I	II.1	II.2	II.3	II.4	Total

Game Theory
December exam 2019

Name: **Group:**

You have two hours and a half to complete the exam. No calculators or any other electronic devices are allowed.

I Short questions (5 points each)

I.1 Consider the following game:

		Player 2 C	Player 2 D
Player 1	C	1, 1	0, 0
	D	x, x	2, 1

Find the Nash equilibria when $0 < x < 1$.

I.2 The only Nash equilibrium in a 2-players, 2-strategy game is an equilibrium in mixed strategies where both players use each of their strategies with some positive probability. Can any of the players have a strictly dominated strategy?

I.3 Find an example of a perfect information dynamic game where the subgame perfect Nash equilibrium is not Pareto optimal.

I.4 Two firms compete *a la* Cournot in a market with demand given by $p = 9 - q$. Both firms' marginal costs are zero, but Firm 2 has a fixed cost of F if it wants to produce. For which values of F the firm will decide not to produce?

I.1 NE in pure strategies: (C, C), (D, D).
In mixed strategies: for Player 1 $q = xq + 2(1-q)$ implies $q = \frac{2}{3-x}$. For Player 2 $p + x(1-p) = 1 - p$ implies $p = \frac{1-x}{2-x}$.
The NE in mixed strategies is: $\left(\left(\frac{1-x}{2-x}[C], \frac{1}{2-x}[D]\right), \left(\frac{2}{3-x}[C], \frac{1-x}{3-x}[D]\right)\right)$.

I.2 No. Otherwise that player would be playing a dominated strategy with positive probability, and would increase her utility by playing that strategy with probability zero.

I.3 The Stackelberg model of oligopoly.

I.4 The interior conditions for a Cournot equilibrium give $q_1 = q_2 = \frac{9}{3} = 3$, with $p = 3$. Firm 2's profits would be $\Pi_2 = 9 - F$. Thus, Firm 2 will only produce if $F \leq 9$.

II. Problems (20 points each)

II.1 Suppose that there are 2 firms in the market offering differentiated products with market demands given by $q_1 = 100 - 2p_1 + p_2$ and $q_2 = 100 - 2p_2 + p_1$, respectively. The strategic variable is the price. Suppose that both firms have a marginal cost of €50.

(a) (2 points) Describe the set of players, the strategy sets, and the profit functions.
(b) (5 points) Compute and draw the best response functions for both firms.
(c) (5 points) Find the Nash equilibria and circle them in the previous graph. Compute the firms' profits.
(d) (5 points) Suppose now that the products are not differentiated so they compete *a la* Bertrand with market demand given by $q = 100 - p$. Which are the Nash equilibria?
(e) (3 points) Still assuming competition *a la* Bertrand as in (d), argue what would change in the equilibrium of the following situations (no computations required, but you can provide them if you want):
    i. Firm 1's marginal cost is still €50, but firm 2's marginal cost is €30.
    ii. A third firm with the same marginal cost of €50 enters the market.

(a) $N = \{\text{Firm 1, Firm 2}\}$.
$S_i = \{p \in [0, \infty)$.
$\Pi_i = (p_i - 50)(100 - 2p_i + p_j)$.

(b) $BR_i(p_j) = \frac{200+p_j}{4}$.

(c) $NE = \left(\frac{200}{3}, \frac{200}{3}\right)$. $\Pi_i = 555.28$.

(d) Both firms charge price equal €50 and make zero profits.

(e) (i) Firm 2 will set his price $p = 50 - \varepsilon$ to attract all the demand since Firm 1 will not be able to decrease its price more than €50.

(ii) Nothing changes.

II.2 Two staff managers in a sorority, the house manager and the kitchen manager, must select a resident assistant from a pool of three candidates, $\{a, b, c\}$. The house manager prefers a to b, and b to c. The kitchen manager prefers b to a, and a to c. The process is as follows: First, the house manager vetoes one of the candidates. Second, the kitchen manager vetoes one of the remaining two candidates. The remaining candidate is elected.

(a) (5 points) Model this situation as an extensive form game. How many subgames are there in the game? How many information sets does each player have? How many strategies are available for each player?
(b) (5 points) Find the subgame perfect Nash equilibria.
(c) (5 points) Are there any Nash equilibria that are not subgame perfect Nash equilibria? If so, give an example. If not, argue why not.
(d) (5 points) Now assume that the kitchen manager can choose between playing as before and being the one playing first. What would she choose?

(a) $S_H = \{a, b, c\}$, $S_K = \{baa, bab, bca, bcb, caa, cab, cca, ccb\}$
The house manager has 3 available strategies, the kitchen manager has 8 available strategies. The game has 4 subgames. The house manager 1 has 1 information set. The kitchen manager has 3 information sets.

(b) The SPNE is $\{b, cca\}$ and it is unique.

(c) Yes. For example, $\{b, bcb\}$ is a Nash equilibrium. Check that there is no unilateral profitable deviation for any of the two players.

(d) The kitchen manager would prefer to play first. His payoff in the new equilibrium ($\{a, ccb\}$) would be 2, whereas the equilibrium payoff in point (a) was equal to 1.

