}{3}), (\frac{50}{3}, \frac{200}{3}), (\frac{100}{3}, \frac{100}{3}) \right\}$$

Stage 1:

	$50(NotFree)$	$100(Free)$
$50(NotFree)$	$0,0$	$\frac{17500}{9},\frac{2500}{9}$
$100(Free)$	$\frac{2500}{9},\frac{17500}{9}$	$\frac{10\,000}{9},\frac{10\,000}{9}$

$$SPNE = \left((50_{(NotFree)},50,\frac{200}{3},\frac{50}{3},\frac{100}{3}),(100_{(Free)},50,\frac{50}{3},\frac{200}{3},\frac{100}{3})\right) \text{ y}$$
$$\left((100_{(Free)},50,\frac{200}{3},\frac{50}{3},\frac{100}{3}),(50_{(NotFree)},50,\frac{50}{3},\frac{200}{3},\frac{100}{3})\right)$$

(b)

Firm A has the same reaction function from part a:

$$q_A(q_B) = \max\left\{\frac{100-q_B}{2},0\right\}$$

Firm B maximizes the convex function:

$$U_B(q_A(q_B),q_B) = \begin{cases} (\frac{3q_B}{2}-50)^2 & if\ q_B \leq 100 \\ q_B^2 & if\ q_B \in [100,200] \end{cases}$$

Thus, in the SPNE, the quantities are $(q_B = 200, q_A = 0)$.

4. (30 points)

Two live music venues, Amadeus and Bachata, are located nearby each other and each has a loyal clientele, estimated to be 100 people per night for Amadeus and 50 for Bachata.

Both venues must decide whether or not to hire a famous musician, that will attract more clients than usual if the other venue does not do so as well. Amadeus can hire for one night the pianist Nizalbe, and Bachata can hire the singer Lizza. If Amadeus hires Nizalbe and Bachata doesn't hire anyone, Amadeus will get 40 extra clients and Bachata will lose 10. Similarly, if Bachata hires Lizza while Amadeus doesn't hire anyone, then it will get 50 extra clients, while Amadeus loses 30. Finally, if both venues hire famous musicians (that is, Amadeus hires Nizalbe, and Bachata hires Lizza) then they will get 20 and 10 extra clients respectively. The benefit of each client is 10 euros for Amadeus and 20 euros for Bachata. Calling N and L the prices of hiring Nizalbe and Lizza respectively, we have that $200 < L < 400$ and $200 < N < 400$.

a) (8 points) Represent this game, and determine what will be the Nash equilibria in pure strategies, depending on the values of N and L.

b) (7 points) Both venues now consider adding to their previous options a new alternative: an open bar. If only one of them introduces an open bar it will get all of the clients, regardless of whether or not the other bar has live music. If both venues have an open bar, the clientes will stay in their preferred venue. The cost of an open bar is 200 euros for Amadeus and 100 euros for Bachata. Find the rationalizable strategy profile for this new game.

c) (15 points) Suppose now that both venues consider competing not just one night, but indefinitely, and that running an open bar has been banned by the health authorities of the country. For which values of intertemporal discount rate δ is it possible to find a SPNE in which both venues decide not to hire any musicians when $L = N = 300$? Write the strategies that allow this equilibrium.

SOLUTIONS:

(a)

Normal form:

	H	NH
H	$\underline{1200-N}, \underline{1200-L}$	$\underline{1400-N}, 800$
NH	$700, \underline{2000-L}$	$1000, 1000$

pure strategy NE: (H, H).

(b)

Normal form:

	H	NH	OB
H	$1200-N, 1200-L$	$1400-N, 800$	$-N, \underline{2900}$
NH	$700, 2000-L$	$1000, 1000$	$0, \underline{2900}$
OB	$\underline{1300}, -L$	$\underline{1300}, 0$	$\underline{800}, \underline{900}$

There is only one rationalizable strategy profile: the unique NE in dominant strategies (OB, OB).

(c)

Suppose now that both players use the following strategy:

-In $t = 1$, play NH

-For $t > 1$:

-Play NH if (NH, NH) was played in every period s, $s = 1, ..., t-1$

-Play H otherwise.

It is easy to see that a deviation is more profitable for the player Bachata. Therefore, let's see what it gets deviating or following the proposed strategy:

Case 1: if Bachata always follows the strategy, it gets:

$$U_2((NH, NH), (NH, NH), ...) = 1000 + 1000\delta + ... = \frac{1000}{(1-\delta)}$$

Case 2: if Bachata deviates (at $t = 1$), it gets:

$$U_2((NH, H), (H, H), (H, H), ...) = 1700 + 900\delta + 900\delta^2 + ... = 1700 + \frac{900\delta}{(1-\delta)}$$

The proposed strategy profile will be a NE iff:

$$\begin{aligned} 1700 + 900\delta/(1-\delta) \quad &\leq \quad 1000/(1-\delta) \Longleftrightarrow 1700(1-\delta) + 900\delta \leq 1000 \Longleftrightarrow \\ &\Longleftrightarrow \quad 700 \leq 800\delta \Longleftrightarrow \frac{7}{8} \leq \delta \end{aligned}$$

In the subgames after some outcome different than (NH, NH), the players play a NE, since they play the NE of the stage game repeatedly. The subgames after a history of (NH, NH)'s are just like the whole game. Thus, the proposed strategy profile is also a NE for these subgames iff $\frac{7}{8} \leq \delta$.

Therefore, if $\frac{7}{8} \leq \delta$, the players are playing a NE in all the subgames and, therefore, the strategy profile is a SPNE.