II.3 Consider the following normal form game:

		Player 2	
		P	H
Player 1	X	12, 12	6, 6
	Y	16, 6	8, 8

(a) (5 points) If the game is played 4 times, which are the subgame perfect Nash equilibria?
(b) (5 points) If the game is played infinitely many times, define a trigger strategy that sustains payoffs (12, 12) in each period. Find the discount factor that makes the trigger strategy a subgame perfect equilibrium.
(c) (6 points) Show that the trigger strategy in (b) is, in fact a subgame perfect Nash equilibrium.
(d) (4 points) Repeat point (b) adding a probability p in each period that the game continues to the next period.

(a) Play (Y, H) unconditionally all periods.

(b) Trigger strategy:

At $t = 1$ play (X, P).
At $t > 1$ play (X, P) if (X, P) was played in all periods $t' < t$,
play (Y, H) otherwise.

$u_1(\text{Trigger strategy, Trigger strategy}) = \frac{12}{1-\delta}$,
$u_1(\text{Dev. in first period, Trigger strategy}) = 16 + \frac{8\delta}{1-\delta}$.
$\frac{12}{1-\delta} \geq 16 + \frac{8\delta}{1-\delta}$ if $\delta \geq 0.5$.

(c) Show that the trigger strategy is a Nash equilibrium:
After the calculations in (b) it is enough to show that the best deviation is a deviation in the first period:
-A deviation in any other period is the same as the deviation in the first period with payoffs computed at the beginning of that period.
-A deviation in more than one period give 6 rather than 8 in any deviating period other than the first, and the same payoff of 8 in non-deviating periods. Thus, it its worse that the one-period deviation.

Show that the trigger strategy is a Nash equilibrium in subgames after a sequence of (X, P) play:
-The subgame is the same as the original game and the trigger strategy requires the same play as in the original game. Thus, the same argument as before works here.

Show that the trigger strategy is a Nash equilibrium in any other subgames:
-The trigger strategy requires the unconditional play of (Y, H) in all periods, i.e., the unconditional play of a NE in all periods. Thus, it is a NE in those subgames.

(d) As before with $u_1(\text{Trigger strategy, Trigger strategy}) = 12 + 12\delta p + \cdots = \frac{12}{1-\delta p}$,
$u_1(\text{Dev. in first period, Trigger strategy}) = 16 + \frac{8\delta p}{1-\delta p}$. The new condition is $\delta p \geq 0.5$.

II.4 Eduardo and Judith attend a prestigious auction in London. Their intention is to bid for a jewel collection of roman art. They are the only bidders. It is well known that Judith is considering bidding either 5,000 or 10,000 euros, and that Eduardo is considering bids of 5,000 or 11,000. The auction is of the closed-envelope, first-price kind. The highest bid gets the collection and, in case of a tie, it will be assigned at random between the bidders with equal probabilities. Both players know that Eduardo values the collection at 15,000, but only Judith knows her own valuation, that is 7,000 with probability p, with $0 < p < 1$ or 12,000 euros with probability $1 - p$.

(a) (4 points) Show the Bayesian game, defining all its elements.
(b) (5 points) Find all Bayesian Nash equilibria in non-weakly dominated pure strategies for all values of p.
(c) (8 points) Repeat (a) and (b) for the case of a second-price auction.
(d) (3 points) Compare the equilibria in both cases (strategies, players' utilities and the revenues for the auctioneer).

(a) Players: {Eduardo, Judith}
Types of Eduardo: {E15}
Types of Judith: {J7, J12}
Beliefs: $p(\text{E15}|\text{J7}) = 1,$
$p(\text{E15}|\text{J12}) = 1,$
$p(\text{J7}|\text{E15}) = p, p(\text{J12}|\text{E15}) = 1 - p.$
Strategies for Eduardo: {5, 11}. Strategies for Judith: {(5, 5), (5, 10), (10, 5), (10, 10)}.
Utilities:

p		Judith 7: 5	Judith 7: 10
Eduardo	5	5, 1	0, -3
	11	4, 0	4, 0

1-p		Judith 12: 5	Judith 12: 10
	5	5, 3.5	0, 2
	11	4, 0	4, 0

(b) For both J7 and J12 action 10 is weakly dominated. There are no other weakly dominated strategies.
After eliminating 10 for J7 and 10 for J12, Eduardo prefers to play 5 for all p. The SPNE is (5, (5, 5)) for all p.

(c) As before except that utilities are given by:

p		Judith 7: 5	Judith 7: 10
Eduardo	5	5, 1	0, 2
	11	10, 0	5, 0

1-p		Judith 12: 5	Judith 12: 10
	5	5, 3.5	0, 7
	11	10, 0	5, 0

For both J7 and J12 action 5 is weakly dominated. For Eduardo 5 is dominated. The SPNE is (11, (10, 10)) for all p.

(d) Players bid more in the second price auction. Utilities are 5, 1, 3.5 for E15, J7 and J12, respectively in (b) and 5, 0, 0 in (c). Thus, Eduardo gets the same utility in both cases, but Judith is worse off in the second. The collection is sold at the price of 10 in (c), more than the price of 5 in (b).